Latitude Dependence of Eddy Variances

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ABSTRACT

The eddy variance of a meteorological field must tend to zero at high latitudes due solely to the nature of spherical polar coordinates. The zonal averaging operator defines a length scale: the circumference of the latitude circle. When the circumference of the latitude circle is greater than the correlation length of the field, the eddy variance from transient eddies is the result of differences between statistically independent regions. When the circumference is less than the correlation length, the eddy variance is computed from points that are well correlated with each other, and so is reduced. The expansion of a field into zonal Fourier components is also influenced by the use of spherical coordinates. As is well known, a phenomenon of fixed wavelength will have different zonal wavenumbers at different latitudes. Simple analytical examples of these effects are presented along with an observational example from satellite ozone data. It is found that geometrical effects can be important even in middle latitudes.

1. Introduction

It is often useful to divide a global meteorological field into a zonally symmetric part, usually taken to be the zonal mean, and a zonally asymmetric part, referred to as the eddies. Alternatively, the field can be expanded in zonal harmonics. In this case the wavenumber zero component contains the zonal mean, and the deviations from zonal symmetry are projected onto the higher wavenumbers. The harmonic amplitudes indicate the degree of zonal asymmetry at different spatial scales.

This formal analysis into zonal-mean and eddy parts has two main justifications. First, the earth's atmosphere exhibits a large degree of zonal symmetry. That is, the magnitude of the zonal mean is usually much larger than the magnitude of the eddies. This is especially true of the time-mean state of the atmosphere, since time averaging largely removes zonally asymmetric transient disturbances. The dominant symmetry, of course, results from the rotation of the earth and the zonally symmetric forcing by solar radiation on time scales longer than the diurnal cycle. However, zonal asymmetries (stationary waves) do exist in the time-mean field, a result of the asymmetrical distribution of the continents and of topographic features on the earth.

Second, the division into zonal-mean and eddy parts can be used to simplify theoretical and numerical models of the atmosphere. For instance, the transient or stationary eddies may be thought of as perturbations to the zonally symmetric basic state. The basic state may be an idealized one, or it may be taken from observations or a more sophisticated model.

The formal division into zonal-mean and eddy fields allows statistics of the two parts of the field to be computed separately. Statistics from the real atmosphere can be compared to predictions by theoretical and numerical models. Frequently used statistics of the zonal-mean field include the time mean and time variance. A number of different eddy statistics may be computed. Oort (1983), for example, has presented latitude–height cross sections of various eddy statistics for different meteorological variables and selected averaging periods. One important parameter of the eddy field is the variance around a latitude circle, or eddy variance, which measures the total deviation from zonal symmetry at all spatial scales. (The transient eddy variance defined by Oort, $[z'^2]$, includes the time variance of the zonal mean.) A simple interpretation of the eddy statistics is that the larger eddy variance in middle latitudes in winter indicates larger amplitude planetary-scale waves and active growth and decay of extratropical cyclonic storms from baroclinic instability.

In this paper we will show that, in the ensemble or time mean, eddy variances decrease near the pole, even when the statistics of the field are uniform over the sphere. In addition, the zonal-harmonic power spectrum for the same field depends on the latitude, though the statistics of the field itself do not. These latitude dependencies result solely from the choice of coordinate system. Spherical coordinates introduce a latitude-dependent length scale that is often forgotten when computing zonal averages: the circumference of the latitude

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circle over which the averaging is carried out. Meteorological fields at high latitudes are known to have fairly well-defined correlation length-scales L. At distances less than L meteorological variables are well correlated; at distances greater than L they are poorly correlated. Qualitatively, the effect on the eddy statistics of a correlated field is as follows. When the circumference of the latitude circle is greater than the correlation distance L, the eddy variance measures the variation between independent regions. When the circumference is less than L, all points on a latitude circle tend to be well correlated with one another and the eddy variance is much reduced. Figure 1 illustrates this concept. The latitude at which this effect becomes important is at a distance L from the pole.

Historically, the scarcity of meteorological observations at high latitudes has prevented reliable analysis into zonal means and eddies. As a result, few such analyses were done at high latitudes and latitude dependencies in the statistical methods themselves may not have been important. In the last ten to fifteen years, however, global general circulation models and polar orbiting satellites have provided simulations or observations all the way to the poles. In these newer global datasets, the use of spherical coordinates to define averaging operators, while formally correct, may create artificial latitude dependencies of eddy variances and zonal power spectra in middle and high latitudes. Hartmann (1976, Fig. 3), for example, shows the eddy variance of temperature and geopotential height in the Southern Hemisphere as a function of latitude and height. Both quantities peak in middle latitudes and fall to zero at the pole. The atmosphere, of course, is not devoid of transient disturbances at the pole. Swanson and Trenberth (1981, Fig. 4) have also plotted the eddy variance of the geopotential height in the Southern Hemisphere. The structure is similar to that found by Hartmann, with the eddy variance falling to zero at the pole.

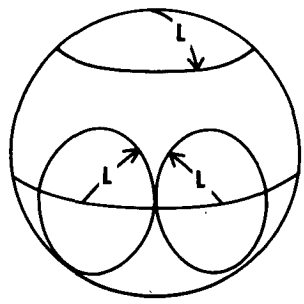

FIG. 1. If a field is well correlated with itself over a distance L, the variance around any latitude circle with radius less than L is small because all points within L of the pole are well correlated. At lower latitudes many independent regions with dimension L exist around a latitude circle. As a result the variance around the latitude circle is larger.

The length scale implicit in zonal averaging can also affect the statistics of zonal-mean quantities. North et al. (1982) have shown that the time variance of zonal means increases toward the pole as 1/cos(latitude) for a field with uniform statistics over the globe. This effect is the converse of that for eddy variances. Zonal means are computed over fewer independent regions as the latitude circles grow smaller. Therefore, sampling fluctuations increase at higher latitudes.

In this paper we will derive analytical expressions for the eddy variance and zonal harmonic spectrum for the simple case of longitudinally invariant statistical fields on a sphere. The latitude dependence of the eddy variance and spectrum are illustrated for several simple forms of the spatial correlation function. In section 4 the theoretical results are compared with observed eddy variance statistics from satellite ozone data.

2. Eddy variance

a. Theory

Because much of the derivation below is analogous to that of North et al. (1982, appendix B), we have followed their notation. Given a field $F(\mathbf{r})$, where $\mathbf{r}$ is the position vector (θ, ϕ), θ is latitude, and ϕ is longitude, the zonal mean is defined as:

$$[F(\theta,\phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta,\phi)d\phi. \tag{1}$$

The latitude dependence θ will be implicit in most of what follows. The deviation from the zonal mean is

$$F^* = F - [F]$$

and the eddy variance is

$$[F^{*2}] = [F^2] - [F]^2.$$

At the pole itself $[F^2] = [F]^2$, so the eddy variance, whether due to stationary or transient eddies, must vanish there.

The theoretical analysis is simplified by considering ensemble rather than time averages. We will assume that we have an infinite ensemble of independent realizations of the state of the earth's atmosphere. The ensemble mean is represented here by angle brackets $\langle\ \rangle$. The ensemble mean of the eddy variance is

$$\langle[F^{*2}]\rangle = \langle[F^2]\rangle - \langle[F]^2\rangle. \tag{2}$$

The eddy variance can be decomposed into transient and stationary parts:

$$\langle[F^{*2}]\rangle = \langle[(F^{*\prime})^2]\rangle + [\langle F^*\rangle^2], \tag{3}$$

where the prime indicates the deviation from the ensemble mean, $F^{*\prime} = F^* - \langle F^*\rangle$. The behavior of the second term, the stationary eddy variance, is briefly discussed at the end of this section. We will be primarily concerned here with the first term, the transient eddy

variance, and so for simplicity we will assume that the ensemble mean $\langle F \rangle$ is zero everywhere. This immediately removes the stationary eddies.

We will investigate how the transient eddy variance depends on the spatial correlation of the field. To simplify the discussion we assume that the point variance $\langle F^2 \rangle$ is independent of longitude, and that the covariance c (or correlation ρ) is a function only of longitudinal separation λ. The theory can be extended to longitudinally inhomogeneous fields, but since the method for doing this is analogous to the procedure followed by North et al. (1982, appendix B) for the variance of zonal means, we will not present a detailed account of it here. The only effect of longitudinal inhomogeneity is that quantities discussed below are replaced by local-variance-weighted averages over longitude.

The ensemble variance at a point is

$$c(0) = \langle F(\phi)F(\phi) \rangle = \langle F^2 \rangle \tag{4}$$

and the covariance between two points separated by longitude λ is

$$c(\lambda) = \langle F(\phi)F(\phi + \lambda) \rangle = \langle F^2 \rangle \rho(\lambda) = c(0)\rho(\lambda). \tag{5}$$

Qualitatively the correlation ρ should have the form shown in Fig. 2. At low latitudes a given point should be well correlated with its nearby neighbors, but poorly correlated with distant points on the same latitude circle. At high latitudes all points are nearby neighbors, so the correlation is high all the way around the latitude circle.

The first term on the right-hand side of (2) can be rewritten by using (4):

$$\langle [F^2] \rangle = [\langle F^2 \rangle] = [c(0)] = c(0). \tag{6}$$

The second term on the right-hand side of (2) is

$$\langle [F]^2 \rangle = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \langle F(\phi)F(\phi') \rangle d\phi d\phi'.$$

Setting $\lambda = \phi - \phi'$ and using (5) gives

$$\langle [F]^2 \rangle = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \langle F^2 \rangle \rho(\lambda) d\phi d\lambda = c(0) f, \tag{7}$$

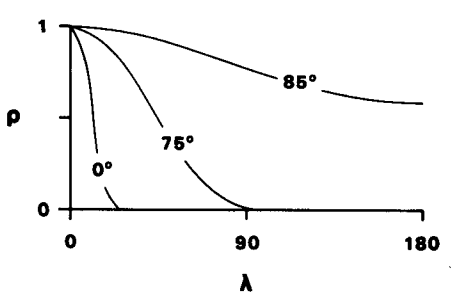

FIG. 2. Expected qualitative form of the correlation ρ as a function of longitudinal separation λ at three different latitudes. The mathematical form of the correlation function is given in (10). The correlation length along a great circle arc is $\pi/8$.

where

$$f = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\lambda) d\lambda \tag{8}$$

is a nondimensional correlation length. It can be interpreted as the fraction of the latitude circle over which the field is significantly correlated. Equation (2) can now be rewritten by using (6) and (7):

$$\langle [F^{*2}] \rangle = c(0)(1 - f). \tag{9}$$

In the simplest case, in which c is independent of latitude, the eddy variance is proportional to $1 - f$. For meteorological variables there is no reason to expect $c(0)$ or the correlation length to vary rapidly near the pole, in which case the $1 - f$ term will dominate in high latitudes and $\langle [F^{*2}] \rangle$ will go to zero. One exception would be "spatial white noise," that is, a field for which the covariance is

$$\langle F(\mathbf{r})F(\mathbf{r}') \rangle = \begin{cases} \langle F^2 \rangle & \text{for } \mathbf{r} = \mathbf{r}' \\ 0 & \text{for } \mathbf{r} \neq \mathbf{r}'. \end{cases}$$

In this extreme case the correlation length is zero and the eddy variance is independent of latitude.

Another exception is the eddy variance of a vector field such as wind velocity. Both the zonal and meridional components of the wind field (if it is reasonably smooth spatially) are constrained to vary nearly sinusoidally (wavenumber 1) with longitude. As a result, the spatial correlation $\rho(\lambda)$ is constrained to behave like $\cos(\lambda)$ near the poles, and the integral scale f in (8) will actually go to zero there (instead of 1, as it does for scalar fields). The eddy variance of vector quantities decomposed into zonal and meridional components are thus not constrained by spherical geometry to vanish near the poles. In particular, eddy kinetic energy need not vanish.

The effect of geometry on the stationary eddy variance, which also contributes to the eddy variance in (3), is less easy to generalize. If the mean field $\langle F(\mathbf{r}) \rangle$ has a Taylor series expansion (i.e., is smoothly behaved) near the poles, then it is easy to show that the stationary eddy variance must go to zero at least as fast as $[(\pi/2) - |\theta|]^2$; that is, quadratically with the angular distance of the latitude circle from the pole.

b. Example

The latitude dependence of $1 - f$ is shown in Fig. 3 for a simple form of the correlation function

$$\rho(\psi) = \begin{cases} [1 + \cos(n\psi)]/2 & \text{for } \psi \leq \pi/n \\ 0 & \text{for } \psi > \pi/n \end{cases} \tag{10}$$

The correlation is assumed to be a function only of great circle distance ψ. The great circle arc ψ between the two points $\mathbf{r}$ and $\mathbf{r}'$ is defined by the relationship $\mathbf{r} \cdot \mathbf{r}' = |\mathbf{r}||\mathbf{r}'| \cos\psi$. The half-width of the correlation

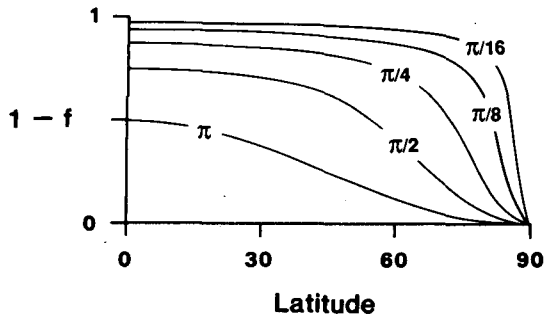

FIG. 3. Latitude dependence of the quantity $1 - f$ for different correlation length scales. The correlation length is given in radians along a great circle arc. The eddy variance is given by (9).

function is π/n. (This artificial correlation function is not analytic at π/n, which may lead to unphysical results. For example, the power spectrum computed from the correlation function may have negative spectral power. In practice, problems with the power spectrum in this case are negligible.) This is the correlation function used to make Fig. 2, with $n = 8$. By defining ρ in terms of ψ instead of λ we describe a field whose statistics are the same everywhere on the earth. At the equator the great circle distance ψ is equal to the longitudinal separation λ. At higher latitudes the great circle distance between two points is less than the longitudinal separation.

Examples are shown in Fig. 3 for $n = 1$, 2, 4, 8 and 16 (half-width of 20 000 km, 10 000 km, etc.). When the correlation function is wider (small n), the effects of the spherical coordinates are felt at lower latitudes. When the half-width is very small, the geometrical effects are only important at high latitudes. The latitude at which the eddy variance begins to depart from the point variance is approximately one half-width from the pole.

3. Zonal harmonic spectra

a. Theory

In this section we will show that the zonal Fourier harmonics of a meteorological field will also depend on latitude, even when the statistics of the field itself do not. (This effect is well known; see Blackmon and White, 1982, for just one example.) A field F that is a function of longitude ϕ can be expanded into zonal Fourier harmonics as

$$F(\phi) = \sum_{m=-\infty}^{\infty} f_m e^{im\phi}.$$

The harmonic amplitudes are given by

$$f_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\phi)e^{-im\phi}d\phi.$$

Because F is real, $f_{-m} = f_m^*$, where f^* is the complex conjugate. The power in wavenumber m, p_m, is $f_m f_m^*$. The power spectrum can also be calculated from the covariance function. In the ensemble mean the power spectrum is

$$\langle p_m \rangle = \langle F^2 \rangle \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(\lambda)e^{-im\lambda}d\lambda.$$

Thus, the average of the correlation function $\rho(\lambda)$ over longitudinal separation λ, (8), determines the total eddy variance, while the Fourier transform of $\rho(\lambda)$ determines the zonal harmonic power spectrum. If $\rho(\psi)$ is uniform over the sphere as in (10), $\rho(\lambda)$ will not be. The Fourier transform of the correlation functions shown in Fig. 2 will be quite different.

b. Example

Zonal power spectra of a field with the cosine correlation function (10) used in the previous section are plotted in Fig. 4 for several different latitudes. The half-width of the cosine curve is $\pi/16$. The power moves rapidly to lower wavenumbers at high latitudes as the size of the latitude circle approaches the correlation length. However, substantial changes in the power spectrum are noticeable even in midlatitudes.

4. Example from real data

To illustrate the existence of these geometrical effects in a real dataset, eddy variance statistics have been calculated for total ozone data from the Total Ozone Mapping Spectrometer (TOMS) on the Nimbus-7 satellite (see Bowman and Krueger, 1985, for a more complete climatology of the first four years of TOMS data). Since we are dealing with real data, time means were calculated instead of (the hypothetical) ensemble means. A subset of the TOMS data was selected, consisting of daily gridded fields of total ozone over the Northern Hemisphere during the month of June for

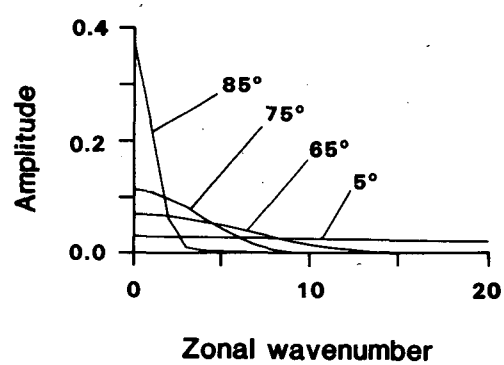

FIG. 4. Power spectrum of the zonal Fourier components of a field with the correlation function given in (10). The half-width of the correlation function is $\pi/16$.

the five years 1979 to 1983. The resolution of the data set is approximately 100 by 100 km. Approximately 15 of the possible 150 days in the selected period are missing at any given location, most of these during the first year.

Figure 5 shows the time means of the eddy variance and the zonal mean of the point variance [$c(0)$]. The eddy variance is low in the tropics (not shown), relatively constant in middle latitudes, and decreases toward zero at the pole, as expected. The dashed line in Fig. 6 clearly shows that the point variance of the field does not decrease at the pole, but remains near the levels reached in midlatitudes. This can also be seen in maps of the point variance (not shown).

The correlation as a function of longitudinal separation is shown in Fig. 6. The observed shape of $\rho(\lambda)$ agrees quite well with the simple model used in sections 2 and 3, though its decrease from 1 for small λ may be closer to linear than quadratic. The correlation length L can be roughly estimated from Fig. 6 to be about the same as that used in Fig. 2, $\pi/8$ radians (22°) along a great circle, or about 2500 kilometers. From the results in Fig. 6, $1-f$ was calculated and is plotted in Fig. 7. As expected, $1-f$ rapidly goes to zero near the pole. Its approach to zero is linear rather than quadratic (cf. Fig. 3) because of the linear behavior of $\rho(\lambda)$ for small λ (Fig. 6). As a consequence the contribution of the transient eddy term in (3) dominates that of the stationary eddy variance (which approaches zero quadratically, as discussed in section 2a) near the pole.

The latitude dependence of the eddy variance of the January 500 mb height field can be inferred from Fig. 4 in North et al. (1982), in which f is plotted as a function of latitude. The rapid rise of f at 75°N suggests that the appropriate correlation length for the 500 mb height field is ~15° of arc or ~1600 km.

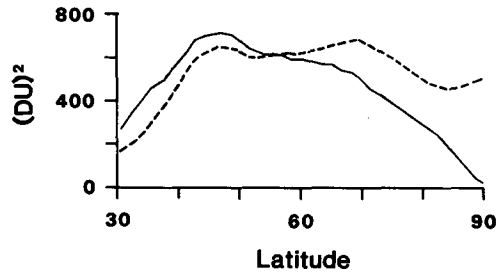

FIG. 5. Time means of the eddy variance (including the stationary eddy variance, solid line) and the zonal mean of the point variance (dashed line) of total ozone in the Northern Hemisphere for June. Total ozone is measured in Dobson units (1 DU = 10^{-3} atm cm), so variances are in $(DU)^2$. Because of the contribution from stationary eddies, the eddy variance is larger than the zonal mean of the point variance south of ~55°N. Daily values from 1979 to 1983 were used to compute the time means. The point variance field is rather flat near the pole, as indicated by the zonal mean of the point variance and by maps of the point variance (not shown). The eddy variance on the other hand goes to zero at the pole, as it must.

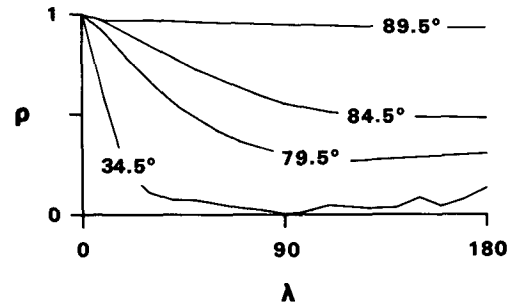

FIG. 6. Zonal-mean correlation ρ as a function of longitudinal separation λ for the total ozone data used in Fig. 5. At high latitudes even points on opposite sides of a latitude circle are well correlated because the actual distance between the points becomes small. Compare with Fig. 2.

5. Conclusions

We have shown that eddy variances can have strong latitude dependencies even when the actual statistics of the field are uniform over the globe. These latitude variations are purely geometrical and arise from the use of spherical polar coordinates. Our results suggest that care should be taken when interpreting the decrease in the eddy variance of many meteorological fields observed on the poleward side of the midlatitude baroclinic zone. For the ozone data shown in the previous section, the point variance is fairly constant north of 50°N, but the eddy variance goes to zero at the pole. Maps of the standard deviation of the wintertime 500 mb height (the square root of the point variance) presented in Oort (1983, Fig. A21) and the latitude profile of f from North et al. (1982) indicate that the same is true for 500 mb heights. That is, the point standard deviation of the 500 mb height is greater than 100 m over the entire polar cap north of 60°N, but, once again, the eddy variance must decrease to zero at the pole. Clearly, eddy variance statistics alone should not be used to argue that significant variability in the atmosphere is confined to the middle latitudes. The point variance and correlation functions, which are inde-

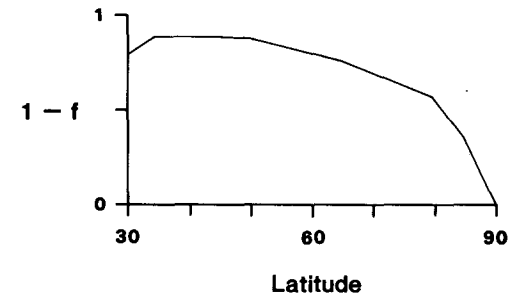

FIG. 7. Latitude dependence of $1-f$ for the total ozone data. As expected from Fig. 6, this quantity goes to zero at the pole. Compare with Figs. 3 and 4.

pendent of the choice of coordinate system, may be better indicators of the level of variability.

We have also shown that zonal harmonic power spectra are influenced by the use of spherical coordinates. At higher latitudes a larger fraction of the total variance can be expected in the low wavenumbers as the latitude circle includes a smaller and smaller part of the earth.

Eddy variances and zonal power spectra are mathematically well-defined quantities. Care should be taken, however, when comparing these quantities between different latitudes.

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REFERENCES

Blackmon, M. L., and G. H. White, 1982: Zonal wavenumber characteristics of Northern Hemisphere transient eddies. *J. Atmos. Sci.*, **39,** 1985–1998.

Bowman, K. P., and A. J. Krueger, 1985: A global climatology of total ozone from the Nimbus 7 Total Ozone Mapping Spectrometer. *J. Geophys. Res.*, **90,** 7967–7976.

Hartmann, D. L., 1976: The structure of the stratosphere in the Southern Hemisphere during late winter 1973 as observed by satellite. *J. Atmos. Sci.*, **33,** 1141–1154.

North, G. R., F. J. Moeng, T. L. Bell and R. F. Cahalan, 1982: The latitude dependence of the variance of zonally averaged quantities. *Mon. Wea. Rev.*, **110,** 319–326.

Oort, A. H., 1983: Global Atmospheric Circulation Statistics, 1958–73, NOAA Prof. Paper 14, National Oceanic and Atmospheric Administration, 180 pp.

Swanson, G. S., and K. E. Trenberth, 1981: Interannual variability in the Southern Hemisphere troposphere. *Mon. Wea. Rev.*, **109,** 1890–1897.