The Latitude Dependence of the Variance of Zonally Averaged Quantities

GERALD R. NORTH, FANTHUNE J. MOENG,[1] THOMAS L. BELL AND ROBERT F. CAHALAN

Laboratory for Atmospheric Sciences, Goddard Space Flight Center, Greenbelt, MD 20771

(Manuscript received 19 March 1981, in final form 14 October 1981)

ABSTRACT

Zonally averaged meteorological fields can have large variances in polar regions due to purely geometrical effects, because fewer statistically independent areas contribute to zonal means near the poles than near the equator. A model of a stochastic field with homogeneous statistics on the sphere is presented as an idealized example of the phenomenon. We suggest a quantitative method for isolating the geometrical effect and use it in examining the variance of the zonally averaged 500 mb geopotential height field.

1. Introduction

Many investigators in recent years have reported climatological data in terms of zonal average statistics. This is in part due to the resulting compaction of the data sets, the smooth variation of most zonally averaged variables with latitude, and the importance of the zonally symmetric aspects of the atmosphere to an understanding of its general circulation. Examples of such compilations are those of NOAA Environmental Research Laboratories in the United States and those of the Soviet groups at Dubninsk and at the Main Geophysical Observatory in Leningrad.

From zonal averages it is a natural step to the examination of the variability of zonally averaged quantities as one measure of the dynamical activity of the atmosphere at different latitudes. For example, Vinnikov (1977), Yamamoto and Hoshiai (1979) and Weare (1979) report variances of zonally averaged temperature. Oort (1977) and Trenberth (1979) report variances of zonally averaged geopotential height for the Northern and Southern Hemispheres, respectively. In all cases the variance increases toward the poles.

Our purpose here is to point out that increase of the variance of a zonally averaged quantity with latitude can often be explained by geometrical considerations based solely on the spherical geometry of the earth. An increase in the variance of zonal means need not imply greater dynamical variability.

In the next section it will be shown that the variance of a zonally averaged stochastic field variable rises toward the poles with an approximate 1/cos(latitude) dependence when the field has statistics that are homogeneous on the sphere. The interpretation of the result is very simple, involving the standard error associated with finite area spatial averages—shorter latitude belts having a larger "sampling error" in estimating the true zonal mean. The result is a spatial analogue of Leith's (1973) work on finite length time averages leading to a kind of "climatic noise." In the final section we illustrate the geometrical effect with data for zonal averages of the 500 mb geopotential height. An appropriate rescaling of the data reveals the extent to which the geometrical effect contributes to the rise with latitude in the variance of the zonally averaged data. A similar rescaling can be applied to other data sets when examining the variance of zonal averages in order to learn what fraction of the change in variance with latitude is due to spherical geometry and what may be genuinely dynamical in origin.

2. A homogeneous noise model

We describe here a model to illustrate the effect of spherical geometry on zonal averages. Points on a sphere are labeled by the radial unit vector $\hat{\mathbf{r}}$. We construct a real, stochastic field $F(\hat{\mathbf{r}})$ on the sphere. The field is expanded in complex[2] spherical harmonics as

$$F(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_l^m(\hat{\mathbf{r}}), \qquad (1)$$

where the spherical harmonics are defined as $Y_l^m(\hat{\mathbf{r}}) = N_{lm} P_l^{|m|}(\sin\theta) e^{im\phi}$, and $P_l^{|m|}$ are associated Legendre polynomials, θ is latitude, ϕ is longitude, and the $N_{lm} = N_{l,-m}$ are real constants chosen so that the

[1] Applied Research Corporation, Landover, MD 20785.

[2] Since the field $F(\hat{\mathbf{r}})$ is real, the use of *complex* spherical harmonics is not strictly necessary, but does permit considerable simplification of the algebra needed to obtain the results below.

0027-0644/82/050319-08$06.00

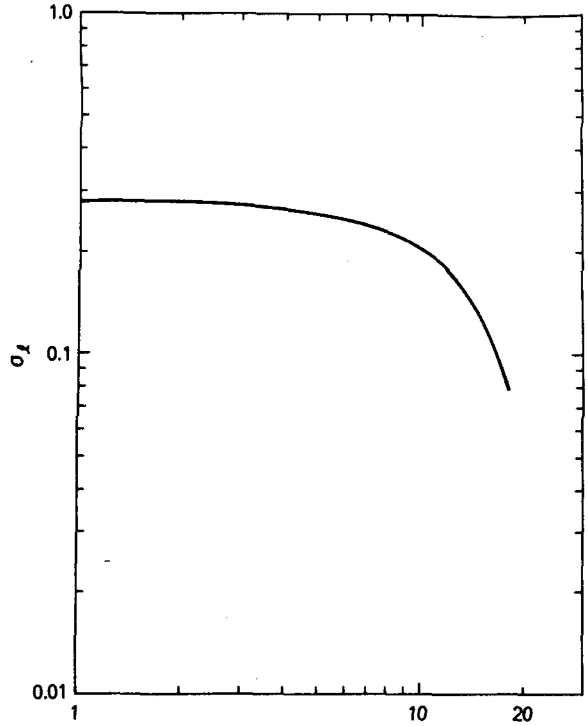

FIG. 1. Spectrum of spherical harmonic components of a random field with spatial correlation function the same as the correlation function of the 500 mb geopotential height field given in Eq. (7). A smooth curve has been drawn through the points.

spherical harmonics are normalized, i.e.,

$$\int d\Omega |Y_l^m|^2 = 1.$$

In particular, $N_{l0} = [(2l + 1)/4\pi]^{1/2}$. See, for example, Morse and Feshbach (1953) for a thorough discussion of the spherical harmonics. The f_{lm} are random complex variables, constrained by the realness of $F(\hat{\mathbf{r}})$ to satisfy $f_{lm} = f^*_{l,-m}$, where asterisks denote complex conjugation.

We show in Appendix A that if the f_{lm} are drawn from a distribution with ensemble statistics $\langle f_{lm} \rangle = 0$ and

$$\langle f^*_{lm} f_{l'm'} \rangle = \sigma_l^2 \delta_{ll'} \delta_{mm'} \qquad (2)$$

(angular brackets denote ensemble averages), then the (second order) statistics of $F(\hat{\mathbf{r}})$ are homogeneous on the sphere; in particular,

$$\langle F(\hat{\mathbf{r}}) \rangle = 0, \qquad (3)$$

$$\langle F(\hat{\mathbf{r}}) F(\hat{\mathbf{r}}') \rangle = \sigma^2 \rho (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'), \qquad (4)$$

where

$$\sigma^2 = (4\pi)^{-1} \sum_{l=0}^{\infty} (2l + 1) \sigma_l^2$$

and the correlation ρ of the field at one point with the field at another point depends only on the great circle distance between the two points. The reverse is also true: Eqs. (1) and (2) follow from Eqs. (3) and (4).[3] The quantity σ_l^2 is the spectrum of the field $F(\hat{\mathbf{r}})$ in the spherical harmonic representation, and is expressed in terms of the covariance of the field in Eq. (A8) of Appendix A. The variance of $F(\hat{\mathbf{r}})$, $\langle F(\hat{\mathbf{r}})^2 \rangle = \sigma^2$, is the same at each point on the sphere. Through Eqs. (1) and (2) we thus have a means of generating a stochastic field with statistics everywhere the same on the sphere.

While no true meteorological field is statistically homogeneous over the globe, a homogeneous model serves as a useful starting point in investigating the statistics of a field. The approximation of local homogeneity and isotropy of statistics is often convenient and is used in many practical applications, such as obtaining the geopotential height field from radiosonde data using optimum interpolation methods (Gandin *et al.*, 1972; Schlatter *et al.*, 1976; Bergman and Bonner, 1976; Julian and Thiébaux, 1975).

We now consider the zonally averaged field

$$[F(\theta)] \equiv \frac{1}{360°} \int_{-180°}^{180°} d\phi \, F(\theta, \phi), \qquad (5)$$

denoted by square brackets. It is shown in Appendix A that the variance of $[F(\theta)]$ may be expressed in terms of the spectrum σ_l^2 as

$$\langle [F(\theta)]^2 \rangle = \sum_{l=0}^{\infty} \sigma_l^2 [(2l + 1)/4\pi][P_l(x)]^2, \qquad (6)$$

where $x = \sin\theta$ and P_l are Legendre polynomials. Note that each term in Eq. (6) is positive and the maximum of each $[P_l(x)]^2$, $l > 0$, is at the poles, $x = \pm 1$. Hence no matter what the variance σ_l^2 in each mode of the model, the variance of zonal averages will be maximum at the poles.

As a specific example consider a stochastic field with the spectrum shown in Fig. 1. We have estimated this spectrum using Eq. (A8) of Appendix A with the spatial autocorrelation function taken from Julian and Thiébaux (1975):

$$R_1(s) = [\alpha J_0(\omega s) + \beta] e^{-\lambda s}, \qquad (7)$$

where $\alpha = 0.99$, $\beta = 0.01$, $\omega = 1.4$, $\lambda = 0.215$, and J_0 is the zero-order Bessel function. $R_1(s)$ represents a reasonable fit to the spatial autocorrelation function of the 500 mb geopotential height field over the United States for separation s in units of 1000 km in the zonal direction. We therefore create a sto-

[3] This is equivalent to the statement that the $Y_l^m(\hat{\mathbf{r}})$ are empirical orthogonal functions (EOF's) for any stochastic field with homogeneous statistics on the sphere (Obukhov, 1947), if the concept of EOF's is extended to include continuous as well as discretely gridded data. A thorough discussion of this may be found in North *et al.* (1982). An application to simple stochastic climate models appears in North and Cahalan (1981).

chastic field with the same statistics in all directions over the entire sphere as the height field has in the zonal direction over the United States. Correlation beyond 1500 km was not included in Eq. (A8) in calculating the spectrum.

The solid curve in Fig. 2 shows a plot of $\langle [F(\theta)]^2 \rangle$ obtained using (6) for the spectrum in Fig. 1. Note the strong peak in the polar region. It must be kept in mind, of course, that (7) is not valid at all latitudes, and so this result should not be considered a global theory of the zonally averaged 500 mb height field, but only as an idealized example of the geometrical effect we are describing.

The variance peaks at the poles lend themselves to a very simple interpretation suggested by elementary standard error theory. Suppose the decorrelation distance for the stochastic field $F(\hat{\mathbf{r}})$ is L [i.e., $\rho(L) \approx 1/e$] so that, roughly speaking, the distance between independent samples is $2L$. For a latitude belt of circumference C $(=2\pi R_E \cos\theta)$, the number of independent samples added in forming the zonal average $[F(\theta)]$ is $N = C/2L$ (N large). In forming the zonal average $[F(\theta)]$ of a single realization we are estimating the mean $\langle F(\theta) \rangle$. The variance of such an estimate is σ^2/N. Hence we would expect

$$\langle [F(\theta)]^2 \rangle \approx \sigma^2 \frac{2L}{2\pi R_E} \frac{1}{\cos\theta}. \qquad (8)$$

A more rigorous derivation of Eq. (8) is given in Appendix B. It is shown there that the approximation (8) is valid for large N when the correlation of the field at one point with the field at another point decreases smoothly with separation, i.e., without large oscillations.

The dashed line in Fig. 2 shows $\langle [F(\theta)]^2 \rangle$ multiplied by $\cos\theta$. As predicted by Eq. (8), it is virtually constant. From Eq. (8) we see that the constant value of the dashed curve is just $2L/2\pi R_E$ [since $\sigma^2 = R_1(0) = 1$], which is just the fraction of the circumference of the earth occupied by one independent sample length (about 2000 km in our example).

The argument leading to Eq. (8) suggests that the solid curve in Fig. 2 may be viewed (since $\sigma^2 = 1$) as a plot of $1/N$, the reciprocal of the number of independent samples contributing to the zonal average at each latitude. We see from Fig. 2 that north of about 80° the number of independent samples is fewer than three, and the dashed curve is beginning to deviate significantly from a constant value. This suggests that our sampling argument for Eq. (8) is reasonably trustworthy for $N \geqslant 3$ or $L \leqslant C/6$. In Appendix B we show that Eq. (8) describes the variance of zonal averages of any field with homogeneous statistics as long as the spatial correlation function of the field does not oscillate over long distances and correlation does not extend around a large fraction of the latitude circle (i.e., $L \ll C$). Consequently our criterion $N \geqslant 3$ is probably useful for a wide class of fields with statistics like those just described.

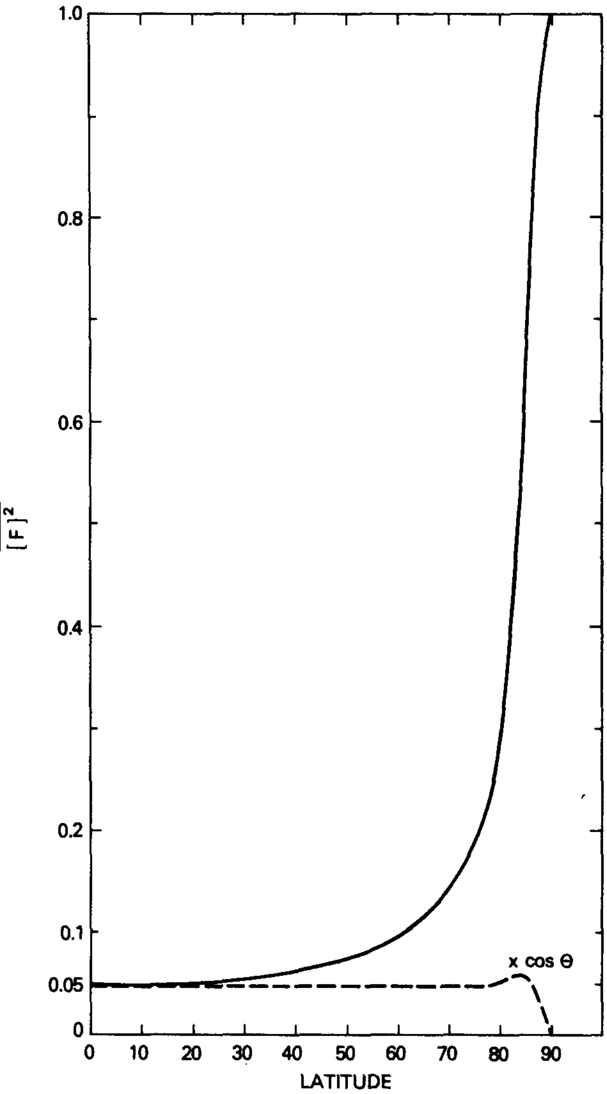

FIG. 2. The solid curve shows the variance of the zonal mean of the random field F computed from Eq. (6) using the spectrum plotted in Fig. 1. The dashed curve shows the solid curve multiplied by $\cos\theta$.

Sometimes data are averaged over a finite width latitude band before being examined for fluctuations. To see what effect averaging over latitude bands has in the model, Eq. (5) can easily be averaged over some interval of θ before the sum (6) is formed. The peaking at the poles is reduced by such a procedure, but not removed unless the band widths increase with latitude so as to contain equal numbers of independent samples.

3. Example using real data

As a further illustration of the geometrical enhancement of the variance of zonal averages we have

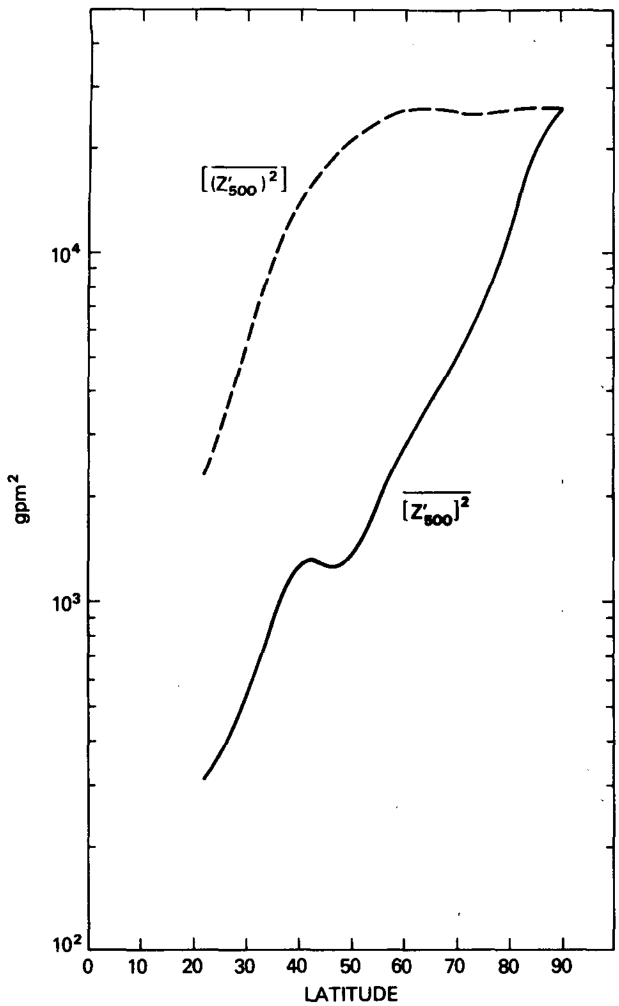

FIG. 3. The solid curve shows the variance of the zonally averaged 500 mb height field computed from daily (1200 GMT) NMC analyses for 15 successive Januaries (1963–77). The dashed curve shows the variances of the 500 mb height at each grid point, averaged around a latitude circle.

examined the daily (1200 GMT) National Meteorological Center (NMC) analyses of the January 500 mb geopotential height for 15 years (1963–77) made available by the National Center for Atmospheric Research (NCAR) tape library. The results are shown in Fig. 3.

The solid curve depicts $\overline{[z'_{500}(\theta)]^2}$, the variance of the zonally averaged geopotential height (primes indicate deviation from the mean, overbars time averages), whereas the dashed curve shows $[\overline{(z'_{500}(\theta, \phi))^2}]$, that is, the local variance at each grid point averaged around the latitude circle (ϕ = longitude). The two curves agree at the pole, as they must. The rapid rise in the grid-point variance $[\overline{(z'_{500})^2}]$ up to about 60°N suggests that much of the rise in the variance of the zonal mean below 60°N is dynamical in origin. However, above 60°N the grid-point variance changes very little, and the factor of 10 rise in the variance of the zonal mean appears to be due largely to the geometrical effect discussed here.

To justify this conclusion more quantitatively, we first attempt to factor out the portion of the rise in the variance attributable to a rise in the level of dynamical activity with latitude by dividing the variance of the zonal mean by the zonally averaged grid-point variance at each latitude, i.e., forming the ratio

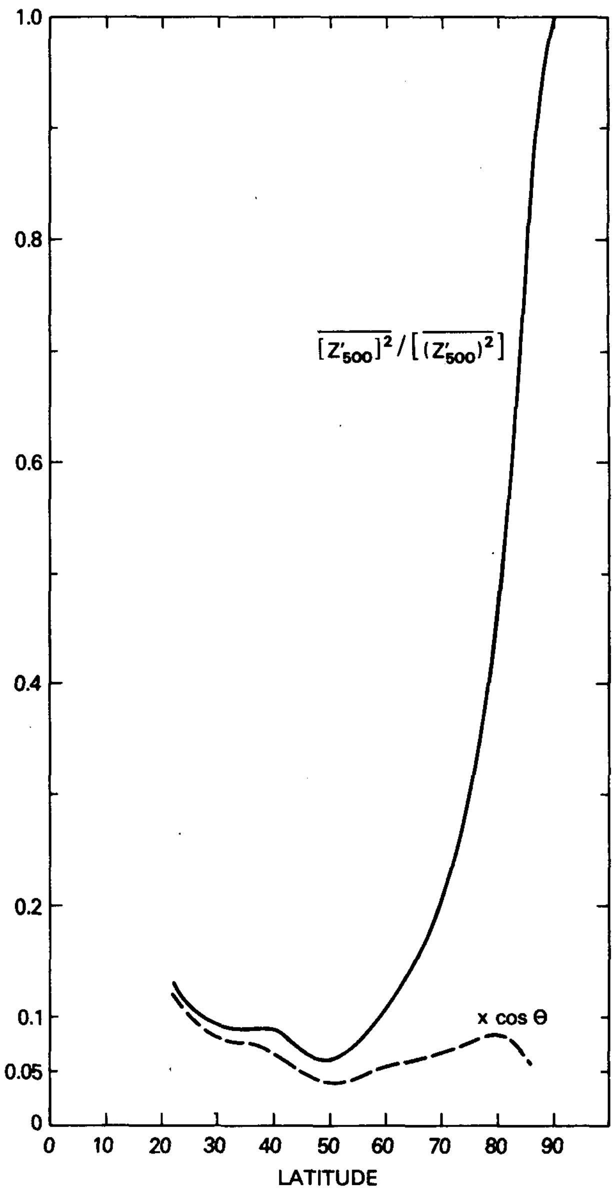

FIG. 4. The solid curve shows the ratio of the variance of zonally averaged geopotential height to the zonally averaged grid-point variance given in Fig. 3. The dashed curve results from multiplying the solid curve by cosθ. As discussed in the text, the dashed curve may be interpreted, below 75°N, as the correlation length typical of each latitude, in units of the earth's circumference.

$\overline{[z'_{500}(\theta)]^2}/[\overline{(z'_{500}(\theta, \phi))^2}]$. This ratio is plotted as the solid curve in Fig. 4. Then, as the results for the stochastic model studied in the previous section suggest, we multiply the ratio by $\cos\theta$ to eliminate the geometrical sampling effect. This is plotted as the dashed curve in Fig. 4. Since the cosine-weighted curve rises only slightly from 60°N to the pole, whereas the solid curve rises by a factor of 10, we conclude that most of the rise in variance of the zonal averages above 60°N is indeed due to the spherical geometry of the earth.

Earlier we were able to interpret the dashed, cosine-weighted curve as the fraction of the earth's circumference occupied by twice the correlation length L [see Eq. (8)]. The more detailed analysis of the variance of zonal averages given in Appendix B shows that this interpretation remains approximately valid even in the presence of inhomogeneity of the statistics, if the length $2L$ is interpreted as a local-variance-weighted, zonally averaged correlation length. This is important, since the grid-point variances vary by a factor of 3 around a latitude belt at 60°N (see, e.g., Blackmon, 1976). For this reason we are still able to interpret the dashed curve in Fig. 4 below 75°N as representing a zonal correlation length (in units of the earth's circumference) typical of that latitude. The length $2L$ is defined more precisely in Eqs. (B16) and (B18) of Appendix B as an integral length scale of the spatial correlation function.

The dip near 50°N in the dashed curve in Fig. 4 is probably more due to the strong negative correlations occurring near that latitude, which reduce the integral length scale, than to a decrease in the distances over which significant correlations extend. The rapid rise in the dashed curve toward the equator reflects the dramatic increase in zonal correlation lengths near the equator which is suggested, for example, by the work of Ghil *et al.* (1979).

4. Concluding remarks

We have shown that zonally averaged quantities can have large variances near the poles for purely geometrical reasons. This geometrical effect explains much of the rise in the variance of the zonally averaged 500 mb height field near the North Pole. Other meteorological fields such as wind fields may show this geometrical effect less clearly because of the strong spatial inhomogeneity in their statistics.

We have also shown that the ratio of the variance of zonal averages to the zonally averaged grid-point variance, $f = \langle[F(\theta)]^2\rangle/[\langle F(\theta, \phi)^2\rangle]$, weighted by the cosine of the latitude, may be interpreted as a zonal correlation length typical of that latitude, for latitudes where the ratio f is smaller than ⅓ (and assuming the spatial correlation function of the field does not oscillate strongly). A plot of this quantity, $f\cos\theta$, can reveal gross changes with latitude in the spatial correlations and may therefore be of more dynamical interest than a plot of the variance of zonal averages.

Many modeling studies indicate an increased sensitivity near the poles to changes in atmospheric CO_2 (Manabe and Wetherald, 1975, 1980) or to other external forcings (e.g., Salmun *et al.*, 1980). The "signal" of climatic change may be larger near the poles but so is the "noise" due both to the geometrical effect studied here and to genuine dynamical variability. The best latitude for detecting climatic change is not where the signal is largest but where the signal-to-noise ratio is largest. Wigley and Jones (1981) suggest, in fact, that this ratio is largest in mid-latitudes.

Acknowledgments. Some of these results were presented during a visit by one of us (GRN) to the USSR under the auspices of the Bilateral Agreement on Environmental Protection. Many conversations and exchanges of climatic data during this and previous visits were helpful to the present study. We especially thank M. I. Budyko, L. S. Gandin, K. Y. Vinnikov, G. V. Gruza and I. L. Karol. We would also like especially to thank D. Gutzler for advice about gaining access to the NMC analyses, B. Doty for help with special computer programs, and W. Baker, D. A. Short and J. Shukla for useful discussions.

APPENDIX A

Stochastic Field with Homogeneous Statistics on a Sphere

Given the expansion of the stochastic field in Eq. (1), and Eq. (2) for the statistics of the coefficients of the expansion, we shall show here that Eqs. (3) and (4) follow.

Eq. (3) follows immediately from the assumption that the ensemble mean of f_{lm} vanishes, i.e., $\langle f_{lm}\rangle = 0$. To prove Eq. (4), we construct the covariance of the field $F(\hat{\mathbf{r}})$:

$$\langle F(\hat{\mathbf{r}})F(\hat{\mathbf{r}}')\rangle = \sum_{lm}\sum_{l'm'}\langle f_{lm}f_{l'm'}\rangle Y_l^m(\hat{\mathbf{r}})Y_{l'}^{m'}(\hat{\mathbf{r}}'), \quad (A1)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sigma_l^2 Y_l^m(\hat{\mathbf{r}})Y_l^{-m}(\hat{\mathbf{r}}'), \quad (A2)$$

using Eq. (2) and $f^*_{l'm'} = f_{l',-m'}$. The addition theorem of spherical harmonics can be written as (see, for example, Morse and Feshbach, 1953)

$$\sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}})Y_l^{-m}(\hat{\mathbf{r}}') = (2l+1)P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')/4\pi, \quad (A3)$$

where P_l is the Legendre polynomial of order l and $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$ is the cosine of the angle between the two unit vectors. Using this theorem in Eq. (A2) gives

$$\langle F(\hat{\mathbf{r}})F(\hat{\mathbf{r}}')\rangle = \sum_{l=0}^{\infty} (2l+1)\sigma_l^2 P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')/4\pi. \quad \text{(A4)}$$

We see from Eq. (A4) that the covariance is a function only of the great circle distance between the two points $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$. The variance $\sigma^2 = \langle F^2\rangle$ at any point $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$ on the sphere is obtained from (A4) using the normalization convention for the Legendre polynomials $P_l(1) = 1$, so that

$$\sigma^2 = \sum_{l=0}^{\infty} (2l+1)\sigma_l^2/4\pi. \quad \text{(A5)}$$

Eq. (4) follows directly from Eqs. (A4), (A5), and the definition of the correlation function $\rho \equiv \langle F(\hat{\mathbf{r}})F(\hat{\mathbf{r}}')\rangle/\langle F^2\rangle$. From Eqs. (4) and (A4) we obtain the explicit expression

$$\rho(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \sum_{l=0}^{\infty} (2l+1)\sigma_l^2 P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')/4\pi\sigma^2 \quad \text{(A6)}$$

for the correlation function.

Given a correlation function $\rho(z)$, $z = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$, for a stochastic field as a function of great circle separation $s = R_E \cos^{-1}z$ between points on the surface of a sphere of radius R_E, we may obtain the "spectrum" σ_l^2 for the field by inverting Eq. (A6) using the orthogonality property of the Legendre polynomials

$$\int_{-1}^{1} dz\, P_l(z)P_{l'}(z) = [2/(2l+1)]\delta_{ll'}. \quad \text{(A7)}$$

Multiplying Eq. (A6) on both sides by a Legendre polynomial, integrating over z, and using (A7), we obtain

$$\sigma_l^2 = 2\pi\sigma^2 \int_{-1}^{1} \rho(z)P_l(z)dz \quad \text{(A8)}$$

for the spherical harmonic spectrum.

To derive Eq. (6) for the variance of the zonally averaged field $[F]$ defined in Eq. (5), we use the fact that the zonal average of any spherical harmonic Y_l^m with $m \neq 0$ vanishes, since $[e^{im\phi}] = 0$. The zonal average of Eq. (1) for the stochastic field F is therefore

$$[F(\theta)] = \sum_{l=0}^{\infty} f_{l0}[(2l+1)/4\pi]^{1/2}P_l(x), \quad \text{(A9)}$$

where $x = \sin\theta$, and we have used $Y_l^0 = [(2l+1)/4\pi]^{1/2}P_l(x)$. Squaring both sides of Eq. (A9) and ensemble averaging to obtain the variance of the zonally averaged field gives

$$\langle [F(\theta)]^2\rangle = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty} \langle f_{l0}f_{l'0}\rangle \times [(2l+1)^{1/2}(2l'+1)^{1/2}/4\pi]P_l(x)P_{l'}(x). \quad \text{(A10)}$$

Eq. (6) follows from this and Eq. (2).

APPENDIX B

Variance of Zonal Averages

We derive an expression for the variance of zonal averages that does not require the assumption of homogeneous statistics on the sphere. The approach is similar to Leith's (1973) treatment of time averages, and permits a more rigorous definition of what is meant by the correlation length L introduced in Eq. (8) of the paper.

We fix our attention on one latitude and define the zonal average of a field F as in Eq. (5). We suppress the latitude variable θ in much of what follows. The variance of the zonal mean of F is obtained by squaring Eq. (5) and averaging over the ensemble, i.e.,

$$\langle [F]^2\rangle = (360°)^{-2} \times \int_{-180°}^{180°} d\phi \int_{-180°}^{180°} d\phi' \langle F(\phi)F(\phi')\rangle. \quad \text{(B1)}$$

If we define the covariance of F as

$$c(\phi;\lambda) \equiv \langle F(\phi)F(\phi+\lambda)\rangle, \quad -180° < \lambda < 180°, \quad \text{(B2)}$$

so that λ is the longitudinal separation of the two points, and define the correlation ρ as

$$\rho(\phi;\lambda) \equiv c(\phi;\lambda)/[c(\phi;0)c(\phi+\lambda;0)]^{1/2}, \quad \text{(B3)}$$

we may write Eq. (B1) using these definitions as

$$\langle [F]^2\rangle = (360°)^{-2} \int_{-180°}^{180°} d\phi \times \int_{-180°}^{180°} d\lambda\, [c(\phi;0)c(\phi+\lambda;0)]^{1/2}\rho(\phi;\lambda). \quad \text{(B4)}$$

We now define the dimensionless correlation "length" $f(\phi)$

$$f(\phi) \equiv (360°)^{-1} \int_{-180°}^{180°} d\lambda\, \frac{c(\phi+\lambda;0)^{1/2}}{c(\phi;0)^{1/2}}\rho(\phi;\lambda). \quad \text{(B5)}$$

The interpretation of $f(\phi)$ depends on the degree of longitudinal inhomogeneity in the statistics of F. In many cases of geophysical interest the ratio $c(\phi + \lambda; 0)^{1/2}/c(\phi; 0)^{1/2}$ varies little from 1 over the longitudinal separations λ for which the correlation ρ is significant. We can therefore approximate (B5) by

$$f(\phi) \approx (360°)^{-1} \int_{-180°}^{180°} d\lambda\, \rho(\phi;\lambda). \quad \text{(B6)}$$

The approximation (B6) is exact for a field with statistics that are homogeneous in longitude (i.e., cylindrically symmetric). (The model described in Section 2 was homogeneous in both longitude and latitude.) To the extent that (B6) is a good approximation and ρ does not oscillate strongly with λ, $f(\phi)$ serves as a measure of the fraction of a latitude circle over which the field F is significantly correlated with itself at longitude ϕ.

Using the definition (B5) to rewrite Eq. (B4), we obtain

$$\langle [F]^2 \rangle = (360°)^{-1} \int_{-180°}^{180°} d\phi\, c(\phi; 0) f(\phi), \quad \text{(B7)}$$

$$= [c]f, \quad \text{(B8)}$$

where

$$[c] \equiv [c(\phi; 0)] \quad \text{(B9)}$$

and

$$f \equiv [c]^{-1} (360°)^{-1} \int_{-180°}^{180°} d\phi\, c(\phi; 0) f(\phi). \quad \text{(B10)}$$

The quantity $[c]$ is the variance of the field at each point averaged around a latitude circle; f is the average around the latitude circle of the correlation "length" $f(\phi)$ weighted by the local variance $c(\phi; 0)$. Note that the solid curves in Figs. 2 and 4 are plots of f versus latitude. The fraction f in Eq. (B8) determines how much lower the variance of the zonally averaged field is than the point variance of the field. We may therefore consider $1/f$ as the number N of independent samples, or effectively uncorrelated segments, entering into a zonal average at a given latitude. Whether or not this is a useful interpretation depends on how homogeneous the statistics of the field are with longitude. Since the use of zonal averaging, to be interesting, presupposes a certain approximation to zonal homogeneity in the statistics of the field, the interpretation suggested above is probably a useful one in all cases where zonal averaging is a plausible format in which to describe the field.

Suppose that approximation (B6) may be used for $f(\phi)$. We change variables from λ to x, with

$$x \equiv C\lambda/360°, \quad \text{(B11)}$$

$$C \equiv 2\pi R_E \cos\theta, \quad \text{(B12)}$$

so that x measures distance along a latitude circle from longitude ϕ and C is the circumference of the latitude circle. Eq. (B6) then becomes

$$f(\phi) = C^{-1} \int_{-C/2}^{C/2} dx\, \rho(\phi; 360°\, x/C) \quad \text{(B13)}$$

or

$$f(\phi) = 2L(\phi)/(2\pi R_E \cos\theta), \quad \text{(B14)}$$

with the definition

$$2L(\phi) \equiv \int_{-C/2}^{C/2} dx\, \rho(\phi; 360°\, x/C). \quad \text{(B15)}$$

If we make the assumption that correlations beyond a certain distance $d \ll C$ may be neglected, then the integral in Eq. (B15) is adequately approximated by

$$2L(\phi) \approx \int_{-d}^{d} dx\, \rho(\phi; 360°\, x/C). \quad \text{(B16)}$$

Eq. (B16) allows us to identify $L(\phi)$ as a correlation length.

Having defined the quantities f and $L(\phi)$ above, we are ready to give a more rigorous justification of Eq. (8). Substituting Eq. (B14) for $f(\phi)$ in Eq. (B10) for f, we obtain

$$f = 2L/(2\pi R_E \cos\theta), \quad \text{(B17)}$$

with

$$L \equiv [c]^{-1} (360°)^{-1} \int_{-180°}^{180°} d\phi\, c(\phi; 0) L(\phi). \quad \text{(B18)}$$

The length L is the average of $L(\phi)$ around the latitude circle, weighted by the local variance $c(\phi; 0)$. To justify Eq. (8), we return to the assumption of homogeneous statistics of F on the sphere. For latitudes where the approximation (B16) is valid, where $L(\phi)$ may be evaluated by integrating over a small fraction of a latitude circle, the integral in (B16) will be nearly independent of latitude since ρ depends only on great circle separation and integration around a small portion of a latitude circle may be adequately approximated by integration along a nearby great circle path. Combining (B16) and (B18), we obtain an expression for L valid everywhere except near the poles [where (B16) is invalid because correlation extends completely around latitude circles]:

$$L \approx \int_0^d ds\, \rho[\cos(s/R_E)]. \quad \text{(B19)}$$

Here $\rho(z)$ is the spatial correlation function of F given in Eq. (4) and s measures distance along a great circle. By substituting Eq. (B17) for f into Eq. (B8) for the variance of $[F]$, we obtain Eq. (8):

$$\langle [F]^2 \rangle = [c] 2L/(2\pi R_E \cos\theta). \quad \text{(B20)}$$

Note that Eq. (B20) is still valid under the much weaker assumption that the statistics of F be approximately homogeneous in longitude only. For example, the variance $[c]$ may differ significantly from one latitude to another, as it does for the 500 mb height field discussed in Section 3. By forming the ratio

$$\langle [F]^2 \rangle \cos\theta/[c] = 2L/(2\pi R_E), \quad \text{(B21)}$$

which produced the dashed curves of Figs. 2 and 4, one obtains L, a measure of the correlation length (B16) zonally averaged according to Eq. (B18). This too may differ from one latitude to another. A constant value of the ratio over a band of latitudes is an indication of relatively homogeneous zonal correlation lengths in those latitudes.

REFERENCES

Bergman, K. H., and W. D. Bonner, 1976: Analysis error as a function of observation density for satellite temperature soundings with spatially correlated errors. *Mon. Wea. Rev.*, **104**, 1308–1316.

Blackmon, M. L., 1976: A climatological spectral study of the 500 mb geopotential height of the Northern Hemisphere. *J. Atmos. Sci.*, **33**, 1607–1623.

Gandin, L. S., R. L. Kagan and A. I. Polischuk, 1972: Estimation of the information content of meteorological observing system. *Tr. Gl. Geofiz. Obs.*, No. 286, 120–140.

Ghil, M., M. Halem and R. Atlas, 1979: Time-continuous assimilation of remote-sounding data and its effect on weather forecasting. *Mon. Wea. Rev.*, **107**, 140–171.

Julian, P. R., and H. J. Thiébaux, 1975: On some properties of correlation functions used in optimum interpolation schemes. *Mon. Wea. Rev.*, **103**, 605–616.

Leith, C. E., 1973: The standard error of time-average estimates of climatic means. *J. Appl. Meteor.*, **12**, 1066–1069.

Manabe, S., and R. T. Wetherald, 1975: The effects of doubling the CO_2 concentration on the climate of a general circulation model. *J. Atmos. Sci.*, **32**, 3–15.

——, and ——, 1980: On the distribution of climate change resulting from an increase in CO_2 content of the atmosphere. *J. Atmos. Sci.*, **37**, 99–118.

Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics*, Part II. McGraw-Hill, 1978 pp.

North, G. R., and R. F. Cahalan, 1981: Predictability in a solvable stochastic climate model. *J. Atmos. Sci.*, **38**, 504–513.

——, T. L. Bell, R. F. Cahalan and F. J. Moeng, 1982: Sampling errors in the estimation of empirical orthogonal functions. *Mon. Wea. Rev.*, **110** (in press).

Obukhov, A. M., 1947: Statistically homogeneous fields on a sphere. *Usp. Mat. Nauk (Adv. Math. Sci.)*, **2**, 196–198.

Oort, A. H., 1977: The interannual variability of atmospheric circulation statistics. NOAA Prof. Pap. No. 8, 76 pp.

Salmun, H., R. F. Cahalan and G. R. North, 1980: Latitude-dependent sensitivity to stationary perturbations in simple climate models. *J. Atmos. Sci.*, **37**, 1847–1879.

Schlatter, T. W., G. W. Branstator and L. G. Thiel, 1976: Testing a global multivariate statistical objective analysis scheme with observed data. *Mon. Wea. Rev.*, **104**, 765–783.

Trenberth, K. E., 1979: Interannual variability of the 500 mb zonal mean flow in the Southern Hemisphere. *Mon. Wea. Rev.*, **107**, 1515–1524.

Vinnikov, K., 1977: On the question of methods of collection and interpretation of data on the changes of surface air temperature of the northern hemisphere for the period 1881–1975. *Meteor. Gidrol.*, **9**, 110–114.

Weare, B. C., 1979: Temperature statistics of short-term climatic change. *Mon. Wea. Rev.*, **107**, 172–180.

Wigley, T. M. L., and P. D. Jones, 1981: Detecting CO_2 induced climatic change. *Nature*, **292**, 205–208.

Yamamoto, R., and M. Hoshiai, 1979: Recent change of the Northern Hemisphere mean surface air temperature estimated by optimum interpolation. *Mon. Wea. Rev.*, **107**, 1239–1244.