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Space–time scaling behavior of rain statistics in a stochastic fractional diffusion model

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Abstract

Statistical properties of rain exhibit the interesting feature that they depend in a nontrivial way on the length and time scales over which rain rate is averaged. A quantitative understanding of this dependence can be utilized to relate statistics at different scales and is important for inter-comparison of rainfall data obtained from measuring devices with differing space–time resolutions. A stochastic dynamical model of rainfall based on a fractional diffusion type kinetic equation introduced earlier by the authors describes fairly well how the second moment statistics of area-averaged rain rate depend on the averaging length L and predicts a power law scaling behavior as $L\rightarrow 0$. The model pictures the correlation of the precipitation field as arising from two-dimensional Lévy flights. The present paper extends the investigation to the full space–time covariance function of the precipitation field. In particular, a scaling regime is identified in which the various second moment statistics of area- and/or time-averaged rain field exhibit invariance under a combined rescaling of the space and time variables—a property known as dynamic scaling, the scaling exponent being identified with the Lévy index. Although the space and time scales resolved in the radar data used to establish the model turn out to be too coarse for the dynamic scaling behavior to be experimentally demonstrated, we predict that it should be observable in high frequency rain gauge data from dense gauge networks.

Keywords: Rain statistics; Space-time scaling; Stochastic models; Fractional diffusion

1. Introduction

Rain formation involves an inter-play of complex dynamical processes in the atmosphere, taking place at many different space and time scales. Such processes include nucleation and growth of raindrops within a cloud followed by their aggregation and break-up due to collisions and random motion due to turbulence in the course of their collective downward fall under gravity. Local physical conditions in the atmosphere that influence precipitation vary rapidly in an unpredictable manner. Consequently, a physically based prediction of the space–time distribution of precipitation is well nigh impossible. Instead, such a description is conveniently sought within the framework of a stochastic

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dynamical model. The instantaneous rain rate at a point on the ground can be regarded as a random variable whose statistical properties are characterized by a small number of empirical model parameters.

A considerable body of evidence now exists to indicate that statistical properties of precipitation fields exhibit self-similar behavior over a broad range of spatial and temporal scales. This manifests itself in scaling properties of the various statistics under a rescaling of the area and time over which rain is averaged. Models that describe statistics of point rainfall have the flexibility of being applicable to situations where precipitation data measured at different spatial and temporal resolutions, such as radar and rain gauge data, have to be compared. Statistics of instantaneous point rain rate can be averaged to any desired space and time scale dictated by the measurement method or numerical climate model. Remote sensing measurements like those carried out by ground-based radar or radar and microwave instruments on board low earth-orbiting satellites, such as the Tropical Rainfall Measuring Mission (TRMM), yield estimates of near-instantaneous area-averaged precipitation rate over large areas at various spatial resolutions. On the other hand, rain gauges directly observe precipitation rate at a point continuously over long periods of time. Comparison between the two types of data is facilitated by a statistical model that can be adapted to arbitrary averaging length and time scales (Bell and Kundu, 2003; Bowman et al., 2003).

In recent years at least two theoretical approaches have been developed to describe the space–time behavior of point rain rate statistics in terms of random processes which incorporate the scaling properties in a natural manner. The first approach describes rainfall as a multiplicative random process generated through exponentiation of an additive process. The multifractal models based on a self-similar multiplicative random cascade process belong to this class. Full space–time models of this type have been investigated by a number of authors (Marsan et al. (1996), Over and Gupta (1996) and Seed et al. (1999)). In the second type of approach the local precipitation rate is modeled by an additive random variable satisfying a stochastic dynamical equation. Such a model based on a simple diffusion type equation was first described by North and Nakamoto (1989). The model considered in this paper generalizes the North-Nakamoto model by introducing an equation involving fractional order spatial derivatives that describe a kind of anomalous diffusion. At small length and time scales the model statistics follow simple power law dependence thus indicating scaling behavior typical of fractals. However, deviation from self-similarity and scaling occurs as the distance and time scales approach the characteristic length and time parameters of the model and the mesoscale processes intervene. The model was first introduced in Bell and Kundu (1996) to fit the statistics of gridded ship-borne radar rainfall data from Global Atmospheric Research Project (GARP) Atlantic Tropical Experiment (GATE) conducted in 1974 in the Eastern Atlantic. More recently, it proved successful in describing second moment statistics of the gridded precipitation data obtained from two ship-borne radars during the Tropical Ocean Global Atmosphere–Coupled Ocean Atmosphere Response Experiment (TOGA-COARE) over the 4-month period November 1992 to February 1993 in the tropical Western Pacific (Kundu and Bell, 2003).

In this paper we demonstrate that at sufficiently small distances and times the statistics generated by the model remain invariant under the combined action of a space–time scale transformation, a property sometimes referred to as *dynamic scaling*. Empirical evidence for such a behavior in isolated storms has been presented by Venugopal et al. (1999) for the logarithmically transformed rain field. We here explore a physical setting in which this aspect of the model prediction could be experimentally verified. In Section 2 we review the model and discuss how the anomalous diffusion can be represented in terms of a simple mathematical picture of two-dimensional random flights. In Section 3 we discuss the scaling properties of the various second moment statistics of area- and/or time-averaged rain rate predicted by the model at sufficiently small space and time scales and obtain the dynamic scaling exponent. Section 4 concludes the paper with some discussions about directions for future work.

2. The model

In this section we describe a stochastic dynamical model that incorporates the self-similar behavior of rain statistics at small scales and derive its underlying dynamical equation from heuristic physical considerations. The statistics are assumed to be homogeneous and isotropic in space and stationary in time.

The rain rate field on the ground is in a sense a two-dimensional section of what in reality is a spatially coordinated process in three dimensions. Formation of rain patterns and their subsequent evolution is modeled as a diffusive process that can be represented in terms of random two-dimensional flights. We envision the space–time non-uniformities of the rain field as being aggregates of localized small-scale fluctuations. The physical scenario can be described as follows: a localized rain fluctuation at a point at time t is associated with the occurrence of others in its vicinity at time $t+\mathrm{d}t$. The physical association comes from the fact that the rain fluctuations originate through interacting dynamical processes at higher altitudes. One can think of this phenomenon as a sequence of instantaneous random jumps in the horizontal plane taking place during the time interval $\mathrm{d}t$, the probability of occurrence of a jump of a specified length vector $\boldsymbol{l}$ being $\gamma \mathrm{d}t$. The probability of m such occurrences in course of a finite time $t>0$ during which the system undergoes a total displacement $\boldsymbol{X}(t)=m\boldsymbol{l}$ is given by the familiar Poisson distribution

$$\Pr[\boldsymbol{X}(t) = m\boldsymbol{l}] \equiv p_m(t) = \exp(-\gamma t)\,(\gamma t)^m/m! \tag{1}$$

The characteristic function of the distribution is given by

$$\Phi_X(\boldsymbol{k},t) \equiv E\left[\exp\{i\boldsymbol{k}\cdot\boldsymbol{X}(t)\}\right] = \exp\left[-\gamma t\,(1-\mathrm{e}^{i\boldsymbol{k}.\boldsymbol{l}})\right] \tag{2}$$

where $E[\ldots]$ denotes the mean over an ensemble with distribution (1). Next consider a more complex process $\boldsymbol{X}(t)$ which is a sum of a continuous infinite number of such independent Poisson processes with the same rate constant γ for which the jump sizes are assumed to be continuously distributed with a (Lévy–Khintchine) measure $\mu(\mathrm{d}\boldsymbol{l})$. Since the jumps of different length vectors are assumed to be independent of one another, the characteristic function of the resulting random process (commonly known as a Lévy process) is simply an infinite product of factors (2) for all $\boldsymbol{l}$, which can be expressed in the form

$$\Phi_X(\boldsymbol{k},t) = \exp\left[-\gamma t \int (1-\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{l}})\mu(\mathrm{d}\boldsymbol{l})\right]. \tag{3}$$

We emphasize that the 'jump' model should be viewed as merely a formal mathematical representation rather than an actual physical description of the underlying cloud microphysical processes that end up producing the space–time correlation in rainfall. Note that the assumption that the rate constant γ is independent of $\boldsymbol{l}$ is not an essential restriction, since any such $\boldsymbol{l}$-dependence could be absorbed within the integration measure $\mu(\mathrm{d}\boldsymbol{l})$.

Since we intend to describe rain statistics that are spatially isotropic, the directions of the jumps are taken to be isotropically distributed, that is, all directions are assumed to be equally likely. In order to incorporate the scaling behavior of rain statistics, the jump sizes are assumed to follow a power law distribution with density $w(l) \sim l^{-(1+\alpha)}$ $(0<\alpha<2)$. The resulting random process is simply a Lévy flight with index α (Mandelbrot, 1982). In order to obtain a normalizable density it is convenient to introduce a sharp lower cut-off $l_0>0$ to the allowed jump lengths. Imposing the normalization condition $\int_0^\infty w(l)\mathrm{d}l = 1$ on the truncated density, we obtain the measure (φ denotes the polar angle)

$$\mu(\mathrm{d}\boldsymbol{l}) = w(l)\mathrm{d}l\,\mathrm{d}\varphi/2\pi \tag{4}$$

with

$$w(l) = \alpha l_0^\alpha l^{-(1+\alpha)}\Theta(l-l_0), \tag{5}$$

where $\Theta(x)$ is the unit step function defined as $\Theta(x)=1$ when $x>0$ and 0 otherwise.

Angular integration in (3) yields

$$\Phi_X(\boldsymbol{k},t) = \exp\left\{-\alpha l_0^\alpha \gamma t \int_{l_0}^\infty [1-J_0(kl)]l^{-(1+\alpha)}\mathrm{d}l\right\}, \tag{6}$$

where $J_0(x)$ denotes the usual Bessel function of order zero. Since $1-J_0(x)=\mathrm{O}(x^2)$ near the origin and the Lévy index α lies in the range $0<\alpha<2$, the radial integral converges if we let $l_0 \to 0$ in the lower limit. Doing so we obtain the simple approximate expression

$$\Phi_X(\boldsymbol{k},t) \approx \exp[-Dk^\alpha t], \tag{7}$$

where

$$D = \alpha b(\alpha)\gamma l_0^\alpha > 0 \tag{8}$$

is the 'generalized' diffusion coefficient and

$$b(\alpha) = \int_0^\infty \mathrm{d}x\, x^{-(1+\alpha)}[1 - J_0(x)] \tag{9}$$

is a numerical constant depending on the index α. For each t the Fourier transform

$$F(\boldsymbol{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \exp[-Dk^\alpha t]\mathrm{d}^2k \tag{10}$$

represents (up to an overall normalization factor) the p.d.f. (probability density function) of a symmetric α-stable Lévy distribution for the random variable $\boldsymbol{X}(t)$ of zero mean, a class of distributions discovered by Paul Lévy in the 1930s (Lévy, 1954). When $\alpha = 2$ one obtains the usual Gaussian normal distribution which represents ordinary diffusion due to Brownian motion.

Unlike the normal distribution, the Lévy distributions possess power law tails proportional to $x^{-(1+\alpha)}$ and the moments of order higher than the first all diverge. The quantity $A = (Dt)^{1/\alpha}$ measures the width of the distribution. As is well known, these distributions share with the normal distribution the property that they are 'stable under addition', i.e. the sum $\boldsymbol{X}(t) = \Sigma_j \boldsymbol{X}_j(t)$ of a number of independent random variables $\boldsymbol{X}_j(t)$ distributed according to the Lévy distribution law of index α also obeys the same law. The width parameter A characterizing the distribution of $\boldsymbol{X}(t)$ is related to the widths a_j of the p.d.f. of the individual components $\boldsymbol{X}_j(t)$ through the formula $A^\alpha = \Sigma_j a_j^\alpha$, which generalizes the additivity of the variance of normally distributed independent variables. Detailed mathematically rigorous expositions of various aspects of the stable distributions can be found in a number of standard references (see for example, Gnedenko and Kolmogorov, 1954; Feller, 1971; Samorodnitsky and Taqqu, 1994; Kahane, 1995).

The restriction $0 < \alpha < 2$ ensures that the function $F(\boldsymbol{x},t)$ is positive-definite as is necessary for it to represent a probability density. We shall shortly see that values of the so-called Lévy index α lying in this range are appropriate for describing realistic rain statistics.

In the present context, the function $F(\boldsymbol{x},t)$ is linearly related to the local instantaneous rain fluctuation field $R'(\boldsymbol{x},t) = R(\boldsymbol{x},t) - \langle R \rangle$ at a point $\boldsymbol{x}$ at time t, (prime denotes deviation from the mean). The corresponding Fourier amplitudes $a(\boldsymbol{k},t)$ are linearly related to the characteristic function $\Phi_X(\boldsymbol{k},t)$ and satisfy the equation

$$\frac{\mathrm{d}a(\boldsymbol{k}, t)}{\mathrm{d}t} = -Dk^\alpha a(\boldsymbol{k}, t).$$

This can be formally interpreted as a continuity equation (in the spatial Fourier domain) expressing conservation of probability, which however must be violated in reality as the precipitation process continues and new groups of fluctuations continually enter the region of the horizontal plane from above. This is incorporated in the model by simply adding a white-noise stochastic force $f(\boldsymbol{k},t)$ which acts as a source/sink term for the rain fluctuation field. Another needed refinement comes from the fact that the relaxation time τ_k of the Fourier modes $a(\boldsymbol{k},t)$ implied by the simple scale-invariant Lévy flight picture has the asymptotic behavior $\tau_k = 1/(Dk^\alpha) \to \infty$ as $k \to 0$. However, physical considerations demand that the long wavelength modes representing large-scale fluctuations be damped out in a finite time. In order to ensure this, we modify the Lévy flight model described above to one heuristically based on the stochastic dynamical equation

$$\frac{\mathrm{d}a(\boldsymbol{k}, t)}{\mathrm{d}t} = -\frac{1}{\tau_k} a(\boldsymbol{k}, t) + f(\boldsymbol{k}, t) \tag{11}$$

with

$$\tau_k = \tau_0 (1 + k^2 L_0^2)^{-\alpha/2} \tag{12}$$

where τ_0 is the limiting value of the relaxation time scale and L_0 is a characteristic length defined by the relation $D = L_0^\alpha/\tau_0$ which in effect separates the large and small scale fluctuations of the rain field and marks the outer limits of a scale-invariant regime of the model statistics. The white-noise forcing term $f(\boldsymbol{k},t)$ is assumed to have zero mean and δ-function covariance

$$\langle f(\boldsymbol{k}, t) f^*(\boldsymbol{k}', t') \rangle = 2\pi F_0 \delta(\boldsymbol{k} - \boldsymbol{k}')\delta(t - t'). \tag{13}$$

Mathematically, the model described by Eqs. (11) and (12) leads to a nonlocal integro-differential equation for the point rain rate in physical space–time, which can be formally expressed in terms of a stochastic differential equation of the diffusion type involving fractional order spatial derivatives. In recent years fractional kinetic equations, representing what is generally termed *anomalous diffusion* or *fractional diffusion,* have found application in the description of a wide variety of physical phenomena. The distribution function (10) for Lévy flights leads to a type of anomalous diffusion that describes faster spreading than ordinary diffusion and is referred to as *superdiffusion*. See, for example, the review articles by Metzler and Klafter (2000) or by Bouchaud and Georges (1990) and references cited therein.

The space–time covariance of the point rain rate field is defined as

$$c(\boldsymbol{\rho},\tau)=\langle R'(\boldsymbol{x},t)R'(\boldsymbol{x}',t')\rangle \tag{14}$$

where $\boldsymbol{\rho}=\boldsymbol{x}-\boldsymbol{x}'$ and $\tau=t-t'$. An explicit calculation yields the expression

$$c(\boldsymbol{\rho},\tau)=g_0\int_0^\infty \mathrm{d}\kappa\,\frac{\kappa}{h(\kappa)}J_0(\kappa\rho_*)\mathrm{e}^{-|\tau_*|h(\kappa)} \tag{15}$$

where $h(\kappa)=(1+\kappa^2)^{\alpha/2}$, $\rho_*=\rho/L_0$, $\rho=|\boldsymbol{\rho}|$, $\tau_*=\tau/\tau_0$ and the multiplicative factor g_0 is given by

$$g_0=\sqrt{\pi/2}F_0\tau_0/L_0^2. \tag{16}$$

The model is thus characterized by four parameters: an overall strength parameter g_0, characteristic time and length parameters τ_0, L_0 and the Lévy index α. As mentioned in the introduction, the model has already been shown to accurately describe the space–time statistics of precipitation in both GATE Phase I (Bell and Kundu, 1996) and in TOGA-COARE (Kundu and Bell, 2003). The GATE Phase I data set consisted of 4-km gridded rain rates in 1716 radar images at roughly 15 min interval over a 280× 280 km^2 region centered at the ship location. The TOGA-COARE data set was much larger and consisted of six subsets (2 ships and 3 monthly 'cruises'). Each subset contained radar images of a 128×128 km^2 square gridded on a 2 km spatial grid at roughly 10 min intervals. For the details of the data analysis and model fitting we refer the reader to the papers mentioned above.

3. Scaling properties of model rain statistics

In this section we examine the scaling behavior of the rainfall model outlined above under the action of a simultaneous rescaling of the space and time variables.

3.1. Dynamic scaling behavior of the point statistics

First we consider the scaling properties of the space–time covariance of the point rain rate. While the exact model defined by Eqs. (11)–(13) is not scale-invariant, there is indeed a 'scaling regime', i.e. a range of values of the space–time variables ρ, τ where the model does exhibit *approximate* scale-invariance. This can be easily seen as follows.

The precipitation spectrum implied by the exact model, which is the Fourier transform of the space–time covariance function $c(\rho,\tau)$, is given by

$$\tilde{c}(k,\omega)=\frac{F_0\tau_0^2}{(\omega\tau_0)^2+(1+k^2L_0^2)^\alpha}. \tag{17}$$

In the limit of large k, ω or more precisely, when

$$(kL_0)^2\gg 1,\quad (\omega\tau_0)^2\gg(kL_0)^{2(\alpha-1)} \tag{18}$$

our model spectrum (17) reduces to the form

$$\tilde{c}^{(\infty)}(k,\omega)=\frac{F_0\tau_0^2}{(\omega\tau_0)^2+(kL_0)^{2\alpha}}, \tag{19}$$

which (up to an overall factor) is invariant with respect to a scale transformation $k\to\lambda^{-1}k$, $\omega\to\lambda^{-\alpha}\omega$ in the Fourier domain:

$$\tilde{c}^{(\infty)}(\lambda^{-1}k,\lambda^{-\alpha}\omega)=\lambda^{2\alpha}\tilde{c}^{(\infty)}(k,\omega). \tag{20}$$

[Throughout this paper the superscript (∞) attached to a quantity denotes a scale-invariant approximation of that quantity in the limit $k\to\infty$, $\omega\to\infty$, corresponding to small space–time scales.]

The Fourier inverse transform of (19) $c^{(\infty)}(\rho,\tau)$, which approximates the true space–time covariance function $c(\rho,\tau)$ in the limit of small ρ, τ, can be expressed in the form

$$c^{(\infty)}(\rho,\tau)=g_0\int_0^\infty \mathrm{d}\kappa\,\kappa^{1-\alpha}J_0(\kappa\rho_*)\mathrm{e}^{-|\tau_*|\kappa^\alpha} \tag{21}$$

and has the scaling behavior

$$c^{(\infty)}(\lambda\rho,\lambda^\alpha\tau)=\lambda^{\alpha-2}c^{(\infty)}(\rho,\tau). \tag{22}$$

From Eq. (22) upon choosing the scale factor λ to be $\lambda=1/\rho_*$, it follows that $c^{(\infty)}(\rho,\tau)$ must have the functional form

$$c^{(\infty)}(\rho,\tau)=g_0\rho_*^{\alpha-2}\phi(\tau_*/\rho_*^\alpha;\alpha). \tag{23}$$

The form of the scaling function $\phi(\xi;\alpha)$ depends explicitly on the index α. Eq. (23) expresses the dynamic scale invariance of our statistical model: the function $c^{(\infty)}(\rho,\tau)$ remains invariant (up to an overall multiplicative factor) under the combined action of the space–time scale transformation

$$\rho_* \to \lambda\rho_*, \quad \tau_* \to \lambda^\alpha \tau_*. \tag{24}$$

The dynamic scaling exponent is simply the Lévy index α. Moreover, it is evident that the limiting behavior of $c^{(\infty)}(\rho,\tau)$ as ρ_*, $\tau_* \to 0$ is *non-uniform*, in the sense that it depends on the manner in which the origin is approached.

The non-uniformity is characterized by the ξ-dependence of the scaling function $\phi(\xi;\alpha)$. The asymptotic behavior of $\phi(\xi;\alpha)$ as function of the scaling variable $\xi = \tau_*/\rho_*^\alpha$ as $\xi \to 0$ and $\xi \to \infty$ are of interest in this context. They are easily determined from the fact that the full space–time covariance function $c(\rho,\tau)$ can be expressed in closed forms when one of its arguments vanish. We recall from Bell and Kundu (1996) that when $\tau = 0$, the spatial covariance function $c(\rho,0)$ be expressed in the form

$$c(\rho,0) = [g_0/\Gamma(1+\nu)]\, C_\nu(\rho_*), \tag{25}$$

where ν is related to the Lévy index α through $\alpha = 2(1+\nu)$, $C_\nu(z) = (z/2)^\nu K_\nu(z)$, $K_\nu(z)$ being the usual modified Bessel function of order ν and $\Gamma(z)$ is the Euler Γ-function. The range $0 < \alpha < 2$ implies $-1 < \nu < 0$. For the rain data sets that we have examined so far, it appears that in fact ν lies in the narrower range $-1/2 < \nu < 0$. For the six TOGA-COARE data sets the exponent ν was found to lie in the range -0.21 to -0.34 with most of the values clustered around $-1/4$, which corresponds to $\alpha = 3/2$. [This special case of the Lévy distribution, known as the Holtsmark distribution, arises naturally in stellar dynamics; see, e.g., Chandrasekhar (1943)].

From the asymptotic behavior of $C_\nu(z)$ as $z \to 0$ when $\nu < 0$, it follows that

$$c(\rho,0) = \frac{1}{2} g_0 \left[\frac{\Gamma(-\nu)}{\Gamma(1+\nu)}\right](\rho_*/2)^{-2|\nu|} + O(1). \tag{26}$$

Comparison with (23) yields the limiting behavior

$$\phi(0;\alpha) = C_1 \equiv \frac{\Gamma(1-\alpha/2)}{2^{\alpha-1}\Gamma(\alpha/2)}. \tag{27}$$

On the other hand, explicit computation yields the formula

$$c(0,\tau) = (g_0/\alpha)|\tau_*|^{-\varepsilon}\Gamma(\varepsilon;|\tau_*|) \tag{28}$$

where $\varepsilon = (2-\alpha)/\alpha = -\nu/(1+\nu)$ and $\Gamma(a;z) = \int_z^\infty t^{a-1}\, e^{-t} dt$ denotes the incomplete gamma function. Since the power law singularity in the time dependence as $\tau_* \to 0$ predicted by (28) must be consistent with the general functional form (23), one readily infers the asymptotic behavior

$$\phi(\xi;\alpha) \xrightarrow[\xi\to\infty]{} C_2 \xi^{-\varepsilon} \tag{29}$$

where $C_2 = (1/\alpha)\Gamma(\varepsilon)$ with $\varepsilon = (2-\alpha)/\alpha > 0$, indicating a power law fall-off as $\xi \to \infty$.

3.2. Scaling behavior of area- and/or time-averaged statistics

The instantaneous point rain rate is a highly singular mathematical quantity. Its statistical properties are not directly accessible to observations, which inevitably involve averaging over a certain area or time interval. Thus the statistics of area- and/or time-averaged rain rate are what the experimental data usually allows one to compute. In what follows we derive the scaling properties of the second moment statistics of space- and time- averaged precipitation fields.

First consider the statistics of area-averaged rain rate

$$R_A(t) = (1/A)\int_A d^2x R(\boldsymbol{x},t)$$

in an $L \times L$ square of area $A = L^2$.The lagged covariance function $c_{AA}(\tau) = \langle R'_A(t)R'_A(t')\rangle$ can be written as an integral over the point covariance function $c(\rho,\tau)$ as

$$c_{AA}(\tau) = (4/L^2)\int_0^L\int_0^L d\rho_1 d\rho_2 (1-\rho_1/L)(1-\rho_2/L)c(\rho,\tau) \tag{30}$$

where $\boldsymbol{\rho} = (\rho_1,\rho_2)$ and $\rho = |\boldsymbol{\rho}|$. In view of (23), as τ_*, $L_* = L/L_0 \to 0$, (30) takes the form

$$c_{AA}^{(\infty)}(\tau) = g_0 L_*^{-|\alpha-2|}\phi_1(\tau_*/L_*^\alpha;\alpha) \tag{31}$$

where $\phi_1(u;\alpha)$ is a dimensionless scaling function defined by

$$\phi_1(u;\alpha) = 4\int_0^1\int_0^1 dz_1 dz_2(1-z_1)(1-z_2) \times (z_1^2+z_2^2)^{-1+\alpha/2}\phi(u(z_1^2+z_2^2)^{-\alpha/2};\alpha). \quad (32)$$

Also, in the scaling limit $L_* \to 0$ the variance of area-averaged rain rate $\sigma_A^2 \equiv c_{AA}(0)$ is readily obtained by setting $\tau=0$ in Eq. (31). It exhibits the now-familiar power law singularity:

$$\sigma_A^{2(\infty)} = g_0\phi_1(0;\alpha)L_*^{-|\alpha-2|}. \quad (33)$$

Setting $u=0$ in (32) and using (27) we have

$$\phi_1(0;\alpha) = \frac{2^{3-\alpha}\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)}\int_0^1\int_0^1 dz_1 dz_2(1-z_1) \times(1-z_2)(z_1^2+z_2^2)^{-1+\alpha/2}.$$

It follows at once that in the scaling limit $\tau_*, L_* \to 0$, the lagged autocorrelation function of area-averaged rain rate $\chi_{AA}(\tau)=c_{AA}(\tau)/c_{AA}(0)$ depends on its two arguments τ and L through the single variable $u=\tau_*/L_*^\alpha$:

$$\chi_{AA}^{(\infty)}(\tau) = \chi_1(\tau_*/L_*^\alpha;\alpha) \quad (34)$$

where $\chi_1(u;\alpha)=\phi_1(u;\alpha)/\phi_1(0;\alpha)$ is a scaling function. Eq. (34) implies invariance of the autocorrelation function under a simultaneous rescaling of τ and L through the scale transformation

$$L \to \lambda L, \quad \tau \to \lambda^\alpha \tau. \quad (35)$$

The scaling function $\chi_1(u;\alpha)$ can be numerically computed. For ease of computation we choose the averaging area to be a circular region of area $A=\pi a^2$, instead of a square. (The exact shape of the averaging area is not an important factor in our considerations). In this case, the lagged covariance function of area-averaged rain rate $c_{AA}(\tau)$ takes the somewhat simpler form

$$c_{AA}(\tau) = (4g_0/a_*^2)\int_0^\infty d\kappa\,\frac{J_1^2(\kappa a_*)}{\kappa h(\kappa)}e^{-|\tau_*|h(\kappa)} \quad (36)$$

where $a_*=a/L_0$, $h(\kappa)=(1+\kappa^2)^{\alpha/2}$ and $J_1(x)$ denotes the usual Bessel function of order one. The shape of the scaling function $\chi_1(u;\alpha)$ predicted by the model for circular areas is shown in Fig. 1 as a function of the scaling variable $u=\tau_*/a_*^\alpha$ for a typical value $\alpha=3/2$, or $\nu=-1/4$.

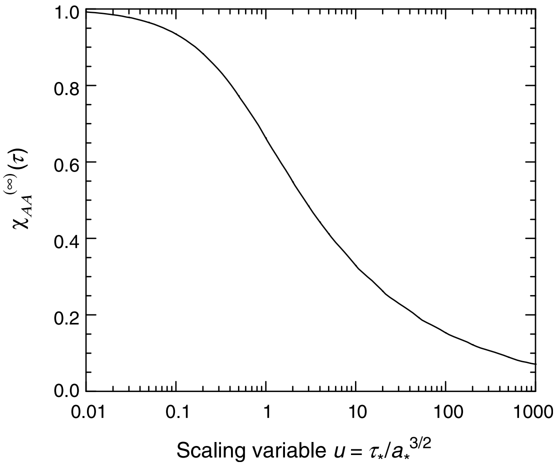

Fig. 1. Plot of the function χ_1 representing the lagged autocorrelation function of area-averaged rain rate in the limit of small τ_*, a_* as function of the scaling variable $u=\tau_*/a_*^{2(1+\nu)}$ for $\nu=-1/4$ computed from the model.

In a similar manner one can derive the scaling properties of the statistics of time-averaged rain rate at a point $R_T(\boldsymbol{x})=(1/T)\int_0^T dt\,R(x,t)$ where T is the averaging time. The covariance of time-averaged rain rate for two points separated by a distance $\rho=|\boldsymbol{x}-\boldsymbol{x}'|$, is given by $c_{TT}(\rho)=\langle R_T'(\boldsymbol{x})R_T'(\boldsymbol{x})\rangle$, which plays an important role in statistical studies of rain gauge data. It can be expressed as an integral over the point covariance function $c(\rho, \tau)$:

$$c_{TT}(\rho) = (2/T)\int_0^T d\tau(1-\tau/T)c(\rho,\tau). \quad (37)$$

In terms of the dimensionless variables $\rho_*=\rho/L_0$, $T_*=T/\tau_0$ in the scaling limit ρ_*, $T_* \to 0$, it takes the functional form

$$c_{TT}^{(\infty)}(\rho) = g_0\rho_*^{\alpha-2}\phi_2(T_*/\rho_*^\alpha;\,\alpha) \quad (38)$$

where $\phi_2(s;\alpha)$ is a scaling function defined as

$$\phi_2(s;\alpha) = 2\int_0^1 dz(1-z)\phi(zs;\alpha). \quad (39)$$

Next we consider the correlation between the time averaged rain rate at two points separated by a distance ρ, defined as $\chi_{TT}(\rho)=c_{TT}(\rho)/c_{TT}(0)$. The denominator represents the variance of time-averaged rain rate and is given in the exact model by

$$\sigma_T^2 \equiv c_{TT}(0) = g_0\left[\frac{1}{\alpha-1}T_*^{-1} - \frac{2}{3\alpha-2}T_*^{-2} + \frac{2}{\alpha}T_*^{-\varepsilon}\Gamma(-2+\varepsilon;T_*)\right], \quad (40)$$

where as before, $\varepsilon=(2-\alpha)/\alpha$. As $T_*\to 0$, it reduces to

$$\sigma_T^2 \approx (2g_0/\alpha)\left[\Gamma(-2+\varepsilon)T_*^{-\varepsilon} - \frac{1}{2\varepsilon} + O(T_*)\right]. \quad (41)$$

This immediately shows that in the scaling limit the function $\chi_{TT}^{(\infty)}(\rho)$ depends on its arguments ρ and T through the combination $s=T_*/\rho_*^{\alpha}$:

$$\chi_{TT}^{(\infty)}(\rho) = \chi_2(s;\alpha) \equiv [(2/\alpha)\Gamma(-2+\varepsilon)]^{-1}s^{-\varepsilon}\phi_2(s;\alpha). \quad (42)$$

Finally, the variance of area-time-averaged rain rate $R_{AT}=(1/AT)\int_0^T \mathrm{d}t \int_A \mathrm{d}^2x\, R(\mathbf{x},t)$, namely, $\sigma_{AT}^2 = \langle R_{AT}'^2\rangle$ is given by

$$\sigma_{AT}^2 = (2/T)\int_0^T \mathrm{d}\tau(1-\tau/T)C_{AA}(\tau). \quad (43)$$

In the scaling limit $L_*, T_*\to 0$, under a combined scale transformation $L\to L'\lambda L$, $T\to T'\lambda^{\alpha}T$, it transforms as $\sigma_{A'T'}^{2(\infty)}=\lambda^{\alpha-2}\sigma_{\mathrm{AT}}^{2(\infty)}$, and consequently has the functional form

$$\sigma_{AT}^{2(\infty)} = g_0 L_*^{-|\alpha-2|}\psi(T_*/L_*^{\alpha};\alpha). \quad (44)$$

3.3. Departure from scaling

The exact model considered in this paper is clearly not scale invariant. Strictly speaking, it is so only in the limit $\rho,\ \tau\to 0$. Indeed marked departure from dynamical scaling occurs as the spatial and temporal separations ρ, τ in the exact space–time covariance function $c(\rho,\tau)$ respectively become comparable to the characteristic length and time parameters of the model, namely L_0, τ_0. The inequalities (18) in the Fourier domain suggest that one might qualitatively expect scaling behavior when $a_*\ll 1$, $u\ll 1/a_*$. For the autocorrelation function of area-averaged rain rate, $\chi_{AA}(\tau)$, the deviation from dynamic scaling can be quantified by examining the ratio $r=\chi_{AA}(\tau)/\chi_{AA}^{(\infty)}(\tau)$ as function of the scaling variable $u=\tau_*/a_*^{2(1+\nu)}$ introduced before for various values of a_*. Results of numerical computation of this quantity are shown in Fig. 2 for the typical value $\nu=-1/4$. The dynamic scaling regime can be identified as the range of u (for a specified $a_*\ll 1$) in which the ratio r introduced above remains close to unity to a specified accuracy, say 10%. The TOGA-COARE data set gives one some idea about the relevant length and time scales. The characteristic length L_0 was found to be in the range 54 to 94 km and the characteristic time τ_0 was roughly within 4.5 and 8.2 h. Consider a typical set of model parameters $\nu=-1/4$, $L_0=50$ km and $\tau_0=6$ h. It is apparent from Fig. 2 that, for an averaging area of radius $a=5$ km ($a_*=0.1$), comparable to the size of TRMM precipitation radar footprints, dynamic scaling is achieved to an accuracy $r\geq 0.9$ when u is less than about 0.5 corresponding to a lag τ less than about 6 min. However, the radar images in TOGA-COARE were available only at roughly 10 min intervals. Consequently, the space–time scales at which the precipitation data could potentially exhibit the predicted dynamic scaling do not seem to be accessible to radar observations. Indeed a scatter-plot of $\chi_{AA}(\tau)$ vs. u for various L at the space–time scales resolved by the TOGA-COARE data shows no clear evidence of dynamic scaling as defined in this paper.

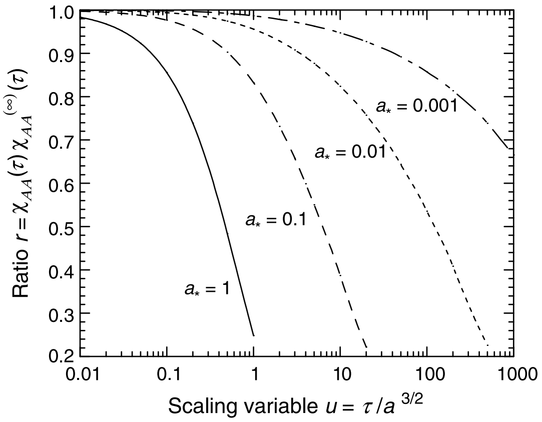

Fig. 2. Plots of the ratio $r=\chi_{AA}(\tau)/\chi_{AA}^{(\infty)}(\tau)$ measuring deviation from scaling as function of u for several values of a_*.

4. Concluding remarks

In this paper we have described a stochastic dynamic model of precipitation obeying a fractional diffusion type equation that leads to power law scaling behavior of the second moment statistics when extrapolated to small length and time scales. In particular, the model predicts dynamic scaling of the lagged autocorrelation function under a combined space–time scale transformation at these scales. However, we found that the space–time resolutions achieved in the gridded radar data sets used to validate the model is inadequate for exploring the dynamic scaling regime because of the large deviation from scaling predicted at these scales. We propose high frequency rain gauge data from a dense gauge network as a possible testing ground for this aspect of the model.

As already mentioned in the introduction, in an interesting paper Venugopal et al. (1999) have presented empirical evidence for dynamic scaling in precipitation statistics at length and time scales similar to ones considered here. We do not believe that their findings necessarily contradict those of this paper since the two studies involve very different rain statistics. The model studied here is intended to apply to an ensemble of precipitation events over a large area of the order of a few hundred kilometers on a side and for a time period of the order of a month, and the fitting procedure utilizes the entire available data. Moreover, the space–time covariance function we compute includes the zeroes of the rain field. On the other hand, the work of Venugopal et al. (1999) seeks to understand spatial and temporal organization of precipitation within individual storms. The statistic they study, namely $\mathrm{Var}[\ln R_A(t+\tau) - \ln R_A(t)]$ regarded as function of τ and A, is culled from the statistically stationary phase of an individual storm event and excludes the grid boxes with zero rain in a radar image. Moreover, since the logarithmic transformation in their statistic emphasizes small rain rates, the measures of variability in the two approaches can lead to very different results. This issue deserves closer study.

Finally, we conclude with a few remarks regarding the accuracy of our statistical model in representing the observed rainfall data. The model generally describes the spatial statistics of the TOGA-COARE and GATE rainfall data quite well (Bell and Kundu, 1996; Kundu and Bell, 2003) and also explains the dependence of the integral correlation time $\tau_A = \int_0^\infty \chi_{AA}(\tau)\mathrm{d}\tau$ on the size of the averaging area L. Once the four model parameters are obtained by fitting the spatial statistics and τ_A estimated from the data, all other second moment statistics, such as the lagged autocorrelation function $\chi_{AA}(\tau)$, are completely determined and provide a rather stringent test of the model. As an example, a comparison between the function $\chi_{AA}(\tau)$ evaluated from the actual data for MIT cruise 3 and the model prediction, with the corresponding best-fit model parameters $L_0 = 64.94$ km, $\tau_0 = 4.5$ h and $\nu = -0.259$, is shown in Fig. 3 for $L = 2$ km grid boxes. Overall the model seems to describe the observed temporal autocorrelation reasonably well except at small time lags where the model apparently underestimates the observation. The other TOGA-COARE data sets and the GATE Phase I data set studied earlier all seem to exhibit this deficiency. One possible way to improve agreement between the model and the observed autocorrelation at small τ would be to allow for an arbitrary power law exponent in the frequency dependence of

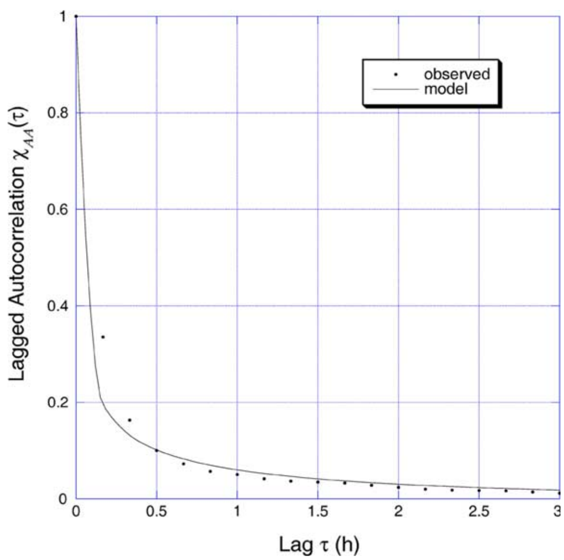

Fig. 3. Plot of the lagged autocorrelation function of area-averaged rain rate $\chi_{AA}(\tau)$ computed from the exact model for the MIT cruise 3 data set and a circular averaging area of radius $a = L/\sqrt{\pi}$, with $L = 2$ km.

the model spectrum (17) and adjust it for a best fit of the function $\chi_{AA}(\tau)$ with observation. Such an extension would correspond to anomalous diffusion described by a fractional kinetic equation involving both fractional space and time derivatives. We plan to investigate this possibility in future work.

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References

Bell, T.L., Kundu, P.K., 1996. A study of the sampling error in satellite rainfall estimates using optimal averaging of data and a stochastic model. J. Climate 9 (6), 1251–1268.

Bell, T.L., Kundu, P.K., 2003. Comparing satellite rainfall estimates with rain gauge data: Optimal strategies suggested by a spectral model. J. Geophys. Res. 108 (D3), 4121 doi:10:1029/2002JD002641.

Bouchaud, J.-P., Georges, A., 1990. Anomalous diffusion in disordered media: Statistical mechanisms and physical applications. Phys. Rep. 195, 127–293.

Bowman, K.P., Phillips, A.B., North, G.R., 2003. Comparison of TRMM rainfall retrievals with rain gauge data from the TAO/TRITON buoy array. Geophys. Res. Lett. 30 (14), 1757 doi:10.1029/2003GL017552.

Chandrasekhar, S., 1943. Stochastic problems in physics and astronomy. Rev. Mod. Phys. 15, 1–89 (1943). Reprintedin: Wax, N. (Ed.), 1943. Selected Papers on Noise and Stochastic processes. Dover Publications, New York, NY.

Feller, W., 1971., second ed. An Introduction to Probability Theory and its Applications, vol II. Wiley, New York.

Gnedenko, B.V., Kolmogorov, A.N., 1954. Limit Distributions for Sums of Independent Random Variables. Addison Wesley, Reading, MA.

Kahane, J.P., 1995. Definition of stable laws, infinitely divisible laws, and Lévy processes. In: Shlesinger, M.F., Zaslavsky, G.M., Frisch, U. (Eds.), Lévy Flights and Related Topics, Lecture Notes in Physics. Springer, New York, pp. 99–109.

Kundu, P.K., Bell, T.L., 2003. A stochastic model of space–time variability of mesoscale rainfall: Statistics of spatial averages. Water Resources Res. 39 (12), 1328 doi:10.1029/2002WR001802.

Lévy, P., 1954. Theorie de l'addition Variables Aléatories. Gauthers Villars, Paris.

Mandelbrot, B.B., 1982. The Fractal Geometry of Nature. Freeman, San Francisco, CA.

Marsan, D., Schertzer, D., Lovejoy, S., 1996. Causal space–time multifractal processes: predictability and forecasting of rain fields. J. Geophys. Res. 101, 26333–263346.

Metzler, R., Klafter, J., 2000. The random walk's guide to anomalous diffusion: A fractional dynamics approach. Phys. Rep. 339, 1–77.

North, G.R., Nakamoto, S., 1989. Formalism for comparing rain estimation designs. J. Atmos. Oceanic Technol. 6, 985–992.

Over, T.M., Gupta, V.K., 1996. A space–time theory of mesoscale rainfall using random cascades. J. Geophys. Res. 101, 26319–26331.

Samorodnitsky, G., Taqqu, M.S., 1994. Stable Non-Gaussian Random Processes. Chapman and Hall, New York.

Seed, A.W., Srikanthan, R., Menabde, M., 1999. A space–time model for design storm rainfall. J. Geophys. Res. 104, 31623–31630.

Venugopal, V., Foufoula-Georgiou, E., Sapozhnikov, V., 1999. Evidence for dynamic scaling in space–time rainfall. J. Geophys. Res. 104, 31599–31610.