Climatic Sensitivity and Fluctuation-Dissipation Relations.

THOMAS L. BELL

Goddard Laboratory for Atmospheric Sciences, NASA/Goddard Space Flight Center Greenbelt, MD 20771

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1. – Introduction.

Weather varies from one day to the next and its changes cannot be predicted very far into the future [1]. The *climate* of an area refers to *probabilities* of meeting with weather of different kinds there. LEITH [2] has suggested a quantitative way to talk about climate and climatic change by introducing the concept of an ensemble. He suggests constructing an ensemble of planets, each identical to the Earth in all respects believed to be important in determining the climate, but differing in the initial conditions for the atmospheres. Averages over the ensemble are assumed to give the same results as long-time averages over a simple planet. In this framework climate refers to the statistics of the ensemble.

In order to make our discussion more concrete, let us concentrate on a single variable describing the atmosphere, the globally averaged surface air temperature $T_0(t)$, which depends on time t. We shall represent ensemble averages by angular brackets $\langle T_0 \rangle$. For a stationary climate, the time-averaged temperature $\overline{T}_0$,

$$\overline{T}_0 \equiv \frac{1}{Y}\int_0^Y \mathrm{d}t\, T_0(t)\,, \tag{1.1}$$

will approach the ensemble average $\langle T_0 \rangle$ for long averaging times Y. The advantage of introducing an ensemble average is that we may discuss climatic change in terms of the time dependence of the nonstationary ensemble-averaged temperature $\langle T_0(t) \rangle$. It is much more difficult to discuss climatic change when climate is defined in terms of time-averaged quantities.

We can write the equations for T_0 schematically as

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = I_0(T_0, x_\alpha) + f_0\,, \tag{1.2}$$

where I_0 depends, in principle at least, on T_0 itself and all of the other variables x_α of the atmosphere, oceans, etc. that affect the temperature T_0 and cause it to vary in time. The variables x_α in turn have equations of the form

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = I_\alpha(T_0, x_\alpha) + f_\alpha \,, \tag{1.3}$$

where again we have represented the interactions of the various variables with each other schematically.

In both eqs. (1.2) and (1.3) we have separated off a « forcing » term f that we are interested in changing to see how the system reacts. The forcing f_0 may, for instance, represent solar heating of the atmosphere, and a change Δf_0 might represent the increased heating due to an increase in the luminosity of the Sun.

Climatic sensitivity is represented by the proportionality constant M relating a climatic change to a small change in forcing:

$$\Delta\langle T_0\rangle = M\,\Delta f_0 \,. \tag{1.4}$$

We assume that climatic change is linear in small changes in the forcing, which is plausible for a stable climate. We can, of course, consider the sensitivity of T_0 to changes in forcing f_α of other variables, and write

$$\Delta\langle T_0\rangle = M_{0\alpha}\Delta f_\alpha \,. \tag{1.5}$$

Climate sensitivity $M_{\alpha\beta}$ is thus a matrix relating shifts in climatic means $\Delta\langle x_\alpha\rangle$ to changes in forcing Δf_β.

The importance of knowing the sensitivity matrix $\mathbf{M}$ to climate research is clear. There is the direct benefit that what we know about changes in the forcing Δf can be translated into its consequences for climate. The classic example of such relationships is the prediction of how much warming of the planet we can expect due to the extra opacity of the atmosphere to infra-red radiation caused by increasing carbon-dioxide concentration.

Another possible benefit lies in the interesting possibility that, if we knew the sensitivity matrix of the atmosphere, we could replace the atmosphere in coupled-ocean-atmosphere models by a « black box » constructed from the sensitivity matrix, so that we could run the model using time steps with a size characteristic of the ocean rather than having to use the much shorter time steps required to integrate properly the full atmospheric equations. Need for some such scheme is exemplified by the discussion of the coupled-ocean-atmosphere problem by DICKINSON [3].

But the sensitivity matrix $\mathbf{M}$ is not easy to obtain. Some information about it can occasionally be obtained when « natural experiments » such as volcanic eruptions occur that affect the radiative equilibrium of the atmosphere. But the climatic changes that result are not easily distinguished from the natural variability of time averages of atmospheric variables due to day-to-day weather changes. This problem is discussed by LEITH [2].

Another approach to learning about **M** is through computer modeling of the climate system. One tries to construct as good a model as possible, incorporating as many of the physical processes determining atmospheric behavior as possible, and integrates the model on a computer. But models of the atmosphere require a lot of simplification of the equations for the climate system, and it is a difficult problem to determine whether the simplifications used are adequate. To test the model, one tries to compare its behavior with the atmosphere's. But quantitative information about the climatic sensitivity of the atmosphere is rather scanty, and tests of the model tend to be limited to reproducing atmospheric behavior on time scales of a week, by comparing weather forecasts to actual weather developments, and to trying to make the model climate agree with the present climate.

Large models of the atmosphere strain the capacity of the most powerful computers, and this makes experimenting with the models difficult, especially for the long runs required to establish climatic means. Determining the sensitivity of the model to a change in forcing Δf_α requires a separate run for each α.

HALL *et al.* [4] have proposed using the adjoint method of sensitivity analysis in order to alleviate the computational task. In this method, a set of modified (adjoint) model equations linearized about a solution of the equations is integrated backwards in time. Certain results from functional analysis [5] allow the solution of the backward adjoint problem to be used in obtaining the sensitivity of the model to many different changes in forcing Δf_α from a single computer integration. The method will be difficult to apply to large general-circulation models of the atmosphere, but is worth further investigation.

Finally, LEITH [6] has suggested a method of determining **M** from observations of natural fluctuations of the atmosphere about the mean. He points out that the average manner in which fluctuations of the atmosphere relax back to the mean should give us some information about the dynamical responsiveness of the atmosphere, and suggests using the *fluctuation-dissipation relation* (FDR) to express this relationship quantitatively.

2. – The fluctuation-dissipation relation.

2·1. *Statement of the relation.* – Leith's paper [6] on the FDR is beautifully written and makes excellent reading, and we shall not attempt here to explore the FDR with as much rigor or in as much detail as he has done, but will only try to review the elements necessary for an understanding of the method.

Let us suppose for simplicity that we need consider only one variable of the atmosphere, T_0. In order to state the FDR, we will first need to define two functions related to the evolution of $T_0(t)$. The first is the *lagged autocorrelation* of the temperature,

$$C(\tau) = \langle T_0'(t+\tau)\, T_0'(t)\rangle / \langle (T_0')^2\rangle \,, \tag{2.1}$$

where we have assumed that the statistics are stationary in time, so that the autocorrelation is a function of lag τ alone and not of t. The primes indicate variations from the climatic mean, $T_0' = T_0 - \langle T_0\rangle$.

The second function needed is the *average response* of T_0 to an infinitesimal heat pulse

$$\Delta f_0(t) = \varepsilon\delta(t - t_0) \tag{2.2}$$

occurring at t_0. To compute this response, consider the perturbed equation obtained by adding Δf_0 to eq. (1.2),

$$\frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}t} = I_0 + f_0 + \varepsilon\delta(t - t_0)\,, \tag{2.3}$$

where $\tilde{T}_0$ is the solution to the perturbed equation. The solution may be written in terms of the response function $g(t; t_0)$ as

$$\tilde{T}_0(t) = T_0(t) + \varepsilon g(t; t_0)\,. \tag{2.4}$$

The average response to a perturbation is obtained from the ensemble mean

$$g(\tau) = \langle g(t; t_0)\rangle\,, \qquad \tau = t - t_0, \tag{2.5}$$

where the stationarity assumption has been used again.

The FDR may be stated now for our simplified, one-variable model:

$$g(\tau) = C(\tau)\theta(\tau)\,, \tag{2.6}$$

where $\theta(\tau)$ is the Heaviside step function. In words, the mean response function is identical to the lagged autocorrelation function for the system. Notice that $g(\tau)$ vanishes for negative τ because of causality.

We may express the response of the system to any small perturbation $\Delta f_0(t)$ as

$$\Delta T_0(t) = \int_{-\infty}^{t} \mathrm{d}t'\, \Delta f_0(t')\, g(t; t') \tag{2.7}$$

and the response to a constant Δf_0 as

$$\Delta T_0(t) = \Delta f_0 \int_{-\infty}^{t} \mathrm{d}t'\, g(t;\, t') \,. \tag{2.8}$$

It follows that the climatic change due to Δf_0 is

$$\Delta \langle T_0 \rangle = \Delta f_0 \int_0^{\infty} \mathrm{d}\tau\, g(\tau) \,, \tag{2.9}$$

and from eq. (1.4) we identify the sensitivity of this elementary model as

$$M = \int_0^{\infty} g(\tau)\, \mathrm{d}\tau \,. \tag{2.10}$$

If the FDR, eq. (2.6), is valid, one can obtain the climate sensitivity directly from the autocorrelation function:

$$M = \int_0^{\infty} C(\tau)\, \mathrm{d}\tau \,. \tag{2.11}$$

The climate sensitivity of a perturbed system obeying the FDR *can be obtained from the correlation time of the undisturbed system.* This is what makes the FDR so interesting.

2·2. *Markov processes and the* FDR. – An example of a system for which the FDR works, and which helps in analyzing some of the issues that need to be considered in using it, is a multivariate first-order Markov process. Its variables x_α satisfy the stochastic equations

$$\dot{x}_\alpha = -\sum_\beta \Lambda_{\alpha\beta} x_\beta + f_\alpha \,, \tag{2.12}$$

where $\mathbf{\Lambda}$ is a matrix of coefficients and $f_\alpha(t)$ is a zero-mean white-noise forcing function with covariance

$$\langle f_\alpha(t)\, f_\beta(t') \rangle = F_{\alpha\beta}\, \delta(t - t') \,, \tag{2.13}$$

$\mathbf{F}$ being a symmetric positive-definite matrix. The solution to eq. (2.12) may be obtained by using an integrating factor, which yields

$$\boldsymbol{x}(t) = \int_{-\infty}^{t} \mathrm{d}t' \exp[-\mathbf{\Lambda}(t - t')]\, \boldsymbol{f}(t') \,. \tag{2.14}$$

From (2.14) we immediately identify the response function

$$\mathbf{g}(t;t') = \exp[-\mathbf{\Lambda}(t-t')]\theta(t-t') . \tag{2.15}$$

Since (2.12) is linear in $\boldsymbol{x}$, this response function is independent of any particular solution. Therefore, the mean response function for the system is

$$\mathbf{g}(\tau) = \exp[-\mathbf{\Lambda}\tau]\theta(\tau) . \tag{2.16}$$

With a little more algebra one can obtain the lagged covariance matrix of the system

$$U_{\alpha\beta}(\tau) \equiv \langle x_\alpha(t+\tau)x_\beta(t)\rangle , \tag{2.17}$$

which may be written

$$\mathbf{U}(\tau) = \exp[-\mathbf{\Lambda}\tau]\,\mathbf{\Gamma}\theta(\tau) + \mathbf{\Gamma}\exp[\mathbf{\Lambda}^{\mathrm{T}}\tau]\theta(-\tau) , \tag{2.18}$$

where $\mathbf{\Gamma}$ is the zero-lag covariance $U_{\alpha\beta}(0)$ and $\mathbf{\Lambda}^{\mathrm{T}}$ is the matrix transpose of $\mathbf{\Lambda}$. Comparing (2.16) and (2.18), we find the relation

$$\mathbf{g}(\tau) = \mathbf{U}(\tau)\mathbf{U}^{-1}(0)\theta(\tau) , \tag{2.19}$$

which is a multivariate generalization of the single-variable expression (2.6). This is the form in which LEITH [6] has expressed the FDR.

We have so far dealt with models assumed to have stationary statistics, but the atmosphere has statistics that vary, for example, with the diurnal and annual cycles. The assumption of stationary statistics can be relaxed in eq. (2.12) by allowing $\Lambda_{\alpha\beta}$ and $F_{\alpha\beta}$ to be time dependent, and a form of the FDR very similar to eq. (2.19) derived appropriate to a system with non-stationary statistics,

$$\langle\mathbf{g}(t+\tau;t)\rangle = \mathbf{U}(\tau;t)\mathbf{U}^{-1}(0;t)\theta(\tau) , \tag{2.20}$$

where the definition of $\mathbf{U}(\tau;t)$ is as in eq. (2.17). Nonstationary versions of the FDR are used in constructing statistical theories of turbulence [7].

3. – Will the FDR work?

The fluctuation-dissipation theorem, which states that the FDR, eq. (2.19), holds for a system, is known to be valid for a wide class of dynamical systems studied in statistical mechanics: namely, systems that 1) satisfy the Liouville equation, which states that an ensemble of systems moves as an incompressible gas in phase space, 2) have quadratic constants of the motion, such as energy, and 3) are in thermal equilibrium [8]. The atmosphere is unfortunately not one of these systems. It does, however, have many characteristics of systems that do obey the FDR, and so, as LEITH [6] argues, the theorem may provide serviceable estimates of the climatic sensitivity. One must keep in mind that, even if the FDR is off by a factor of 2, it may still provide useful information where none is otherwise available.

Some of the arguments that the FDR may be applicable to the atmosphere are:

1) The statistics of the atmosphere on large spatial scales and for time scales small compared with a year are probably not too far from Gaussian, if for no other reason than that the central-limit theorem implies that large-scale averages of independent smaller-scale phenomena will tend to be normally distributed. Very little has been done to test systematically the statistics of the large scales [9]. Some model results will be presented later in this section. (The Gaussianity of geopotential height on small scales has been studied by WHITE [10]. Deviations from normality are small but detectable in some regions of the northern hemisphere.)

2) The time-dependent behavior of the atmosphere on large scales can be represented by multivariate Markov processes. This is perhaps more conjecture than the result of observation, since very little research has been done yet to test it systematically. Results of a model study described later in this section are encouraging.

3) Closure models for some statistical properties of turbulence in the atmosphere con be constructed that satisfy a FDR and seem to be able to do a creditable job of representing the statistics [11].

There are indications that the FDR may serve less well for small scales. There is, first, evidence from turbulence closure models [11, 12] that on small scales the correlation times may over-estimate response times by more than a factor of 2. There is also reason to believe that from a practical point of view the FDR will be difficult to use for obtaining small-scale sensitivities because of our inability to collect enough of the necessary data. This will be discussed in more detail later.

The best evidence for the usefulness of the FDR may come from tests of the method using atmospheric models. We present a few results of such tests here, some of which are not yet completed.

With regard to how well the statistics of a model are fitted by a Markov process, we consider first a version of the 2-level Held-Suarez general-circulation model [13]. The model used here has no oceans, no moisture transport and mean annual solar heating (*i.e.* no diurnal or seasonal cycle), in order to reduce the number of variables and time scales involved in carrying out the test.

The amplitudes of large-scale temperature fluctuations are extracted from the data generated by the model, using

$$\bar{\theta}_l \equiv (2\pi)^{-1}\int \mathrm{d}\Omega\, P_l(\sin\lambda)\,\bar{\theta}(\hat{r})\,, \tag{3.1}$$

where $\bar{\theta}(\hat{r})$ is the vertically averaged potential temperature of the atmosphere at point $\hat{r}$ on the sphere, P_l is a Legendre polynomial, λ is latitude, and integration is over the surface of the sphere, $\mathrm{d}\Omega$ being the surface element of the unit sphere. The variable $\bar{\theta}_0$ represents globally averaged temperature of the model, and $\bar{\theta}_2$ represents roughly the equator-to-pole temperature difference averaged over the two hemispheres.

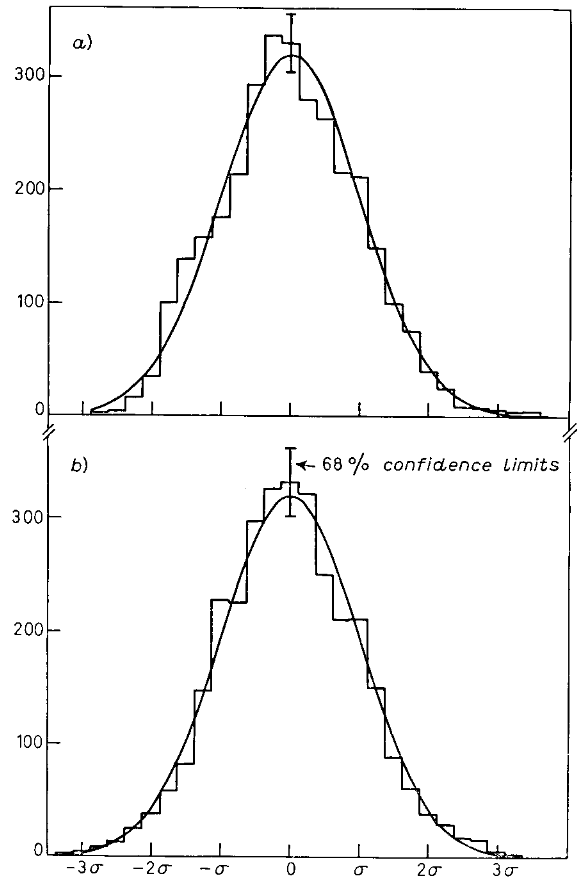

Fig. 1. – Histograms of two of the Legendre amplitudes defined in eq. (3.1) of vertically averaged potential temperature for the Held-Suarez model described in the text: Gaussian curves are fitted to the histograms. *a*) P_2, 3200 days, mean = − 13.79, σ = 0.68; *b*) P_0, 3200 days, mean = 53.168, σ = 0.074.

Figure 1 shows histograms of these two variables from a 3200-day run of the model. Gaussian curves are fitted to the histograms. The variables are well approximated by Gaussian variables. The tendency to have more days at large positive deviations of $\bar{\theta}_2$ than at large negative deviations is probably a reflection of the increased stability of the atmosphere when the meridional gradient of temperature is reduced.

Figure 2 shows an attempt to represent the lagged correlation statistics of the variables $\bar{\theta}_2$ using a Markov model. The solid curve shows the autocorrelation function of the variable from data, which is, of course, only imperfectly known because it is estimated from a finite-length time series. Two Markov

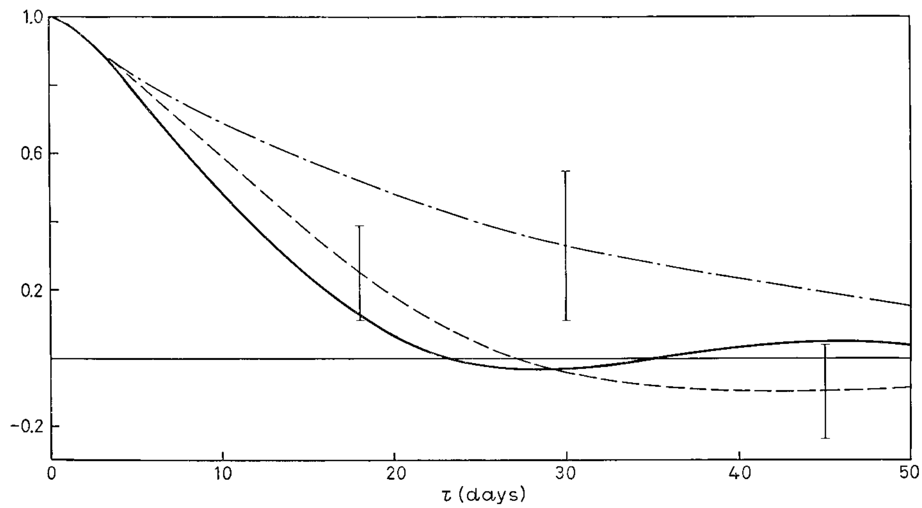

Fig. 2. – Lagged autocorrelation of the amplitude $\bar{\theta}_2$ defined in eq. (3.1) (solid curve) and autocorrelations for a first-order Markov process (2.12) fitted to the data using 2 variables $\bar{\theta}_2$ and $\bar{\theta}_4$ (dash-dotted curve) and using 7 variables $\bar{\theta}_2$, $\bar{\theta}_4$, $\bar{\theta}_6$, $\hat{\theta}_0$, $\hat{\theta}_2$, $\hat{\theta}_4$, $\hat{\theta}_6$ (dashed curve), where $\hat{\theta}$ is the vertical gradient of potential temperature in the Held-Suarez model. Error bars show 68% confidence limits.

models of the form shown in eq. (2.12) are tried, one in which only 2 variables $\bar{\theta}_2$ and $\bar{\theta}_4$ are included, and another in which 7 variables $\bar{\theta}_2$, $\bar{\theta}_4$, $\bar{\theta}_6$, $\hat{\theta}_0$, $\hat{\theta}_2$, $\hat{\theta}_4$, $\hat{\theta}_6$ are used, where $\hat{\theta}_l$ represents the Legendre amplitude of the vertical gradient of temperature. The variable $\bar{\theta}_0$ was left out of the Markov models because its fluctuations are small and so tied to the fluctuations of $\bar{\theta}_2$ that it was not useful to include it.

The 68% confidence limits are shown on the model fits. The sampling error size for the correlation function of $\bar{\theta}_2$ is similar to the 68% confidence limits of the 7-variable model fits; deviations of the correlation function from 0 for lags τ beyond 20 days are not statistically significant. The 7-variable fit, while not perfect, is probably good to 25% or so. Cross-correlations, not shown here, are similarly well fitted.

Note how much error in our knowledge of the true behavior of the correlations is present because of sampling error, even for a relatively long run of the model. When the FDR is used to estimate the response characteristics of the model from the correlation functions, the estimates will suffer the same level of uncertainty. A test of the validity of the FDR for this model is under way but not yet completed.

We shall touch briefly on a test of the FDR for a much simpler model where the FDR proves to function quite well as a predictor of the model's sensitivity even though the model was studied in a regime far from where the fluctuation-dissipation theorem can be proved. Details may be found elsewhere [14].

The model was constructed using the barotropic vorticity equation for incompressible flow on a nonrotating plane. The equation governing the vorticity $\zeta = (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \hat{e}_z$, the vertical component of the curl of the velocity, is

$$\frac{\partial \zeta}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \zeta = F - D \,, \tag{3.2}$$

where the forcing and dissipation terms F and D will be specified in a moment. If a Fourier mode expansion of vorticity is introduced,

$$\zeta(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k}, t) \exp[i\boldsymbol{k} \cdot \boldsymbol{x}] \,, \qquad \boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y) \,, \tag{3.3}$$

after assuming spatial periodicity, the equations of motion of the amplitudes $\tilde{\zeta}(\boldsymbol{k}, t)$ can be obtained using (3.2). These equations are truncated, so that only terms involving Fourier modes with $k^2 = |\boldsymbol{k}|^2 = 1$, 2, 4 and 5 (with $L = 2\pi$) are allowed to be nonzero. This results in 20 coupled nonlinear equations. The forcing term F and the dissipation term D on the right-hand side of (3.2) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\zeta}(\boldsymbol{k}) = \text{nonlinear terms} + F(\boldsymbol{k}) - \nu(\boldsymbol{k}) \zeta(\boldsymbol{k}) \tag{3.4}$$

for the Fourier amplitude equations. The study described here sets $F(\boldsymbol{k}) = 0$ for all $\boldsymbol{k}$ except $F(0, 1) = 1$. The « viscosities » were given values

$$\nu(k^2 = 1) = \nu(k^2 = 5) = 1 \,, \tag{3.5a}$$

$$\nu(k^2 = 2) = \nu(k^2 = 4) = -0.6 \,. \tag{3.5b}$$

The modes with $k^2 = 2$, 4 were made artificially unstable by the choice (3.5b)

in order to make the model behave turbulently. This is discussed in more detail in ref. [14]. The resulting model is turbulent, but because of the forcing and dissipation terms the FDR cannot be proved for it.

The response functions and the correlation functions for two of the model variables are plotted in fig. 3. They should be equal if eq. (2.6) were satisfied. The response function is determined from an ensemble average and sampling error estimates (due to the finite size of the ensemble) are shown as 68 % confidence limit error bars. The uncertainty diverges nearly exponentially with τ because of the turbulent behavior of the model. It is clear that, while the FDR is not exact, it is satisfied to within 25 %.

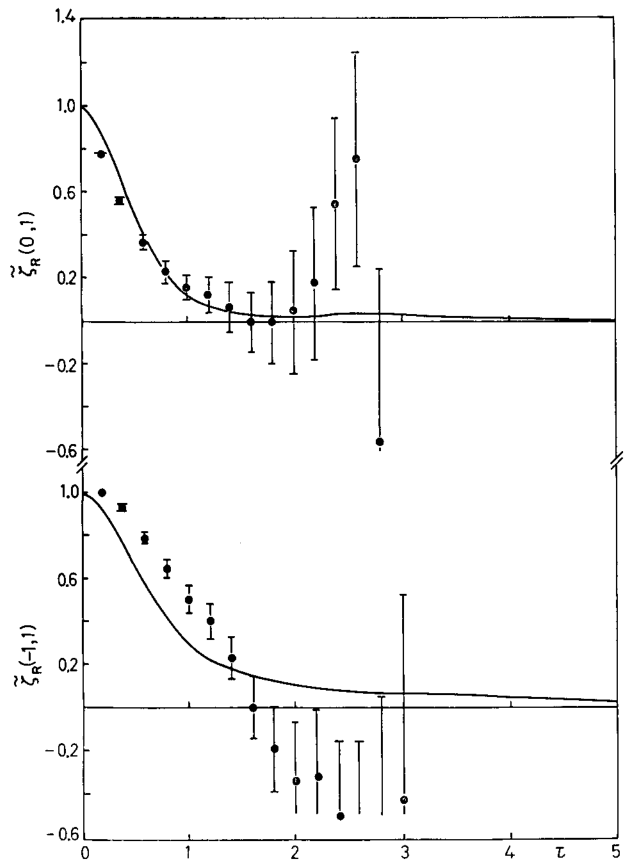

Fig. 3. – Lagged autocorrelation functions $C(\tau)$ (smooth curves) and response functions $g(\tau)$ (dots) for 2 of the 20 variables for the model described in eqs. (3.2)-(3.6). These functions are graphed for the real part of $\tilde{\zeta}(0, 1)$, defined in eq. (3.3), in the lower half of the figure, and for the real part of $\tilde{\zeta}(-1, 1)$ in the upper half. Error bars on the response function are 68% confidence limits determined from sampling errors due to the finite size of the ensemble used in estimating the ensemble averages in eq. (2.5). For further details see ref. [14].

The tests just described explored the validity of eq. (2.6) at each lag τ. To test whether the climatic sensitivity of the model is accurately predicted by eq. (2.11), the strength of $F(0, 1)$ in eq. (3.4) was increased from 1.0 to 1.5. A change

$$\Delta\langle\tilde{\zeta}(0, 1)\rangle = 0.28 \pm 0.05$$

was observed in the climatic mean of variable $\tilde{\zeta}(0, 1)$. The uncertainty in the change is due to sampling error from the finite length of the computer run used to obtain the new climatic mean of the model.

From eq. (1.4), this result corresponds to a value of the climatic sensitivity of the model

$$M = 0.56 \pm 0.10 \,. \tag{3.6}$$

The value of the integral of $C(\tau)$ for variable $\tilde{\zeta}(0, 1)$ estimated from fig. 3 is

$$\int_0^\infty C(\tau)\, \mathrm{d}\tau = 0.65 \pm 0.05 \,. \tag{3.7}$$

Estimate (3.7) of the sensitivity M of the model from the FDR using eq. (2.11) agrees with the climatic sensitivity (3.6) actually observed for the variable to within the confidence limits of the estimates.

Similar tests [14] for three other variables of the model showed no statistically significant departures from equality (2.11) predicted by the FDR. The integrated form of the FDR, eq. (2.11), thus seems to be obeyed quite well by this model, even though the dissipation terms in the equations of motion drive it well away from the regime where a fluctuation-dissipation theorem can be proved.

4. – Sampling problems.

Even if we were sure the FDR is exact for the atmosphere, there are practical problems with using it due to the unavailability of data needed to obtain the covariance matrices in (2.19). There is also the obvious problem that the number of variables that may in principle be used to describe fully the atmosphere is enormous. General-circulation models of the atmosphere can integrate equations for $10^5 \div 10^6$ variables. One must find some means of limiting *a priori* the variables that need to be considered.

LEITH [6] suggests using empirical orthogonal functions (EOF's), the eigenfunctions of the covariance matrix

$$\mathbf{U}(0)\boldsymbol{\psi}^{(k)} = \lambda_k \boldsymbol{\psi}^{(k)} \,, \tag{4.1}$$

where $\psi^{(k)}$ is the k-th eigenfunction and λ_k is the corresponding eigenvalue, ordered so that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant \lambda_{k+1} \geqslant \ldots$. EOF's are orthonormal:

$$\psi^{(k)} \cdot \psi^{(l)} = \delta_{lm} . \tag{4.2}$$

They can be used as a basis to represent the original set of variables $\boldsymbol{x}$ as

$$\boldsymbol{x}(t) = \sum_k \xi_k(t) \psi^{(k)} , \tag{4.3}$$

$$\xi_k = \psi^{(k)} \cdot \boldsymbol{x} . \tag{4.4}$$

The variance of ξ_k is λ_k.

EOF's have the useful property that a large portion of the total variance of the system $\sum_\alpha U_{\alpha\alpha}$ can be expressed in terms of a small subset of the eigenfunctions, because of the relation

$$\sum_\alpha U_{\alpha\alpha} = \sum_k \lambda_k \tag{4.5}$$

and the empirical fact that the eigenvalues λ_k tend to diminish rapidly with increasing k. It is not uncommon to find that over 90 % of the variance can be represented by the first dozen EOF's in expansion (4.3). Moreover, the EOF's in this subset tend to describe large-scale variations and have the longest time scales. They, therefore, tend to have large sensitivities, if the FDR is any guide (see eq. (2.11)).

The FDR, eq. (2.19), has a particularly simple form when expressed in terms of EOF amplitudes:

$$\tilde{g}_{kl}(\tau) = \tilde{U}_{kl}(\tau)/\lambda_l , \tag{4.6}$$

$\tau \geqslant 0$, where the tildes denote quantities expressed in the EOF basis. As LEITH [15] points out, determining the response of EOF variable ξ_k to a perturbation of variable ξ_l using the FDR (4.6) requires knowledge of the covariance of those two variables alone. In contrast, using the FDR expressed in terms of the original variables is considerably more complicated, since it requires knowing the full covariance matrix of all of the variables in order to invert the 0-lag matrix in eq. (2.19). Model tests of the validity of the FDR would be much easier to carry out in this basis, since the tests could focus on a subset of EOF's and escape the cumbersome collection of statistics for thousands of variables.

However, one does not know the EOF's of the atmosphere exactly, owing both to the lack of data for all variables and to sampling errors due to the short time histories of atmospheric data. It may be that EOF amplitudes describing large-scale variability of the atmosphere may escape these problems. Further investigation is needed here.

The usefulness of EOF's in any practical study of climate sensitivity may depend on how well the forcing $\Delta\boldsymbol{f}$ can be expressed in terms of the $\boldsymbol{\psi}^{(k)}$. If the forcing of a single variable x_α is Δf_α, representing some extra heating near a grid point, perhaps, the EOF representation of this forcing would be

$$\Delta\tilde{f}_k = \psi_\alpha^{(k)} \Delta f_\alpha \; ; \tag{4.7}$$

that is, the forcing amplitude in the EOF representation, $\Delta\tilde{f}_k$, is proportional to the α-th component of EOF $\boldsymbol{\psi}^{(k)}$. But it is an empirical fact that the components of EOF's tend to be about $1/\sqrt{p}$ in size, for a p-variable system, *independent of* k, and so a localized perturbation of variable x_α will tend to appear in the EOF representation evenly spread over all EOF's. The EOF representation is, therefore, more appropriate to studies of sensitivities to large-scale influences, since they are likely to be represented by just a few EOF's.

One can also argue that sensitivities to small-scale disturbances may be difficult to obtain from the FDR because of sampling problems due to insufficient amounts of data for determining the covariance matrices needed. As a simple model designed to illustrate the kinds of problems that may appear, let us suppose that the true lagged covariance matrix of a system may be written

$$\tilde{U}_{jk}(\tau) = \lambda_j \delta_{jk} \exp\left[- |\tau|/\tau_j\right] , \tag{4.8}$$

where we have expressed the statistics for the EOF amplitudes rather than for the original variables x_α. The zero-lag covariance matrix is necessarily diagonal, by definition of EOF's, and we assume that it remains diagonal for all lags τ. Different correlation times τ_j are allowed for different EOF's. It is observed in analyses of atmospheric models that eigenvalues λ_k decrease with k almost exponentially, and much faster than time scales τ_k.

Suppose now that we have a data set covering a time period $0 \leqslant t \leqslant T$ and we try to estimate the covariance function (4.8) from the data. For $j \neq k$, the estimates $\tilde{\mathbf{U}}^{(e)}$ of $\tilde{\mathbf{U}}$ will differ from zero by an amount

$$\tilde{U}_{jk}^{(e)}(\tau) = 0 \pm (\lambda_j \lambda_k)^{\frac{1}{2}} \left[2\tau_j \tau_k / \left((\tau_j + \tau_k) T\right)\right]^{\frac{1}{2}} \tag{4.9}$$

for $j \neq k$ and large lag τ, a result that can be found in standard textbooks on statistics [16].

The estimate of the sensitivity matrix $\tilde{M}_{jk}$ is obtained by integrating expression (2.19) over τ, as in eq. (2.11). However, we cannot integrate very far in τ, because our data allow us to estimate covariances only for $\tau < T$, in principle, and usefully only for $\tau \ll T$. We must decide somehow where to stop the integration over τ, but wherever we stop we shall have accumulated sampling errors in the integral dictated by the size of errors in (4.9). A more sophisticated approach might be to fit the data to a Markov process and estimate $\tilde{\mathbf{M}}$ from the Markov fit. This reduces the sampling errors somewhat over the brute-force approach just described, but, of course, does not eliminate them. A laborious but straightforward calculation gives an estimate of the errors in what we obtain for $\tilde{M}_{jk}$ of order

$$\tilde{M}_{jk}^{(e)} = \tau_j \delta_{jk} \pm (\lambda_j/\lambda_k)^{\frac{1}{2}} [\tau_j \tau_k (\tau_j + \tau_k)/T]^{\frac{1}{2}} . \tag{4.10}$$

Sampling errors generate off-diagonal elements in the sensitivity matrix!

Suppose now that we wanted the sensitivity of EOF $\boldsymbol{\psi}^{(j)}$ to a perturbation $\Delta\tilde{f}_k$, $j \ll k$, for which $\lambda_j \gg \lambda_k$ and $\tau_j \gg \tau_k$. Result (4.10) becomes approximately

$$\tilde{M}_{jk}^{(e)} \approx \pm \tau_j [(\lambda_j/\lambda_k)(\tau_k/T)]^{\frac{1}{2}} . \tag{4.11}$$

But, as mentioned earlier, the ratio τ_k/λ_k increases rapidly with k for the atmosphere. Consequently, if we try to estimate the sensitivity of EOF $\boldsymbol{\psi}^{(j)}$ to a *localized perturbation*, for which $\Delta\tilde{f}_j$ tends to be of the same order of magnitude as $\Delta\tilde{f}_k$ (as we mentioned at the beginning of this section), the sampling noise from the contribution $\tilde{M}_{jk}\Delta\tilde{f}_k$ will tend to swamp the *true* contribution $\tilde{M}_{jj}\Delta\tilde{f}_j = \tau_j \Delta\tilde{f}_j$ by a factor $[(\lambda_j/\lambda_k)(\tau_k/T)]^{\frac{1}{2}}$ unless T is enormous. This argument is by no means rigorous, but it explains sampling problems encountered in trying to obtain statistically stable results for the Held-Suarez model study.

5. – Additional remarks.

The FDR may give us access to the climate sensitivity of the atmosphere without having to construct elaborate climate models, although confidence in the accuracy of the FDR will probably require testing how well it works with models.

LEITH [15] has suggested two areas where the FDR may also be useful for studying and improving models of the atmosphere. The first is more in the realm of weather forecasting than in climate modeling. Suppose, as is likely, that the climatic mean of a forecast model differs from the true climate. Every time atmospheric data are used to initialize the forecast model, part of the time evolution of the models will consist in a drift from the climatic mean of the atmospheric data to the climatic mean preferred by the model. This climatic drift will generate a systematic bias in the model forecast. The bias could be removed if we knew what it was. The mean response function $\mathbf{g}(\tau)$ represents exactly the information we need to remove the bias, and, if the FDR is valid, can be obtained from the covariance statistics of the model.

The second area where the FDR can help is in suggesting tests of climate models to probe how well the model's climatic sensitivity agrees with the real atmosphere's. Even if the FDR were to prove an inaccurate gauge of climatic sensitivity (and there is no reason as yet to suppose this), the FDR suggests that the failure of a model to generate values of $\mathbf{U}(\tau)\mathbf{U}^{-1}(0)$ that agree with values derived from atmospheric data is good cause for concern about the ability of the model to estimate climatic sensitivity well.

6. – Conclusion.

We have discussed how the fluctuation-dissipation relation might be useful to the study of climate dynamics and reviewed some of the reasons for believing it might be an accurate guide to climatic sensitivity of the atmosphere, at least on large scales. Much hard work remains to be done on solving the sampling problems associated with making actual estimates of climate sensitivity from real data, but is justified by the unique contribution such estimates would represent to research on how and why the climate changes.

* * *

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