Cluster Expansion in a Renormalizable Theory: The Elastic Form Factor*

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The method of cluster decomposition, well known in statistical mechanics and recently applied to φ^3 field theory by Chang, Yan, and Yao and by Campbell and Chang, is here extended to the calculation of the electromagnetic form factor in a renormalizable neutral pseudoscalar theory. In particular the set of so-called uncrossed "rainbow" diagrams are analyzed in the limit of spacelike momentum transfer squared, q^2, much larger than the pion or proton mass squared, μ^2 or M^2. In this limit the Dirac form factor behaves as $F_1(q^2)=B(g^2, \mu^2/m^2, M^2/m^2)(-q^2/m^2)^{-A(g^2)}$ where g is the $\pi^0 p$ coupling and m is an arbitrary scale factor. The functions A and B are shown to arise in momentum space from "volume" and "surface" effects, respectively. They are given as a power series in g^2. The first two terms in A are calculated explicitly, giving $A=(g^2/32\pi^2)+\frac{5}{2}(g^2/32\pi^2)^2+\cdots$.

I. INTRODUCTION

The only theoretical structure so far developed which has all the basic principles such as superposition, analyticity, and so on, is quantum field theory. A compelling question is what the asymptotic behavior of the theory may imply about high-momentum or short-distance aspects of hadrons. Since only perturbative solutions exist, one must analyze infinite series. In most studies one keeps only the leading asymptotic term in each order of perturbation theory, while neglecting similar terms in higher orders. The hope is that nonleading terms will not affect the qualitative conclusions

The classic example of this approach is the analysis of "ladder" diagrams in φ^3 theory with s, the center-of-mass energy squared, taken asymptotic.[1] The leading term of an n-rung ladder is proportional to $(\ln s)^{n-1}$ and the sum is proportional to $s^{\alpha(t)}$ where α represents a Regge trajectory function for the exchanged ladder. It was known for some time that inclusion of all nonleading terms, proportional to $(\ln s)^m$, $m< n-1$, still leads to Regge behavior with the only modification being in the trajectory function. Recently Chang, Yan, and Yao[2] showed that the nonleading terms are associated with regions of integration in which adjacent longitudinal momenta on the sides of the ladder are comparable. Campbell and Chang[3] found that these "correlations" provide a physical mechanism for producing the Regge behavior via the cluster-decomposition method of statistical mechanics. They also applied the technique to production processes,[4] and their conclusions support and extend the gas model of Feynman and Wilson.[5]

The intention of this paper is to apply the cluster method to a certain calculation in a renormalizable theory. In particular, a set of diagrams for the electromagnetic form factor of the proton with neutral pions is chosen. These are called "rainbow" diagrams and are shown in Fig. 1. The external protons are on-mass-shell and the photon momentum squared is asymptotic in the spacelike direction.[6] That is, $-q^2 \gg m^2$ where m is an arbitrary parameter of the order of the proton or pion mass squared, M^2 or μ^2. The sum of terms leading in $\ln(-q^2/m^2)$ was one of many results given in an excellent study by Appelquist and Primack.[7] They found for the rainbows in leading order:

$$\exp\left[-\frac{q^2}{32\pi^2}\ln\left(-\frac{q^2}{m^2}\right)\right]=\left(-\frac{q^2}{m^2}\right)^{-(g^2/32\pi^2)},$$

where g is the $\pi^0 p$ coupling constant. They also found that the sum of next-to-leading terms is $\ln(-q^2/m^2)$ larger. The present paper verifies these results, although the sum of next-to-leading terms is numerically different. Inclusion of all nonleading terms provides a simple modification on the leading terms.

The results of this work can be summarized as follows:

(a) The rainbow diagrams represent a conserved current, so only the usual Dirac and Pauli form factors appear. The Dirac form factor F_1 is expected to be larger than the Pauli term F_2 by $O(m^2/-q^2)$ and only F_1 is calculated.

(b) The 3-axis is chosen along the initial proton momentum. All 4-vectors are written in component form as $a=(a_+, \vec{a}, a_-)$ where $a_\pm = a_0 \pm a_3$ and $\vec{a}$ is in the 12 plane, transverse to the initial proton. The minus integrations are performed, fixing the minus components by mass-shell conditions. The plus integrations are then finite, but the unrenormalized transverse integrals are logarithmically divergent. However, it is found that the only important integration regions are those in which the magnitudes of the transverse momenta are ordered in a certain way. The innermost pion carries the largest transverse momentum.

(c) Each divergent transverse integral requires a single subtraction involving products of lower-order amplitudes. It is found that the subtractions cancel contributions from all integration regions except those in which the magnitudes of transverse momenta are in the opposite order to that mentioned in (b). Then the outermost pion has the largest transverse momentum, and all transverse integrations are bounded by $\vec{q}^2$.

(d) The integrand, appropriately defined including all subtractions, has a factorization property analogous to that of the integrand of the partition function in statistical mechanics. Along with the properties described above, this is sufficient to show by the cluster-decomposition method that the Dirac form factor of the rainbow has the form:

$$F_1(q^2)=\exp\left[\sum_{i=1}^{\infty}\left(\frac{g^2}{32\pi^2}\right)^i C_i(q^2)\right],$$

where

$$C_i(q^2)=a_i\ln(-q^2/m^2)+b_i(\mu^2/m^2,\,M^2/m^2)+O(m^2/(-q^2))\,.$$

The q^2 dependence comes from the "interior region," where the transverse pion momenta squared are much less than $\vec{q}^2$ and much greater than m^2. In particular, a_i arises when any i adjacent pions have comparable transverse momenta, or are "correlated." It is independent of both M^2 and μ^2. Secondly, b_i arises when any i adjacent pions have transverse momenta squared comparable to $\vec{q}^2$ or m^2, representing a "surface effect" dependent on the masses.

(e) The leading terms arise from completely independent integrations, giving

$$a_1=-1$$

in agreement with Ref. 7. Correlations of any two pions contributes

$$a_2=-\tfrac{5}{2}\,.$$

Thus the nonleading terms represent correlations which are indeed important when g is $O(1)$ as in strong interactions.

The outline of the paper is as follows: Section II gives the proof that the rainbow current is conserved and some notation is introduced. In Sec. III the lowest-order diagrams are examined, and the transverse dependence and renormalization are analyzed in detail. Section IV shows how the low-order results are generalized to the N-pion rainbow, with special emphasis on renormalization. Section V gives the cluster decomposition and proof of exponentiation. Finally the results and possible extensions are discussed in Sec. VI.

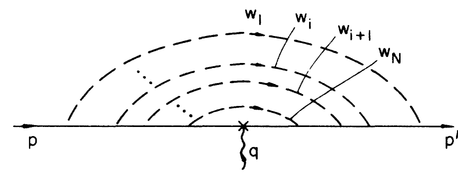

FIG. 1. The general $\pi^0 p$ "rainbow" diagram studied here. Only the Dirac form factor is extracted.

Appendix A shows how the current conservation comes about in terms of Feynman parameters for the two-pion case. Appendix B shows how the one-, two-, and three-pion numerators are calculated in the formalism of this paper. Appendix C gives the calculation of a_1 and a_2.

II. KINEMATICS AND NOTATION

As mentioned in the Introduction the set of diagrams studied here are the neutral pion "rainbow" diagrams shown in Fig. 1. The initial proton of 4-momentum p emits a number of pions, the ith pion having 4-momentum w_i. It then interacts with the electromagnetic current, picking up 4-momentum q and reabsorbs the pions in opposite order. The final proton momentum is $p'=p+q$.

It is important to note that one does not in general expect these diagrams to produce a conserved current. In other words, the current is generally conserved only when it is inserted in all possible ways on a continuous charged line.[8] This generates a broader class of diagrams than the rainbows, and a typical one is shown in Fig. 2. However, when the external protons are on-shell the symmetry of the rainbows gives current conservation, as will now be shown.

A. Rainbow Form Factors

The electromagnetic vertex function, for any value of p'^2, p^2, or q^2, must transform as a 4-vector under the Lorentz group and parity. Under these conditions, one can construct twelve independent quantities from the available vectors and Dirac matrices. These can be cast in the following convenient form[9]:

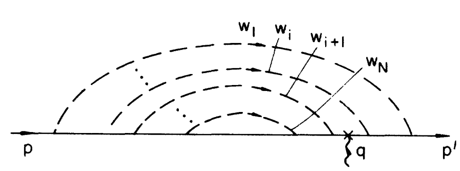

FIG. 2. A typical diagram one must add to that of Fig. 1 in order to ensure gauge invariance for any p^2, p'^2, and q^2.

$$\Gamma_\mu(p',p)=\frac{\not p'+M}{2M}(\mathfrak{F}_1\gamma_\mu+i\mathfrak{F}_2\sigma_{\mu\nu}q^\nu+\mathfrak{F}_3q_\mu)\frac{\not p+M}{2M}$$
$$+\frac{-\not p'+M}{2M}(\mathfrak{F}_4\gamma_\mu+i\mathfrak{F}_5\sigma_{\mu\nu}q^\nu+\mathfrak{F}_6q_\mu)\frac{-\not p+M}{2M}$$
$$+\frac{\not p'+M}{2M}(\mathfrak{F}_7\gamma_\mu+i\mathfrak{F}_8\sigma_{\mu\nu}q^\nu+\mathfrak{F}_9q_\mu)\frac{-\not p+M}{2M}$$
$$+\frac{-\not p'+M}{2M}(\mathfrak{F}_{10}\gamma_\mu+i\mathfrak{F}_{11}\sigma_{\mu\nu}q^\nu+\mathfrak{F}_{12}q_\mu)\frac{\not p+M}{2M}, \quad (2.1)$$

where $\mathfrak{F}_i=\mathfrak{F}_i(p'^2,p^2,q^2)$. Clearly only three terms contribute when this is sandwiched between positive- or negative-energy spinors. One can check that the general rainbow vertex, without the spinors, transforms as the bare vertex under charge conjugation. In other words,

$$C^{-1}\Gamma_\mu(p',p)C=-\Gamma_\mu^T(-p,-p'). \quad (2.2)$$

This holds for any diagram which is symmetric about the current insertion. It implies conditions on the form factors in Eq. (2.1). The one of interest here is

$$\mathfrak{F}_3(p'^2,p^2,q^2)=-\mathfrak{F}_3(p^2,p'^2,q^2) \quad (2.3)$$

which implies that $\mathfrak{F}_3$ vanishes when $p'^2=p^2$. Thus one has

$$\bar u(p')\Gamma_\mu(p',p)u(p)$$
$$=\bar u(p')[F_1(q^2)\gamma_\mu+iF_2(q^2)\sigma_{\mu\nu}q^\nu/2m]u(p). \quad (2.4)$$

Thus the rainbow current is conserved and one need not include diagrams such as the one shown in Fig. 2.

The functions F_1 and F_2 are the usual Dirac and Pauli form factors, respectively. It is expected that $F_2/F_1\sim O(m^2/(-q^2))$ since a factor of q is extracted in the definition of F_2 and it must be a function of q^2. This can be verified in low orders. Only F_1 will be calculated here.

B. Infinite-Momentum Technique

The components of a typical 4-vector a_μ will be written as $(a_+,\vec a,a_-)$ which are related to the usual components by $a_\pm=a_0\pm a_3$ and $\vec a=(a_1,a_2)$.[10] The invariant product takes the form

$$a\cdot b=\tfrac12(a_+b_-+a_-b_+)-\vec a\cdot\vec b. \quad (2.5)$$

Firstly, a Lorentz boost along the z axis with rapidity ξ causes the change

$$a\to(a_+e^\xi,\vec a,a_-e^{-\xi}) \quad (2.6)$$

which clearly leaves Eq. (2.5) invariant. Secondly, Lorentz transformations in the transverse plane form a Euclidean subgroup. Such a transformation is generally complicated, but if a and b are placed on-shell by setting $a_-=(\vec a^2+m_a{}^2)/a_+$, and similarly for b_-, it takes the form

$$a\to\left(a_+,\vec a+a_+\vec c,\frac{(\vec a+a_+\vec c)^2+m_a{}^2}{a_+}\right) \quad (2.7)$$

and similarly for b, where $\vec c$ is a constant transverse vector.

Finally, the Dirac matrices in this representation have the following properties:

$$\{\gamma_+,\gamma_-\}=4,$$
$$\{\gamma_\pm,\vec\gamma\}=0, \quad (2.8)$$
$$\gamma_\pm{}^2=0.$$

C. Choice of Frame

The particular reference frame chosen here has its 3 axis defined by the initial proton direction, and the scale of plus components defined by $p_+=1$. Furthermore, since q^2 is spacelike the frame description can be completed by setting $q_+=0$. With the initial and final protons on-shell this implies

$$p=(1,\vec 0,M^2),$$
$$q=(0,\vec q,\vec q^{\,2}), \quad (2.9)$$
$$p'=(1,\vec q,\vec q^{\,2}+M^2).$$

In order to determine the form factors in Eq. (2.4) it is sufficient to calculate a single component of Γ_μ. A convenient choice is found to be $\mu=+$. Then F_1 can be extracted from $\bar u(p')\Gamma_+u(p)$ by multiplying on the left by $\gamma_+u(p')$, on the right by $\bar u(p)$ and summing over spins. Since $p_+=1$ and $q_+=0$ one obtains upon taking the trace

$$F_1(q^2)=\tfrac18\,\mathrm{Tr}[\gamma_+(\not p'+M)\Gamma_+(p',p)(\not p+M)]. \quad (2.10)$$

A final notation is to emphasize the different roles played by the longitudinal and transverse pion momenta. After integration over the minus components it will be seen that the plus components lie between $p_+=1$ and $q_+=0$. They represent the fractional longitudinal momenta as in the "parton model".[11] As a reminder of this the plus component of the ith pion is called x_i, so that

$$w_i=(x_i,\vec w_i,w_{i-}). \quad (2.11)$$

III. LOW-ORDER CALCULATIONS

In this section the three lowest-order amplitudes are examined explicitly. Integrations over the w_{i-} components are performed and the properties of the transverse integrations are analyzed. Special attention is paid to renormalization. The insights obtained here are generalized to all orders in Sec. IV.

A. One-Pion Diagram

According to Eq. (2.10) the amplitude in this order can be written[12]

$$F_1^{(1)} \equiv \frac{-ig^2}{(2\pi)^4}\int d^4 w_1 \frac{\frac{1}{8}\mathrm{Tr}[\gamma_+(\not p' + M)\gamma_5(\not p' - \not w_1 + M)\gamma_+(\not p - \not w_1 + M)\gamma_5(\not p + M)]}{(w_1^2 - \mu^2 + i\epsilon)[(p - w_1)^2 - M^2 + i\epsilon][(p' - w_1)^2 - M^2 + i\epsilon]}, \tag{3.1}$$

where $d^4 w_1 = \frac{1}{2} dx_1 d^2 w_1 dw_{1-}$.

Integration Over w_{1-}

In the frame defined in Eq. (2.10) the denominator factors can be expressed as

$$\begin{aligned} & x_1\left(w_{1-} - \frac{\vec{w}_1^2 + \mu^2}{x_1} + \frac{i\epsilon}{x_1}\right), \\ & (1 - x_1)\left(M^2 - w_{1-} - \frac{\vec{w}_1^2 + M^2}{1 - x_1} + i\frac{\epsilon}{1 - x_1}\right), \\ & (1 - x_1)\left[\vec{q}^2 + M^2 - w_{1-} - \frac{(\vec{q} - \vec{w}_1)^2 + M^2}{1 - x_1} + i\frac{\epsilon}{1 - x_1}\right]. \end{aligned} \tag{3.2}$$

The contour integral of w_{1-} will vanish if the poles lie on the same side of the real axis. This implies that $\mathrm{sgn}(x_1) = \mathrm{sgn}(1 - x_1)$, or

$$0 < x_1 < 1. \tag{3.3}$$

Under this condition, the contour can be closed in the lower half-plane, picking up the pion pole. Around the semicircle $w_{1-} = Re^{i\theta}$ with $R \to \infty$. The proton propagators contribute two factors of $R\gamma_+$ in the numerator, but these are cancelled by the current, since $\gamma_+^2 = 0$. Thus the integral damps out as $1/R^2$ on the semicircle, and only the residue at the pole contributes. Then Eq. (3.1) takes the form

$$\begin{aligned} F_1^{(1)} = \frac{-g^2}{2(2\pi)^3}\int dx_1 d^2 w_1 \, \tfrac{1}{8}\mathrm{Tr}[\gamma_+(\not p' + M)\gamma_5(\not p' - \not w_1 + M)\gamma_+(\not p - \not w_1 + M)\gamma_5(\not p + M)] \\ \times\left\{x_1(1 - x_1)\left(M^2 - \frac{\vec{w}_1^2 + \mu^2}{x_1} - \frac{\vec{w}_1^2 + M^2}{1 - x_1}\right)(1 - x_1)\left[\vec{q}^2 + M^2 - \frac{\vec{w}_1^2 + \mu^2}{x_1} - \frac{(\vec{q} - \vec{w}_1)^2 + M^2}{1 - x_1}\right]\right\}^{-1}, \end{aligned} \tag{3.4}$$

where in the numerator it is understood that $w_{1-} = (\vec{w}_1^2 + \mu^2)/x_1$.

The denominator in Eq. (3.4) could have been written down immediately from Weinberg's infinite-momentum rules.[13] It can be rewritten as

$$x_1\left[\left(1 + \frac{X_1}{x_1}\right)\vec{w}_1^2 + (1 - X_1)M^2 + \frac{X_1}{x_1}\mu^2\right]\left[\left(1 + \frac{X_1}{x_1}\right)\vec{w}_1'^2 + (1 - X_1)M^2 + \frac{X_1}{x_1}\mu^2\right], \tag{3.5}$$

where

$$X_1 \equiv 1 - x_1, \quad \vec{w}_1' \equiv \vec{w}_1 - x_1\vec{q}. \tag{3.6}$$

X_1 is the plus component of the proton after the pion is emitted. The symmetry in $\vec{w}_1$ and $\vec{w}_1'$ is a result of invariance under (2.7), $\vec{q}$ being the only constant transverse vector available. This symmetry must also appear in the numerator.

As shown in Appendix A the numerator may be rewritten as

$$-\tfrac{1}{4}\mathrm{Tr}\{[(x_1\not p' - \not w_1) - x_1 M][(\not w_1 - x_1\not p) - x_1 M]\}.$$

The combination of 4-vectors appearing in each factor has zero plus component, so their dot product involves only the transverse components. Equation (3.4) then takes the form

$$F_1^{(1)} = \frac{g^2}{2(2\pi)^3}\int_0^1 dx_1 \int d^2 w_1 \left[\frac{\vec{w}_1 \cdot \vec{w}_1' + x_1^2 M^2}{x_1\left(\frac{\vec{w}_1^2}{x_1} + x_1 M^2 + \frac{1 - x_1}{x_1}\mu^2\right)\left(\frac{\vec{w}_1'^2}{x_1} + x_1 M^2 + \frac{1 - x_1}{x_1}\mu^2\right)}\right]. \tag{3.7}$$

Renormalization

The transverse integral in (3.7) is logarithmically divergent as $\vec{w}_1^{\,2}\to\infty$. The finite part is defined as usual by subtracting the amplitude with the external lines on-shell, which here means $\vec{q}=\vec{0}$.[14] This is represented diagrammatically in Fig. 3 and gives

$$\tilde{F}_1^{(1)} \equiv \frac{g^2}{2(2\pi)^3}\int_0^1 x_1 dx_1 \int d^2 w_1 \left\{ \frac{\vec{w}_1\cdot\vec{w}_1' + x_1^2 M^2}{[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2][\vec{w}_1'^{\,2}+x_1^2M^2+(1-x_1)\mu^2]} - \frac{\vec{w}_1^{\,2}+x_1^2M^2}{[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2]^2}\right\}. \tag{3.8}$$

One can add and subtract $(1-x_1)\mu^2$ in both numerators and combine terms to give

$$\tilde{F}_1^{(1)} = \frac{g^2}{2(2\pi)^3}\int_0^1 x_1 dx_1 \int d^2 w_1 \left\{ \frac{x_1\vec{q}\cdot\vec{w}_1'}{[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2][\vec{w}_1'^{\,2}+x_1^2M^2+(1-x_1)\mu^2]} + (1-x_1)\mu^2 \frac{x_1^2\vec{q}^{\,2}+2x_1\vec{q}\cdot\vec{w}_1}{[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2]^2[\vec{w}_1'^{\,2}+x_1^2M^2+(1-x_1)\mu^2]}\right\}. \tag{3.9}$$

Letting $d^2w_1\to d^2w_1'$ simplifies the angular integrations. The second term is $O(m^2/\vec{q}^{\,2})$. Letting $y=\vec{w}_1'^{\,2}+x_1^2M^2+(1-x_1)\mu^2$ one obtains after angular integration

$$\tilde{F}_1^{(1)} = \frac{g^2}{2(2\pi)^3}\int_0^1 x_1 dx_1 \tfrac{1}{2}\pi \int_{x_1^2M^2+(1-x_1)\mu^2}^{\infty}\frac{dy}{y}\left\{1-\frac{y+x_1^2\vec{q}^{\,2}}{\left[y^2-(2x_1^2\vec{q}^{\,2})y+(x_1^2\vec{q}^{\,2})^2\left(1+4\,\frac{x_1^2M^2+(1-x_1)\mu^2}{x_1^2\vec{q}^{\,2}}\right)\right]^{1/2}}\right\} + O(m^2/\vec{q}^{\,2}). \tag{3.10}$$

In the limit $\vec{q}^{\,2}\gg M^2,\ \mu^2$ this becomes simply

$$\tilde{F}_1^{(1)} = -\left(\frac{g^2}{32\pi^2}\right)\ln(\vec{q}^{\,2}/m^2) + O(m^2/\vec{q}^{\,2}). \tag{3.11}$$

Analysis of $\vec{w}_l$ Integration

In light of the simple result shown in Eq. (3.11) it is natural to rewrite Eq. (3.8) as

$$\tilde{F}_1^{(1)} \equiv \left(\frac{g^2}{32\pi^2}\right)G_1,$$
$$G_1 \equiv \frac{2}{\pi}\int dx_1 \int \frac{d^2w_1}{\vec{w}_1^{\,2}} g_1(\vec{q},\vec{w}_1,m). \tag{3.12}$$

where it is understood that $x_1>0$. Here the "renormalized integrand" g_1 is given in terms of the "unrenormalized integrand" f_1 by[15]

$$g_1(\vec{q},\vec{w}_1,m) \equiv f_1(\vec{q},\vec{w}_1,m) - f_1(\vec{0},\vec{w}_1,m)$$
$$\equiv \theta(1-x_1)\vec{w}_1^{\,2}\left\{\frac{\vec{w}_1\cdot\vec{w}_1'+x_1^2M^2}{(1/x_1)[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2][\vec{w}_1'^{\,2}+x_1^2M^2+(1-x_1)\mu^2]} - \frac{\vec{w}_1^{\,2}+x_1^2M^2}{(1/x_1)[\vec{w}_1^{\,2}+x_1^2M^2+(1-x_1)\mu^2]^2}\right\}. \tag{3.13}$$

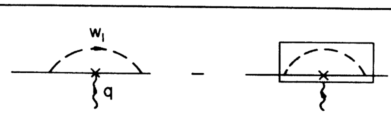

FIG. 3. The finite contribution in $O(g^2)$.

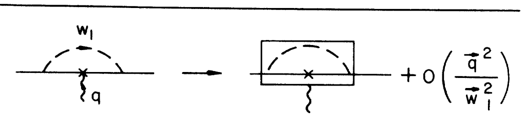

FIG. 4. Behavior in the region $\vec{w}_1^{\,2}\gg\vec{q}^{\,2}$.

TABLE I. Behavior of one-pion integrand in transverse-momentum space.

Transverse region	$f_1(\vec{q}, \vec{w}_1, m)$	$f_1(\vec{0}, \vec{w}_1, m)$	$g_1(\vec{q}, \vec{w}_1, m)$
$\vec{w}_1^{\,2} \ll m^2$	ϵ_1	ϵ_1	ϵ_1
$\vec{w}_1^{\,2} \approx m^2$	ϵ_1, ϵ_2	1	1
$\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg m^2$	ϵ_2, ϵ_3	1	1
$\vec{w}_1^{\,2} \approx \vec{q}^{\,2}$	1	1	1
$\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$	1	1	ϵ_4

This equation is represented in Fig. 3. It can be checked that f_1 and g_1 are at most $O(1)$, as expected from Eq. (3.11).

To take advantage of the fact that $\vec{q}^{\,2} \gg m^2$ one can partition the transverse integration as follows

$$\int \frac{d^2 w_1}{\vec{w}_1^{\,2}} = \int_{-\pi}^{\pi} \frac{d\theta}{2} \left(\int_0^{\epsilon_1 m^2} + \int_{\epsilon_1 m^2}^{m^2/\epsilon_2} + \int_{m^2/\epsilon_2}^{\epsilon_3 \vec{q}^{\,2}} + \int_{\epsilon_3 \vec{q}^{\,2}}^{\vec{q}^{\,2}/\epsilon_4} + \int_{\vec{q}^{\,2}/\epsilon_4}^{\infty} \right) \frac{dw_1^{\,2}}{\vec{w}_1^{\,2}}, \tag{3.14}$$

where the ϵ_i are arbitrarily small positive constants. The order of magnitude of f_1 and g_1 in these regions is given in Table I. [As mentioned in footnote 15, $x_1 \sim O(1)$.] First note that[16]

$$f_1 \sim O(1) \text{ only when } \vec{w}_1^{\,2} \gtrsim (\vec{q}^{\,2}, m^2). \tag{3.15a}$$

[$\vec{w}_1^{\,2} \gtrsim \vec{q}^{\,2}$ is intended to mean $\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$ or $\vec{w}_1^{\,2} \approx \vec{q}^{\,2}$, as in the last two integration regions in (3.14).] Secondly, in the region $\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$ the integral over f_1 diverges, but the subtraction cancels the contribution from this region because

$$f_1(\vec{q}, \vec{w}_1, m) \to f_1(\vec{0}, \vec{w}_1, m) + O(\vec{q}^{\,2}/\vec{w}_1^{\,2}). \tag{3.15b}$$

This property is represented in Fig. 4. These properties of f_1 imply for g_1, as shown in the last column of Table I, the following properties:

$$g_1 \sim O(1) \text{ only when } \vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim m^2; \tag{3.16a}$$

when $\vec{q}^{\,2} \gg \vec{w}_1^{\,2}$

$$\begin{aligned} g_1(\vec{q}, \vec{w}_1, m) &\to -f_1(\vec{0}, \vec{w}_1, m) + O(\vec{w}_1^{\,2}/\vec{q}^{\,2}) \\ &\equiv g_1(\vec{0}, \vec{w}_1, m) + O(\vec{w}_1^{\,2}/\vec{q}^{\,2}). \end{aligned} \tag{3.16b}$$

Neglecting μ^2/M^2 and letting $m^2 = M^2$ one has

$$\begin{aligned} G_1 &= \frac{2}{\pi} \int_0^1 x_1 dx_1 \int_{-\pi}^{\pi} \tfrac{1}{2} d\theta \left\{ \int_{\epsilon_1 M^2}^{M^2/\epsilon_2} \frac{(-1) dw_1^{\,2}}{\vec{w}_1^{\,2} + x_1^{\,2} M^2} + \int_{M^2/\epsilon_2}^{\epsilon_3 \vec{q}^{\,2}} \frac{(-1) dw_1^{\,2}}{\vec{w}_1^{\,2}} + \int_{\epsilon_3 \vec{q}^{\,2}}^{\vec{q}^{\,2}/\epsilon_4} \left[\frac{\vec{w}_1 \cdot (\vec{w}_1 - x_1 \vec{q})}{(\vec{w}_1 - x_1 \vec{q})^2} - 1 \right] \frac{dw_1^{\,2}}{\vec{w}_1^{\,2}} \right\} + O(M^2/\vec{q}^{\,2}) \\ &= \frac{2}{\pi} \int_0^1 x_1 dx_1 \pi \left(-\ln \left| \frac{1}{\epsilon_2 x_1^{\,2}} \right| - \ln \left| \frac{\epsilon_2 \epsilon_3 \vec{q}^{\,2}}{M^2} \right| - \ln \left| \frac{x_1^{\,2}}{\epsilon_3} \right| \right) + O(M^2/\vec{q}^{\,2}) \\ &= [a_1 \ln(\vec{q}^{\,2}/M^2) + b_1] + O(M^2/\vec{q}^{\,2}), \end{aligned} \tag{3.17}$$

where $a_1 = -1$ and $b_1 = 0$. Note that the final answer is independent of the ϵ_i. The $\ln(\vec{q}^{\,2}/M^2)$ comes when $\vec{w}_1^{\,2}$ is free to vary from M^2/ϵ_2 to $\epsilon_3 \vec{q}^{\,2}$. The constant from the region $\vec{w}_1^{\,2} \approx M^2$ cancels with that from $\vec{w}_1^{\,2} \approx \vec{q}^{\,2}$, so that $b_1 = 0$.

The important insight here is that $\ln(\vec{q}^{\,2}/M^2)$ comes only from the region $\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg m^2$ where g_1 is independent of $\vec{q}^{\,2}$ and m^2. Such regions in which the transverse ordering is disjoint will be called "independent" regions of the transverse phase space. It should be emphasized that this association of $\ln(\vec{q}^{\,2}/m^2)$ with the independent region is a result of the properties in (3.16) and the transverse symmetry. That is, first, in the region $\vec{w}_1^{\,2} \approx M^2$ one has (3.16) so that g_1, and the integral, are independent of $\vec{q}^{\,2}$. Second, in the region $\vec{w}_1^{\,2} \approx \vec{q}^{\,2}$ the symmetry allows g_1 to be written as a function of $\vec{w}_1/|\vec{q}|$ and so the integral is again independent of $\vec{q}^{\,2}$. The symmetry is a general property, and it will be seen that Eqs. (3.15) and (3.16) can also be generalized.

B. Two-Pion Diagram

The contribution of the two-pion rainbow to F_1 is

$$F_1^{(2)} \equiv \left[\frac{-ig^2}{(2\pi)^4}\right]^2 \int d^4 w_1 \int d^4 w_2 \, \tfrac{1}{8}\mathrm{Tr}[\gamma_+(\not p' + m)\gamma_5(\not p' - \not w_1 + M)\gamma_5(\not p' - \not w_1 - \not w_2 + M)\gamma_+(\not p - \not w_1 - \not w_2 + M) \times \gamma_5(\not p - \not w_1 + M)\gamma_5(\not p + M)] \times \{(w_1^2 - \mu^2 + i\epsilon)(w_2^2 - \mu^2 + i\epsilon)[(p - w_1)^2 - M^2 + i\epsilon)][(\not p' - w_1)^2 - M^2 + i\epsilon] \times [(p - w_1 - w_2)^2 - M^2 + i\epsilon][(p' - w_1 - w_2)^2 - M^2 + i\epsilon]\}^{-1} . \tag{3.18}$$

The denominator factors can again be written as in Eqs. (3.2) and the contour integrals over w_{1-} and w_{2-} performed. Two factors of w_{1-} and of w_{2-} are cancelled by the γ_+ in the numerator, so around the semicircle both integrals damp out as $(1/R^2)$. In order for the poles to be on opposite sides of the real axis one must require

$$0 < x_1 < 1, \quad 0 < x_2 < 1 - x_1 . \tag{3.19}$$

The denominator again takes the form of Weinberg's rules. It can be rewritten similarly to Eq. (3.5) to explicitly display the symmetry. The numerator can be written as a trace involving only transverse components as shown in Appendix B. In the notation of Eqs. (3.12), (3.13), and (B7) one finds

$$F_1^{(2)} \equiv [g^2/2(2\pi)^3]^2 \int dx_1 \int dx_2 \int \frac{d^2 w_1}{\vec{w}_1^{\,2}} \int \frac{d^2 w_2}{\vec{w}_2^{\,2}} f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) , \tag{3.20}$$

where $x_{1,2} > 0$,

$$f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \equiv \theta(1 - x_1)\theta(1 - x_1 - x_2)\vec{w}_1^{\,2}\vec{w}_2^{\,2}\, \tfrac{1}{4}\mathrm{Tr}\left\{\left[(\vec{\not w}_1' - x_1 M)\left(\frac{1 - x_2}{x_1}\vec{\not w}_1' + \vec{\not w}_2' + M\right) - \frac{1 - x_1 - x_2}{x_1}\mu^2\right][\vec{w}_{1,2}' \to \vec{w}_{1,2}]\right\} \times \left\{x_1\left(\frac{\vec{w}_1^{\,2}}{x_1} + x_1 M^2 + \frac{1 - x_1}{x_2}\mu^2\right)[\vec{w}_1 \to \vec{w}_1']\, x_2\left[\frac{1 - x_2}{x_1}\vec{w}_1^{\,2} + \frac{1 - x_1}{x_2}\vec{w}_2^{\,2} + 2\vec{w}_1 \cdot \vec{w}_2 + (x_1 + x_2)M^2 + (1 - x_1 - x_2)\left(\frac{1}{x_1} + \frac{1}{x_2}\right)\mu^2\right][\vec{w}_{1,2} \to \vec{w}_{1,2}']\right\}^{-1} , \tag{3.21}$$

$$\vec{w}_{1,2}' \equiv \vec{w}_{1,2} - x_{1,2}\vec{q} , \tag{3.22}$$

and the Dirac matrices in the right-hand part of the trace are ordered opposite to the left-hand part, as in Eq. (B10).

Properties of the Unrenormalized Integrand

To find the order of magnitude of f_2 in various integration regions, one can ignore the cross terms since $\vec{w}_i \cdot \vec{w}_j \sim O(\vec{w}_{i,j}^{\,2})$ [and $x_{1,2} \sim O(1)$ as seen in footnote 15]. Thus one finds that f_2 behaves at most as

$$f_2 \sim \{[1 + |\vec{q}|/|\vec{w}_1| + m^2/\vec{w}_1^{\,2}][1 + |\vec{q}|/|\vec{w}_2| + \vec{w}_1^{\,2}/\vec{w}_2^{\,2} + m^2/\vec{w}_2^{\,2}]\}^{-1} \sim O(1) \text{ only when } \vec{w}_2^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim (\vec{q}^{\,2}, m^2) . \tag{3.23a}$$

Regions other than those ordered as in (3.23a) will contribute to $O(m^2/-q^2)$.

Secondly, one can verify that (3.21) has the following properties: when $\vec{w}_2^{\,2} \gg \vec{w}_1^{\,2}$

$$f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \to f_1(\vec{q}, \vec{w}_1, m) f_1(\vec{0}, \vec{w}_2, 0) + O(\vec{w}_1^{\,2}/\vec{w}_2^{\,2}) ; \tag{3.23b}$$

when $\vec{w}_1^{\,2} \gg (\vec{q}^{\,2}, m^2)$

$$f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \to f_1(\vec{0}, \vec{w}_1, \vec{w}_2, 0) + O(\vec{q}^{\,2}/\vec{w}_1^{\,2}) . \tag{3.23c}$$

In (3.23b), $f_1(\vec{0}, \vec{w}_2, 0)$ is the integrand for the one-pion loop with $p_+ = 1 - x_1 = p_+'$ and $\vec{q}^{\,2} = 0 = m^2$ (so that $p_- = 0 = p_-'$). By Lorentz invariance [see (2.6)] the x_2 integration must be independent of p_+ when $p_- = 0$. Thus by letting $x_2 \to x_2/(1 - x_1)$ one finds that $f_1(\vec{0}, \vec{w}_2, 0)$ becomes the usual one-pion integrand evaluated in this limit.

Renormalization

The transverse integrals in Eq. (3.20) are logarithmically divergent as $\vec{w}_{1,2}^{\,2} \to \infty$. The finite part of the $\vec{w}_2$ integral is defined as usual by subtracting the integrand with the external lines of the w_2 subintegration

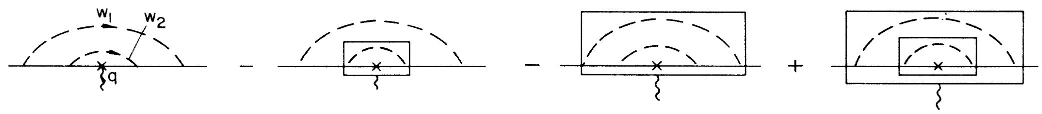

FIG. 5. The finite contribution in $O(g^4)$.

on-shell. In order to keep the limits on the x_2 integral the same as in (3.19) one lets

$$p - w_1 \to (1 - x_1, \vec{0}, M^2/1 - x_1), \quad q^2 = 0. \tag{3.24}$$

The resulting integrand then has the form: $f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) - f_1(\vec{q}, \vec{w}_1, m) f_1(\vec{0}, \vec{w}_2, m)$, where $f_1(\vec{0}, \vec{w}_2, m)$ is the integrand for the one-pion loop with $p_+ = 1 - x_1 = p_+'$ and $\vec{q}^{\,2} = 0$. Finally, the finite part of the w_1 integral is defined by subtracting at $\vec{q}^{\,2} = 0$. This gives a renormalized integrand of the form

$$g_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) = \{[f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) - f_1(\vec{q}, \vec{w}_1, m) f_1(\vec{0}, \vec{w}_2, m)] - [f_2(\vec{0}, \vec{w}_1, \vec{w}_2, m) - f_1(\vec{0}, \vec{w}_1, m) f_1(\vec{0}, \vec{w}_2, m)]\}. \tag{3.25}$$

This is represented diagrammatically by Fig. 5.

The properties of f_2 in (3.23b) and (3.23c) can be represented as in Fig. 6 and Fig. 7. Along with the property of f_1 in Fig. 4 they imply the following behavior for g_2: (i) when $\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$ the first and third terms cancel and the second and fourth terms cancel. (ii) When $\vec{w}_2^{\,2} \gg \vec{w}_1^{\,2}$ the first and second terms cancel and the third and fourth terms cancel. Therefore one finds

$$g_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \sim O(1) \ \text{ only when } \ \vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim \vec{w}_2^{\,2} \gtrsim m^2 \tag{3.26a}$$

and otherwise $g_2 \ll 1$ and gives $O(m^2/-q^2)$. Comparing this with (3.23a) one sees that the effect of the subtractions is just to reverse the transverse ordering. Just as for the one-pion diagrams the transverse pion momenta are bounded by m^2 and $\vec{q}^{\,2}$.

In order to show as in the one-pion case that the $\ln(\vec{q}^{\,2}/m^2)$ factors are associated with the independent regions in the transverse phase space one needs properties analogous to (3.16b). When $\vec{q}^{\,2} \gg (\vec{w}_1^{\,2} \gtrsim \vec{w}_2^{\,2} \gtrsim m^2)$ only the last two terms in g_2 can be $O(1)$, so that when $\vec{q}^{\,2} \gg (\vec{w}_1^{\,2} \gtrsim \vec{w}_2^{\,2} \gtrsim m^2)$

$$g_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \to g_2(\vec{0}, \vec{w}_1, \vec{w}_2, m) + O(\vec{w}_1^{\,2}/\vec{q}^{\,2}), \tag{3.26b}$$

where

$$g_2(\vec{0}, \vec{w}_1, \vec{w}_2, m) \equiv -f_2(\vec{0}, \vec{w}_1, \vec{w}_2, m) + f_1(\vec{0}, \vec{w}_1, m) f_1(\vec{0}, \vec{w}_2, m).$$

Similarly, when $(\vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2}) \gg (\vec{w}_2^{\,2} \gtrsim m^2)$ only the second and fourth terms can be $O(1)$, so that when $(\vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2}) \gg (\vec{w}_2^{\,2} \gtrsim m^2)$

$$g_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \to g_1(\vec{q}, \vec{w}_1, 0) g_1(\vec{0}, \vec{w}_2, m) + O(\vec{w}_2^{\,2}/\vec{w}_1^{\,2}), \tag{3.26c}$$

where $g_1(\vec{0}, \vec{w}_2, m) \equiv -f_1(\vec{0}, \vec{w}_2, m)$. This is the first example of a general "factorization" property. Finally, one has when $(\vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim \vec{w}_2^{\,2}) \gg m^2$

$$g_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) \to g_2(\vec{q}, \vec{w}_1, \vec{w}_2, 0) + O(m^2/\vec{w}_2^{\,2}). \tag{3.26d}$$

The $\vec{w}_1$ and $\vec{w}_2$ integrations can be partitioned just as in Eq. (3.14), and only the following regions will contribute when $q^2 \gg m^2$: (i) $(\vec{q}^{\,2} \approx \vec{w}_1^{\,2} \approx \vec{w}_2^{\,2}) \gg m^2$; (ii) $(\vec{q}^{\,2} \approx \vec{w}_1^{\,2}) \gg (\vec{w}_2^{\,2} \approx m^2)$; (iii) $\vec{q}^{\,2} \gg (\vec{w}_1^{\,2} \approx \vec{w}_2^{\,2} \approx m^2)$; (iv) $(\vec{q}^{\,2} \approx \vec{w}_1^{\,2}) \gg \vec{w}_2^{\,2} \gg m^2$; (v) $\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg (\vec{w}_2^{\,2} \approx m^2)$; (vi) $\vec{q}^{\,2} \gg (\vec{w}_1^{\,2} \approx \vec{w}_2^{\,2}) \gg m^2$; and (vii) $\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg \vec{w}_2^{\,2} \gg m^2$. In region (i) g_2 can be written as a function of $\vec{w}_1/|\vec{q}|$ and $\vec{w}_2/|\vec{q}|$ and because of the transverse symmetry the integral becomes independent of $\vec{q}^{\,2}$. In region (ii) the result in (3.26c) implies that the $\vec{w}_2$ integral is explicitly independent of $\vec{q}^{\,2}$ and the transverse symmetry allows one to scale out $\vec{q}$ in the $\vec{w}_1$ integral. Because

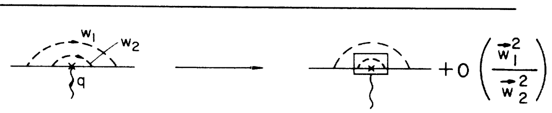

FIG. 6. Behavior in the region $\vec{w}_2^{\,2} \gg \vec{w}_1^{\,2}$.

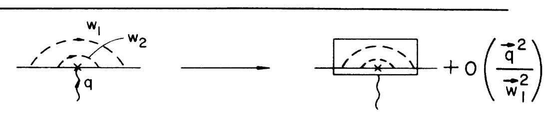

FIG. 7. Behavior in the region $\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$.

of (3.26b) region (iii) is explicitly independent of $\vec{q}^{\,2}$. Therefore only the last four regions contribute any q^2 dependence.

Leading and Next-to-Leading Terms

For simplicity one neglects μ^2/M^2 and lets $m^2 = M^2$. Letting $\tilde{F}_2 \equiv (g^2/32\pi^2)^2 G_2$ as in Eq. (3.12), region (iv) makes the following contribution to G_2:

$$\left(\frac{2}{\pi}\right)^2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_{-\pi}^{\pi} \frac{d\theta}{2} \int_{\epsilon_3 \vec{q}^{\,2}}^{\vec{q}^{\,2}/\epsilon_4} \frac{dw_1^2}{\vec{w}_1^{\,2}} \left[\frac{\vec{w}_1 \cdot (\vec{w}_1 - x_1\vec{q})}{(\vec{w}_1 - x_1\vec{q})^2} - 1\right]\left[\frac{-\pi x_1 x_2}{(1-x_1)^2} \int_{M^2/\epsilon_2'}^{\epsilon_3' \vec{w}_1^{\,2}} \frac{dw_2^2}{\vec{w}_2^{\,2}}\right] + O(M^2/\vec{q}^{\,2})$$
$$= 4 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{(1-x_1)^2}\left[\ln\left(\frac{q^2}{M^2}\right) \ln\left(\frac{x_1^2}{\epsilon_3}\right)\right] + O(1). \tag{3.27}$$

Note that the coefficient of $\ln(q^2/M^2)$ is the same as appeared in this region of $\vec{w}_1$ in the one-pion case [see Eq. (3.17)]. The dependence on the ϵ_i and ϵ_i' will cancel, and this will be shown here for the leading and next-to-leading terms.

The contribution from regions (v), (vi), and (vii), where $\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg M^2$, is

$$\left(\frac{2}{\pi}\right)^2 \pi \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_{M^2/\epsilon_2}^{\epsilon_3 \vec{q}^{\,2}} \frac{dw_1^2}{\vec{w}_1^{\,2}} \frac{x_1 x_2}{(1-x_1)^2} \left(\int_{\epsilon_1' M^2}^{M^2/\epsilon_2'} \frac{d^2 w_2}{\vec{w}_2^{\,2} + \left(\frac{x_2}{1-x_1}\right)^2 M^2} + \int_{M^2/\epsilon_2'}^{\epsilon_3' \vec{w}_1^{\,2}} \frac{d^2 w_2}{\vec{w}_2^{\,2}} \right.$$
$$\left. - \int_{\epsilon_3' \vec{w}_1^{\,2}}^{\vec{w}_1^{\,2}/\epsilon_4'} \frac{d^2 w_2}{\vec{w}_2^{\,2}} \left\{ \frac{\vec{w}_2^{\,2}\left(\vec{w}_2 + \frac{1-x_2}{x_1}\vec{w}_1\right)^2}{\left[\vec{w}_2^{\,2} + \frac{x_2}{1-x_1} 2\vec{w}_1 \cdot \vec{w}_2 + \frac{x_2(1-x_2)}{x_1(1-x_1)}\vec{w}_1^{\,2}\right]^2} - 1 \right\} \right) + O(M^2/\vec{q}^{\,2}). \tag{3.28}$$

The first $\vec{w}_2$ integral is explicitly independent of $\vec{w}_1^{\,2}$. The last one can be written as a function of $\vec{v}_2 \equiv \vec{w}_2/|\vec{w}_1|$ and is then also independent of $\vec{w}_1^{\,2}$. Thus the leading term can come only from the second term, in which $\vec{q}^{\,2} \gg \vec{w}_1^{\,2} \gg \vec{w}_2^{\,2} \gg m^2$. So each factor of $\ln(\vec{q}^{\,2}/m^2)$ is indeed associated with an independent region in the transverse phase space.

The only nontrivial integral in (3.28) is the one from the region $\vec{w}_1^{\,2} \approx \vec{w}_2^{\,2}$. The angular integration of the first term in curly braces leaves the following integration over $\vec{v}_2^{\,2} \equiv \vec{w}_2^{\,2}/\vec{w}_1^{\,2}$:

$$\pi \int_{\epsilon_3'}^{1/\epsilon_4'} dv_2^2 \left(\frac{\vec{v}_2^{\,4} + \left[\left(\frac{1-x_2}{x_1}\right)^2 - 3\frac{x_2(1-x_2)}{x_1(1-x_1)}\right]\vec{v}_2^{\,2} + \left(\frac{1-x_2}{x_1}\right)^3 \frac{x_2}{1-x_1}}{\left\{\vec{v}_2^{\,4} + 2\frac{x_2}{1-x_1}\left(\frac{1-x_2}{x_1} - 2\frac{x_2}{1-x_1}\right)\vec{v}_2^{\,2} + \left[\frac{x_2(1-x_2)}{x_1(1-x_1)}\right]^2\right\}^{3/2}} \right).$$

The second two terms in the denominator can be added and subtracted in the numerator to give

$$\pi \int_{\epsilon_3'}^{1/\epsilon_4'} dv_2^2 \left(\left\{\vec{v}_2^{\,4} + 2\frac{x_2}{1-x_1}\left(\frac{1-x_2}{x_1} - 2\frac{x_2}{1-x_1}\right)\vec{v}_2^{\,2} + \left[\frac{x_2(1-x_2)}{x_1(1-x_1)}\right]^2\right\}^{-1/2} \right.$$
$$\left. + \frac{\left(\frac{1-x_2}{x_1} - 4\frac{x_2}{1-x_1}\right)\left(\frac{1-x_2}{x_1} - \frac{x_2}{1-x_1}\right)\vec{v}_2^{\,2} + \left(\frac{1-x_2}{x_1}\right)^2 \frac{x_2}{1-x_1}\left(\frac{1-x_2}{x_1} - \frac{x_2}{1-x_1}\right)}{\left\{\vec{v}_2^{\,4} + 2\frac{x_2}{1-x_1}\left(\frac{1-x_2}{x_1} - 2\frac{x_2}{1-x_1}\right)\vec{v}_2^{\,2} + \left[\frac{x_2(1-x_2)}{x_1(1-x_1)}\right]^2\right\}^{3/2}} \right).$$

The dependence on x_1 and x_2 considerably simplifies after the $\vec{v}_2^{\,2}$ integration. The sum of (3.27) and (3.28) then becomes

$$G_2 = 4\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{x_1 x_2}{(1-x_1)^2}\left(\ln\left(\frac{\vec{q}^{\,2}}{M^2}\right)\ln\left(\frac{x_1^{\,2}}{\epsilon_3}\right) + \int_{M^2/\epsilon_2}^{\epsilon_3 \vec{q}^{\,2}} \frac{dw_1^{\,2}}{\vec{w}_1^{\,2}} \left\{ \ln\left|\frac{1}{\epsilon_2'\left(\frac{x_2}{1-x_1}\right)^2}\right| + \ln\left(\epsilon_2'\epsilon_3'\frac{\vec{w}_1^{\,2}}{M^2}\right) - \ln\left|\frac{1}{\epsilon_4'\,\frac{x_2}{x_1}\,\frac{1-x_1-x_2}{(1-x_1)^2}}\right| - \left[\frac{(1-x_1)(1-x_2)}{x_1x_2} - 2\right] + \ln\left|\frac{1}{\epsilon_3'\epsilon_4'}\right| \right\} \right) + O(1)\,. \tag{3.29}$$

Let $x_2 \to x_2/(1-x_1)$ and note that ϵ_2', ϵ_3', and ϵ_4' cancel. Integration of the $\ln(\vec{w}_1^{\,2}/M^2)$ term gives $1/2![\ln(\epsilon_3\vec{q}^{\,2}/M^2)]^2$ and the ϵ_3 dependence cancels [to $O(\ln(\vec{q}^{\,2}/M^2))$] with that of the first term in Eq. (3.29). Finally, the $\ln x_1^{\,2}$ from $\vec{q}^{\,2} \approx \vec{w}_1^{\,2}$ (first term) cancels with the $\ln x_2^{\,2}$ from $\vec{w}_2^{\,2} \approx m^2$ (second term). This is the same cancellation which occurred in (3.17). The $\ln(\vec{q}^{\,2}/M^2)$ term thus comes only from $\vec{w}_1^{\,2} \approx \vec{w}_2^{\,2}$, and one obtains

$$G_2 = \frac{1}{2!}\left(a_1 \ln\frac{\vec{q}^{\,2}}{m^2} + b_1\right)^2 + \left(a_2 \ln\frac{\vec{q}^{\,2}}{m^2} + b_2\right) + O(m^2/\vec{q}^{\,2})\,, \tag{3.30}$$

where a_1 and b_1 appear in Eq. (3.17), $a_2 = -\frac{5}{2}$ and $b_2 \sim O(1)$.[17]

Since the leading terms are known to exponentiate, G_2 can always be written in the form of Eq. (3.30). The significant thing, however, is the way in which these terms arise. The generalization of Eqs. (3.26) and (3.30) to the next order in g^2 provides a nontrivial indication that nonleading terms will also exponentiate.

C. The Three-Pion Diagram

Performing the minus integrations one finds as usual that only the poles at $w_{i-} = (\vec{w}_i^{\,2} + \mu^2)/x_i$ contribute, with the conditions

$$0 < x_1 < 1\,, \quad 0 < x_2 < 1 - x_1\,, \quad 0 < x_3 < 1 - x_1 - x_2\,. \tag{3.31}$$

One can define f_3 analogously to Eq. (3.21), with a denominator given by Weinberg's rules and the numerator written as in Appendix B [Eq. (B.12)]. The following properties can then be explicitly shown:

$$f_3 \sim O(1) \;\text{ only when }\; \vec{w}_3^{\,2} \gtrsim \vec{w}_2^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim (\vec{q}^{\,2}, m^2)\,; \tag{3.32a}$$

when $\vec{w}_3^{\,2} \gg \vec{w}_2^{\,2}$

$$f_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to f_2(\vec{q}, \vec{w}_1, \vec{w}_2, 0) f_1(\vec{0}, \vec{w}_3, 0) + O(\vec{w}_2^{\,2}/\vec{w}_3^{\,2})\,, \tag{3.32b}$$

when $\vec{w}_2^{\,2} \gg \vec{w}_1^{\,2}$

$$f_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to f_1(\vec{q}, \vec{w}_1, m) f_2(\vec{0}, \vec{w}_2, \vec{w}_3, 0) + O(\vec{w}_1^{\,2}/\vec{w}_2^{\,2})\,, \tag{3.32c}$$

when $\vec{w}_1^{\,2} \gg (\vec{q}^{\,2}, m^2)$

$$f_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to f_3(\vec{0}, \vec{w}_1, \vec{w}_2, \vec{w}_3, 0) + O(\vec{q}^{\,2}/\vec{w}_1^{\,2})\,. \tag{3.32d}$$

As mentioned in connection with (3.23b), integration over the functions with $\vec{q}^{\,2} = 0 = m^2$ is independent of the p_+ of the external proton. In (3.32b) $f_1(\vec{0}, \vec{w}_2, 0)$ corresponds to an external $p_+ = 1 - x_1$ and in (3.32c) $f_2(\vec{0}, \vec{w}_2, \vec{w}_3, 0)$ to $p_+ = 1 - x_1 - x_2$. These factorization properties are represented in Figs. 8, 9, and 10.

Each of the transverse integrals has a logarithmic ultraviolet divergence. In order to keep the limits on the x_i the same as in (3.31) the w_3 and w_2 subtractions are performed at the points

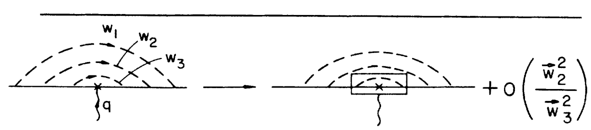

FIG. 8. Behavior in the region $\vec{w}_3^{\,2} \gg \vec{w}_2^{\,2}$.

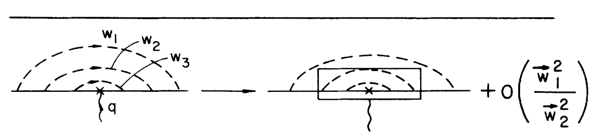

FIG. 9. Behavior in the region $\vec{w}_2^{\,2} \gg \vec{w}_1^{\,2}$.

$$p - w_1 - w_2 \to \left(1 - x_1 - x_2, \vec{0}, \frac{M^2}{1 - x_1 - x_2}\right), \quad (3.33)$$

$$p - w_1 \to \left(1 - x_1, \vec{0}, \frac{M^2}{1 - x_1}\right), \quad (3.34)$$

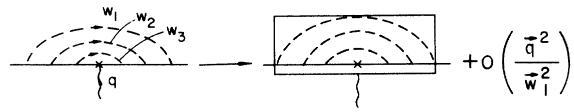

FIG. 10. Behavior in the region $\vec{w}_1^{\,2} \gg \vec{q}^{\,2}$.

respectively. The renormalized three-pion integrand is then given by

$$g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \equiv \{[f_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) - f_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) f_1(\vec{0}, \vec{w}_3, m)] - f_1(\vec{q}, \vec{w}_1, m)[f_2(\vec{0}, \vec{w}_2, \vec{w}_3, m) - f_1(\vec{0}, \vec{w}_2, m) f_1(\vec{0}, \vec{w}_3, m)]\} - \{\vec{q} = 0\}. \quad (3.35)$$

To see that the appropriate cancellations occur in Eq. (3.35) to reverse the ordering given in (3.32a) one must understand how to group the terms appropriately. For example, when $\vec{w}_3^{\,2} \gg \vec{w}_2^{\,2}$ the first and second pairs of terms are each $\ll 1$.

Diagrammatically, g_3 is a sum of three-pion diagrams with any number (up to three) of concentric boxes drawn around any loop. Each box in a given term has a factor of (-1) associated with it. A term can only be $O(1)$ if transverse momenta not separated by a box are ordered as in (3.32a).

If any of the inequalities in (3.32a) hold, one can group terms as follows: Each term in which the momenta on either side of the inequality are not separated by a box cancels with the term which is identical except for an additional box between pions of unequal momenta. Thus one finds

$$g_3 \sim O(1) \text{ only when } \vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim \vec{w}_2^{\,2} \gtrsim \vec{w}_3^{\,2} \gtrsim m^2. \quad (3.36a)$$

Furthermore, if any of the inequalities in (3.36a) holds, only terms in which the unequal momenta are separated by a box can be $O(1)$. Thus one has when $\vec{q}^{\,2} \gg \vec{w}_1^{\,2}$

$$g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to g_3(\vec{0}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) + O(\vec{w}_1^{\,2}/\vec{q}^{\,2}), \quad (3.36b)$$

when $\vec{w}_1^{\,2} \gg \vec{w}_2^{\,2}$

$$g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to g_1(\vec{q}, \vec{w}_1, 0) g_2(\vec{0}, \vec{w}_1, \vec{w}_2, m) + O(\vec{w}_2^{\,2}/\vec{w}_1^{\,2}), \quad (3.36c)$$

when $\vec{w}_2^{\,2} \gg \vec{w}_3^{\,2}$

$$g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to g_2(\vec{q}, \vec{w}_1, \vec{w}_2, 0) g_1(\vec{0}, \vec{w}_3, m) + O(\vec{w}_3^{\,2}/\vec{w}_2^{\,2}), \quad (3.36d)$$

when $\vec{w}_3^{\,2} \gg m^2$

$$g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) \to g_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, 0) + O(m^2/\vec{w}_3^{\,2}). \quad (3.36e)$$

As a result of the properties in (3.36) and the transverse symmetry, each factor of $\ln(\vec{q}^{\,2}/m^2)$ in the integration over g_3 will be associated with an independent region in the transverse phase space. That is, the leading term will come when all the inequalities in (3.36a) hold. The next-to-leading term comes when any two are approximately equal, and so on. One can show that

$$G_3 = \frac{1}{3!}[a_1 \ln(\vec{q}^{\,2}/m^2) + b_1]^3 + [a_1 \ln(\vec{q}^{\,2}/m^2) + b_1][a_2 \ln(\vec{q}^{\,2}/m^2) + b_2] + [a_3 \ln(\vec{q}^{\,2}/m^2) + b_3] + O(m^2/\vec{q}^{\,2}), \quad (3.37)$$

where a_1 and b_1 appear in Eq. (3.17) and a_2 and b_2 appear in Eq. (3.30). This result is strong indication that not only leading but also nonleading terms will exponentiate. The next section shows that properties (3.36) can be generalized to all orders and the succeeding section gives the proof of exponentiation.

IV. THE GENERAL RESULT

In this section the properties found in low orders for the unrenormalized integrand are generalized. Furthermore, it is shown that these imply similar properties for the renormalized integrand.

A. The Unrenormalized Integrand

Each w_{i-} integration can be done by contour methods. The contribution on the semicircle damps out. The contours can be closed in the lower half-plane, picking up the pole at $w_{i-} = (\vec{w}_i^{\,2} + \mu^2)/x_i$. In order that the poles are not on the same side of the real axis one must require

$$0 < x_i < X_{i-1}, \tag{4.1}$$

where X_{i-1} is the + component carried by the proton when it emits the ith pion. It is given by

$$X_{i-1} \equiv 1 - x_1 - x_2 - \cdots - x_{i-1}. \tag{4.2}$$

After the γ_5's are eliminated in the numerator, one obtains the result

$$F_1^{(N)} \equiv \left(\frac{g^2}{32\pi^2}\right)^N \prod_{i=1}^{N} \int dx_i \int \frac{d^2 w_i}{\vec{w}_i^{\,2}} f_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m), \tag{4.3}$$

where it is understood that $x_i > 0$. The unrenormalized integrand has the form

$$f_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \equiv \theta(X_1) \cdots \theta(X_N) \vec{w}_1^{\,2} \cdots \vec{w}_N^{\,2} \left[\frac{\mathfrak{N}_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m)}{D_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m)}\right]. \tag{4.4}$$

The function D_N in Eq. (4.4) is given by Weinberg's rules.[13] A typical denominator factor associated with an $(x_0 + x_3)$-ordered state after the current insertion is

$$\left\{\vec{q}^{\,2} + M^2 - \left[\frac{\vec{w}_1^{\,2} + \mu^2}{x_1} + \cdots + \frac{\vec{w}_i^{\,2} + \mu^2}{x_i} + \frac{(\vec{w}_1 + \cdots + \vec{w}_i - \vec{q})^2 + M^2}{X_i}\right]\right\}.$$

This can be rewritten in the form

$$-\frac{1}{X_i}\left[\left(1 + \frac{X_i}{x_1}\right)\vec{w}_1'^{\,2} + \cdots + \left(1 + \frac{X_i}{x_i}\right)\vec{w}_i'^{\,2} + \sum_{j \neq k} 2\vec{w}_j' \cdot \vec{w}_k' + (1 - X_i)M^2 + X_i\left(\frac{1}{x_1} + \cdots + \frac{1}{x_i}\right)\mu^2\right],$$

where $\vec{w}_j' \equiv \vec{w}_j - x_j \vec{q}$, $j = 1, \ldots, i$. States before the current give the same form with $\vec{q} = 0$. Thus including the factors $x_i X_i^{\,2}$

$$D_N = \prod_{i=1}^{N} \left\{ x_i \left[\left(1 + \frac{X_i}{x_1}\right)\vec{w}_1^{\,2} + \cdots + \left(1 + \frac{X_i}{x_i}\right)\vec{w}_i^{\,2} + \sum_{j \neq k} 2\vec{w}_j \cdot \vec{w}_k + (1 - X_i)M^2 + X_i\left(\frac{1}{x_1} + \cdots + \frac{1}{x_i}\right)\mu^2\right] \times [\vec{w}_j \to \vec{w}_j', \ j = 1, \ldots, i] \right\}. \tag{4.5}$$

Removing the γ_5's from $\mathfrak{N}_N$ gives an over-all factor of $(-1)^N$ which has been included in the first factor of Eq. (4.3). Then one has

$$\mathfrak{N}_N = \tfrac{1}{8}\mathrm{Tr}\{\gamma_+(\not{p}' + M)(\not{p}' - \not{w}_1 - M) \cdots (\not{p}' - \not{w}_1 - \cdots - \not{w}_N \pm M)\gamma_+(\not{p} - \not{w}_1 - \cdots - \not{w}_N \pm M) \cdots (\not{p} - \not{w}_1 - M)(\not{p} + M)\}, \tag{4.6}$$

where the w_{i-} integrations have set $w_i^{\,2} = \mu^2$. Because of the invariance under (2.8), $\mathfrak{N}_N$ must be symmetric under $\vec{w}_i \leftrightarrow \vec{w}_i'$ just as D_N is.

Although each propagator factor in Eq. (4.6) has terms of the form $[(\vec{w}_i^{\,2} + \mu^2)/x_i]\gamma_+$ one can show that each contributes only linearly in $|\vec{w}_i|$ and μ.[18] This is essentially because $\gamma_\pm^{\,2} = 0$. One can then demonstrate that

$$f_N \sim O(1) \ \text{ only when } \ \vec{w}_N^{\,2} \gtrsim \vec{w}_{N-1}^{\,2} \gtrsim \cdots \gtrsim \vec{w}_1^{\,2} \gtrsim (\vec{q}^{\,2}, m^2) \tag{4.7}$$

and if $\vec{w}_i^{\,2} \gg \vec{w}_j^{\,2} (j > i)$, $f_N \sim O(\vec{w}_j^{\,2}/\vec{w}_i^{\,2})$.

It is easy to check explicitly that as any $x_i \to 0$ one has $f_N \sim x_i$. This is to be expected, since pseudoscalar theory is not infrared divergent. Thus $x_i \sim O(1)$. It can then be seen from Eq. (4.5) that when $\vec{w}_{i+1}^{\,2} \gg \vec{w}_i^{\,2}$

$$D_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \to D_i(\vec{q}, \vec{w}_1, \ldots, \vec{w}_i, m) D_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, 0) + O(\vec{w}_i^{\,2}/\vec{w}_{i+1}^{\,2}), \tag{4.8}$$

where D_{N-i} is the denominator with the external $p_+ = X_i$.

In the region specified by (4.7) and (4.8) the large variables appear in $\mathfrak{N}_N$ only in the inner $N - i$ factors on either side of γ_+. One can show that the product of these factors behaves as $\mathfrak{N}_{N-i}\gamma_+$ in this region,[19] so that when $\vec{w}_{i+1}^{\,2} \gg \vec{w}_1^{\,2}$

$$\mathfrak{N}_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \to \mathfrak{N}_i(\vec{q}, \vec{w}_1, \ldots, \vec{w}_i, m) \mathfrak{N}_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, 0) + O(\vec{w}_i^{\,2}/\vec{w}_{i+1}^{\,2}), \tag{4.9}$$

where $\mathfrak{N}_{N-i}$ is the numerator with the external $p_+ = X_i$. Combining Eqs. (4.8) and (4.9) one has when $\vec{w}_{i+1}^{\,2} \gg \vec{w}_i^{\,2}$

$$f_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \to f_i(\vec{q}, \vec{w}_1, \ldots, \vec{w}_i, m) f_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, 0) + O(\vec{w}_i^{\,2}/\vec{w}_{i+1}^{\,2}), \tag{4.10}$$

where f_{N-i} is the integrand with $p_+=X_i$. Since it also has $p_-=0$ in this region, the integral of it must be independent of p_+ by Lorentz invariance.

B. Renormalization

It will now be seen that the properties of f_N in Eqs. (4.7) and (4.10) are sufficient to show that: (i) subtractions just reverse the ordering; and (ii) when any inequalities hold in the new ordering the integrand factorizes.

Each of the transverse integrals in Eq. (4.3) is logarithmically divergent. One defines the finite part of the $\vec{w}_N$ integral by subtracting the integrand with the momenta flowing into the w_N-loop on-shell. The resulting integrand is then subtracted with the external lines of the w_{N-1}-loop on-shell to define the $\vec{w}_{N-1}$ integral, and so on. The result is

$$\tilde{F}_1^{(N)} \equiv \left(\frac{g^2}{32\pi^2}\right)^N \prod_{i=1}^N \int dx_i \int \frac{d^2 w_i}{\vec{w}_i^{\,2}} g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m), \tag{4.11}$$

where the renormalized integrand can be written as

$$g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \equiv \sum_{k=0}^{N} f_{N-k}(\vec{q}, \vec{w}_1, \ldots, \vec{w}_{N-k}, m) h_k(\vec{0}, \vec{w}_{N-k+1}, \ldots, \vec{w}_N, m), \tag{4.12}$$

with $f_0 \equiv 1$ and

$$h_k(\vec{0}, \vec{w}_{N-k+1}, \ldots, \vec{w}_N, m) \equiv \sum_{\{n_i\}} [(-1)^s f_{n_1}(\vec{0}, \vec{w}_{N-k+1}, \ldots, m) \cdots f_{n_s}(\vec{0}, \ldots, \vec{w}_N, m)]. \tag{4.13}$$

The set of integers $\{n_i\}$ is defined by requiring $\sum_{i=1}^{s} n_i = k$. In order to keep the x_i limits as in (4.1) the protons are placed on-shell by

$$p - w_1 - \cdots - w_i \to (X_i, \vec{0}, M^2/X_i). \tag{4.14}$$

Diagrammatically, the above equations can be represented by the sum of N-pion rainbows with all possible combinations of boxes drawn around the loops, and a factor of (-1) associated with each box. A typical term is shown in Fig. 11. The n_i of Eq. (4.13) is the number of pions in the ith box, and s is the number of boxes in each term.

Note that each subtraction doubles the number of terms. Consider two adjacent loops labelled i and $i+1$ and divide the terms of g_N exactly in half as follows: The first set contains all diagrams in which loops i and $i+1$ are not separated by a box. The second set can be obtained from the first by simply drawing an additional box between loops i and $i+1$ in each term. Each term can only be $O(1)$ in the regions shown in (4.7). When $\vec{w}_{i+1}^{\,2} \gg \vec{w}_i^{\,2}$ one has (4.10), so that each term in the first set will cancel with its associated term in the second set. This is true for any i including $\vec{w}_0 \equiv \vec{q}$. Thus one has

$$\begin{aligned} &g_N \sim O(1) \;\text{ only when }\; \vec{q}^{\,2} \gtrsim \vec{w}_1^{\,2} \gtrsim \cdots \gtrsim \vec{w}_N^{\,2} \gtrsim m^2; \\ &\text{when } \vec{w}_j^{\,2} \gg \vec{w}_i^{\,2},\; j>i \;\text{ then }\; g_N \sim O(\vec{w}_i^{\,2}/\vec{w}_j^{\,2}). \end{aligned} \tag{4.15}$$

Regions other than those in (4.15) thus contribute $O(m^2/\vec{q}^{\,2})$.

Secondly, when $\vec{w}_i^{\,2} \gg \vec{w}_{i+1}^{\,2}$ all the terms in the first set above are $O(\vec{w}_{i+1}^{\,2}/\vec{w}_i^{\,2})$. Summing up all terms with a box between loops i and $i+1$, one finds that when $\vec{w}_i^{\,2} \gg \vec{w}_{i+1}^{\,2}$,

$$g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \to g_i(\vec{q}, \vec{w}_1, \ldots, \vec{w}_i, 0) g_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, m) + O(\vec{w}_{i+1}^{\,2}/\vec{w}_i^{\,2}), \tag{4.16}$$

where

$$g_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, m) \equiv h_{N-i}(\vec{0}, \vec{w}_{i+1}, \ldots, \vec{w}_N, m). \tag{4.17}$$

The following points should be emphasized here:

(1) It is essential that the integral over the second factor in (4.10) be independent of any variables in the first factor, including the x_i, in order for it to cancel the appropriate term in Eq. (4.12). In the present notation, this occurs because when $\vec{w}_{i+1}^{\,2} \gg \vec{w}_i^{\,2}$ the only dependence on any $x_{k<i+1}$ in f_{N-i} appears in $p_+=X_i$, which can be scaled out because $p_-=0$. With the on-shell condition written as in (4.14) the cancellation can be seen without rescaling.

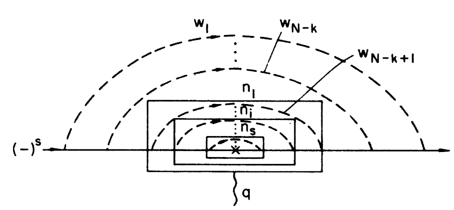

FIG. 11. Typical subtraction term.

(2) Note that one must consider *all* of the subtraction terms in order to see that g_N has the simple properties. The particular terms which cancel to give (4.15) and the particular terms which combine to give (4.16) depend on which transverse pion momenta are in widely separated regions.

(3) The pions are emitted like bremmstrahlung and then reabsorbed. One might expect that as the proton "slows down" the succeeding pions emitted will have smaller longitudinal and transverse momenta. This physical ordering for the x_i is provided by the w_{i-} integrations and for the $\vec{w}_i$ by the renormalization.

(4) It is important that the N-pion integrand factors into functions which are exactly the i-pion and $(N-i)$-pion integrands, evaluated in the appropriate regions. This fact allows one to interpret the $\ln(\vec{q}\,^2/m^2)$ terms in $\tilde{F}_1^{(N)}$ physically and gives a simple result for F_1, as shown in the next section.

V. CLUSTER EXPANSION FOR RAINBOWS

In this section it is shown that the properties of the integrand in this model of the Dirac form factor allow one to apply the cluster-decomposition method of statistical mechanics. This provides a physical interpretation of the q^2 dependence and a mechanism for exponentiation of F_1.

A. Cluster Functions and Sum Over N

The finite contribution of the N-loop rainbow diagram to F_1 is defined as in Eq. (4.11):

$$\tilde{F}_1^{(N)} \equiv \left(\frac{g^2}{32\pi^2}\right)^N G_N\,, \qquad G_N \equiv \prod_{i=1}^{N} \int \frac{2dx_i}{\pi} \int \frac{d^2 w_i}{\vec{w}_i^{\,2}}\, g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m), \tag{5.1}$$

where g_N is defined in Eq. (4.12). The cluster decomposition is more conveniently carried out after symmetrization over all pairs of $x_i, \vec{w}_i$. So one defines the symmetrized functions as follows:

$$g_N^s(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) \equiv \sum_{P(N)} g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m)\,, \qquad G_N^s \equiv \prod_{i=1}^{N} \int \frac{2dx_i}{\pi} \int \frac{d^2 w_i}{\vec{w}_i^{\,2}}\, g_N^s(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m)\,. \tag{5.2}$$

Here $P(N)$ means all permutations of the N indices. The integral of each term must give the same result, so that

$$G_N^s = N!\, G_N\,. \tag{5.3}$$

The properties of g_N imply the following properties for g_N^s: (i) Property (4.15) implies that $g_N^s \sim O(1)$ only when $\vec{q}\,^2 \gtrsim (\vec{w}_1^2, \ldots, \vec{w}_N^2) \gtrsim m^2$. (ii) Let $\{\vec{w}_i\}$ be any set of i transverse pion momenta, and $\{\vec{w}_{N-i}\}$ the remaining ones. Then property (4.16) implies that when all momenta in the first set are much larger than those in the second, then one has

$$\begin{aligned} g_N^s(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) &\to \sum_{P(i)} \sum_{P(N-i)} g_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m) + O(\vec{w}_{N-i}^{\,2}/\vec{w}_i^{\,2}) \\ &\to g_i^s(\vec{q}, \{\vec{w}_i\}, 0)\, g_{N-i}^s(\vec{0}, \{\vec{w}_{N-i}\}, m) + O(\vec{w}_{N-i}^{\,2}/\vec{w}_i^{\,2})\,, \end{aligned} \tag{5.4}$$

where each factor is evaluated in the appropriate region, and $\vec{w}_i^{\,2}$ is a minimum in the first set and $\vec{w}_{N-i}^{\,2}$ a maximum in the second.

One can now do a cluster expansion of the g_N^s in exact analogy to the standard statistical mechanics treatment of the partition integrand.[20] The derivation is given here for completeness. One defines the cluster functions $c_N(\vec{q}, \vec{w}_1, \ldots, \vec{w}_N, m)$ through the following set of equations:

$$\begin{aligned} g_1^s(\vec{q}, \vec{w}_1, m) &= c_1(\vec{q}, \vec{w}_1, m)\,, \\ g_2^s(\vec{q}, \vec{w}_1, \vec{w}_2, m) &= c_1(\vec{q}, \vec{w}_1, m) c_1(\vec{q}, \vec{w}_2, m) + c_2(\vec{q}, \vec{w}_1, \vec{w}_2, m)\,, \\ g_3^s(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m) &= c_1(\vec{q}, \vec{w}_1, m) c_1(\vec{q}, \vec{w}_2, m) c_1(\vec{q}, \vec{w}_3, m) \\ &\quad + c_1(\vec{q}, \vec{w}_1, m) c_2(\vec{q}, \vec{w}_2, \vec{w}_3, m) + c_1(\vec{q}, \vec{w}_2, m) c_2(\vec{q}, \vec{w}_1, \vec{w}_3, m) \\ &\quad + c_1(\vec{q}, \vec{w}_3, m) c_2(\vec{q}, \vec{w}_1, \vec{w}_2, m) + c_3(\vec{q}, \vec{w}_1, \vec{w}_2, \vec{w}_3, m)\,, \end{aligned} \tag{5.5}$$

where the Nth equation defines c_N and is of the form

$$g_N^s(\vec{q},\vec{w}_1,\ldots,\vec{w}_N,m)=\sum_{\{m_i\}}\sum_P\{[c_1(\vec{q},\ ,m)\cdots c_1(\vec{q},\ ,m)][c_2(\vec{q},\ ,\ ,m)\cdots c_2(\vec{q},\ ,\ ,m)]\cdots[c_N(\vec{q},\ ,\ldots,m)]\}, \tag{5.6}$$

where the first square brackets enclose m_1 factors, the second m_2 factors,..., and the last m_N factors. Here the m_i are integers ≥ 0 and are summed over all sets $\{m_i\}$ in which

$$\sum_{i=1}^N im_i=N. \tag{5.7}$$

The second sum in Eq. (5.6) is over all *distinct* ways of filling the blanks with the $\vec{w}_i$.

The above equations can be solved successively for the c_i to yield

$$\begin{aligned}
&c_1(\vec{q},\vec{w}_1,m)=g_1^s(\vec{q},\vec{w}_1,m),\\
&c_2(\vec{q},\vec{w}_1,\vec{w}_2,m)=g_2^s(\vec{q},\vec{w}_1,\vec{w}_2,m)-g_1^s(\vec{q},\vec{w}_1,m)g_1^s(\vec{q},\vec{w}_2,m),\\
&c_3(\vec{q},\vec{w}_1,\vec{w}_2,\vec{w}_3,m)=g_3^s(\vec{q},\vec{w}_1,\vec{w}_2,\vec{w}_3,m)-g_2^s(\vec{q},\vec{w}_1,\vec{w}_2,m)g_1^s(\vec{q},\vec{w}_3,m)\\
&\qquad -g_2^s(\vec{q},\vec{w}_1,\vec{w}_3,m)g_1^s(\vec{q},\vec{w}_2,m)-g_2^s(\vec{q},\vec{w}_2,\vec{w}_3,m)g_1^s(\vec{q},\vec{w}_1,m)\\
&\qquad +2g_1^s(\vec{q},\vec{w}_1,m)g_1^s(\vec{q},\vec{w}_2,m)g_1^s(\vec{q},\vec{w}_3,m),\\
&\cdots .
\end{aligned} \tag{5.8}$$

The property (5.4) implies that when any two arguments of c_N are in the region $\vec{w}_i^{\,2}\gg\vec{w}_j^{\,2}$ then $c_N\sim O(\vec{w}_j^{\,2}/\vec{w}_i^{\,2})$. As is well known in statistical mechanics this implies that integration over c_N can only give one factor of the "volume," from integration over the "center of gravity." The "volume" element here is $d(\ln\vec{w}_i^{\,2})$ and when any $\vec{w}_i^{\,2}\gg\vec{w}_j^{\,2}$ the integral is damped as $\exp(-\ln\vec{w}_i^{\,2})$. Thus the "integrated cluster function" behaves as

$$\begin{aligned}
C_N&\equiv\frac{1}{N!}\prod_{i=1}^N\int\frac{2dx_i}{\pi}\int\frac{d^2w_i}{\vec{w}_i^{\,2}}\,c_N(\vec{q},\vec{w}_1,\ldots,\vec{w}_N,m)\\
&=a_N\ln(\vec{q}^{\,2}/m^2)+b_N+O(m^2/\vec{q}^{\,2}).
\end{aligned} \tag{5.9}$$

It will be seen in the next subsection that a_N comes from the "interior region" $\vec{q}^{\,2}\gg(\vec{w}_1^{\,2},\ldots,\vec{w}_N^{\,2})\gg m^2$ and is independent of the masses.

When Eq. (5.6) is integrated over all the x_i and $\vec{w}_i$ each term in the $\sum_P$ contributes equally. For a given i one can permute the factors of c_i or the arguments of a given c_i without obtaining a distinct term. The number of terms with this symmetry is well known to be

$$\frac{N!}{m_1!\cdots m_N!\,(1!)^{m_1}\cdots(N!)^{m_N}}\,.$$

Thus one obtains

$$\begin{aligned}
G_N^s&=\sum_{\{m_i\}}\left[N!\prod_{i=1}^N\frac{1}{m_i!\,(i!)^{m_i}}\int\frac{2dx_i}{\pi}\int\frac{d^2w_i}{\vec{w}_i^{\,2}}(c_i\cdots c_i)\right]\\
&=N!\sum_{\{m_i\}}\prod_{i=1}^N\left(\frac{1}{m_i!}C_i^{\,m_i}\right).
\end{aligned} \tag{5.10}$$

Combining Eqs. (5.1) and (5.3) and using the restriction (5.7) one has

$$\tilde{F}_1^{(N)}=\sum_{\{m_i\}}\prod_{i=1}^N\frac{1}{m_i!}\left[\left(\frac{g^2}{32\pi^2}\right)^iC_i\right]^{m_i}. \tag{5.11}$$

The Born term contributes $\tilde{F}_1^{(0)}=1$. Summing over all N with the restriction (5.7) is equivalent to summing each m_i independently. Thus the rainbow form factor has the form

$$\begin{aligned}
F_1(q^2)&=\prod_{i=1}^\infty\left\{\sum_{m_i=0}^\infty\frac{1}{m_i!}\left[\left(\frac{g^2}{32\pi^2}\right)^iC_i(\vec{q}^{\,2})\right]^{m_i}\right\}\\
&=\exp\left[\sum_{i=1}^\infty\left(\frac{g^2}{32\pi^2}\right)^iC_i(\vec{q}^{\,2})\right].
\end{aligned} \tag{5.12}$$

The physical meaning of the derivation of Eq. (5.12) is as follows: The amplitude of a particular diagram is analogous to the partition function of a canonical ensemble of pions in a transverse momentum space of volume equal to $\ln(\vec{q}^2/m^2)$. Equation (5.4) can be interpreted as saying that the interaction has a "finite range." The ensemble can therefore be analyzed in terms of all possible ways in which the pions can cluster together, with no interaction between the clusters. This "cluster decomposition" is restricted by the number of particles available. However, in the sum over all orders, analogous to a grand canonical ensemble, one has any number of clusters of a given type. The "grand partition function" thus breaks into a product of functions for the noninteracting clusters, which is Eq. (5.12).

The amplitude of a particular diagram given in Eq. (5.10) is rather complicated and can be misleading. For example, if one sums up the "next-to-leading" terms from each diagram, one obtains

$$\left[a_2\left(\frac{g^2}{32\pi^2}\right)^2 \ln\left(\frac{\vec{q}^2}{m^2}\right)\right]\left(\frac{\vec{q}^2}{m^2}\right)^{a_1(g^2/32\pi^2)}, \tag{5.13}$$

where a_1 and a_2 are defined by Eq. (5.9). This is *larger* than the "leading" terms by a factor of $\ln(\vec{q}^2/m^2)$.[21]

"Interior Region"

In order to better understand Eq. (5.9), consider the various contributing regions of integration. Since $\vec{q}^2 \gg m^2$ and $c_N \to 0$ unless the pion momenta are clustered together in momentum space, only the following regions contribute:

(a) $\vec{q}^2 \gg (\vec{w}_1^2 \approx \cdots \approx \vec{w}_N^2) \approx m^2$,
(b) $\vec{q}^2 \approx (\vec{w}_1^2 \approx \cdots \approx \vec{w}_N^2) \gg m^2$,
(c) $\vec{q}^2 \gg (\vec{w}_1^2 \approx \cdots \approx \vec{w}_N^2) \gg m^2$.

The contribution of region (a) is explicitly independent of $\vec{q}^2$. In region (b) one can write $\vec{w}_i = |\vec{q}|(\vec{w}_i/|\vec{q}|)$ and $\vec{w}_i' = |\vec{q}|[(\vec{w}_i/|\vec{q}|) - x_i\hat{q}]$ where $\hat{q}$ is a unit vector. It is then found that all the $\vec{q}^2$ dependence cancels out.[22] The $\vec{q}^2$ dependence can therefore come only from region (c) which is called the "interior region."

In the interior region any particular transverse pion momentum, say $\vec{w}_i^2$, is integrated from $(1/\epsilon)m^2$ to $\epsilon'\vec{q}^2$. Any other $\vec{w}_k^2$ may for example be integrated from $\epsilon_{k-1}\vec{w}_{k-1}^2$ to $(1/\epsilon_{k-1}')\vec{w}_{k-1}^2$ (where $\epsilon, \epsilon', \epsilon_k, \epsilon_k' \ll 1$ and $\vec{w}_0^2 \equiv \vec{w}_N^2$ in c_N). The C_N in this region are thus eventually determined by integrals of the form

$$\left(\int_{\epsilon_N \vec{w}_N^2}^{\vec{w}_N^2/\epsilon_N'} \frac{dw_1^2}{\vec{w}_1^2} \int_{\epsilon_1 \vec{w}_1^2}^{\vec{w}_1^2/\epsilon_1'} \frac{dw_2^2}{\vec{w}_2^2} \cdots \int_{\epsilon_{i-2}\vec{w}_{i-2}^2}^{\vec{w}_{i-2}^2/\epsilon_{i-2}'} \frac{dw_{i-1}^2}{\vec{w}_{i-1}^2}\right) \int_{m^2/\epsilon}^{\epsilon'\vec{q}^2} \frac{dw_i^2}{\vec{w}_i^2} \times \left\{\int_{\epsilon_i \vec{w}_i^2}^{\vec{w}_i^2/\epsilon_i'} \frac{dw_{i+1}^2}{\vec{w}_{i+1}^2} \cdots \int_{\epsilon_{N-1}\vec{w}_{N-1}^2}^{w_{N-1}^2/\epsilon_{N-1}'} \frac{dw_N^2}{\vec{w}_N^2}\right\} g_N(\vec{0}, \vec{w}_1, \ldots, \vec{w}_N, 0), \tag{5.14}$$

where g_N takes the form of h_N given in Eq. (4.13), and is independent of $\vec{q}^2$. Note that in this form one must integrate in the following order: $\vec{w}_{i-1}^2, \vec{w}_{i-2}^2, \ldots, \vec{w}_1^2, \vec{w}_N^2, \ldots, \vec{w}_{i+1}^2, \vec{w}_i^2$. However the result of the first $N-1$ integrations is independent of $\vec{w}_i^2$. To see this, let $\vec{v}_k \equiv \vec{w}_k/|\vec{w}_{k-1}|$, so that

$$\vec{w}_k = \vec{v}_k(|\vec{v}_{k-1}| \cdots |\vec{v}_{i+1}|)|\vec{w}_i|.$$

With this substitution in Eq. (5.14) the integrand and the limits become independent of $\vec{w}_i^2$. Also, only the $N-1$ relative angles appear, so one angular integration can be done trivially in the interior. Now c_N involves various permutations of the g_N. One can always scale out $\vec{w}_1$ say, although it will appear in a different position in each term. The $\vec{w}_1$ integration then gives just $[\pi\ln(\epsilon\epsilon'\vec{q}^2/m^2)]$. The dependence on ϵ and ϵ' will cancel when regions (a) and (b) are included. The remaining coefficient in $C_N(\vec{q}^2)$ should be independent of the ϵ_i and ϵ_i' and has the form

$$a_N = \frac{\pi}{N!}\int\frac{2dx_1}{\pi}\cdots\int\frac{2dx_N}{\pi}\int\frac{d^2v_2}{\vec{v}_2^2}\cdots\int\frac{d^2v_N}{\vec{v}_N^2}\tilde{c}_N(\vec{0}, \vec{v}_2, \ldots, \vec{v}_N, 0), \tag{5.15}$$

where $\tilde{c}_N$ is c_N rewritten in terms of the $\vec{v}_i$.

Note that a_N is independent of both M^2 and μ^2. Physically one might expect correlations of pions to depend on μ^2. In the φ^3 ladder diagrams contributions from longitudinal correlations depend on the mass of the "internal" particles.[3] In the present case the pion mass appears as a shift of origin, i.e., in the form $\vec{w}_i^2 + \mu^2$, and the transverse pion correlations do not depend on it. The constants a_1 and a_2 are calculated from Eq. (5.15) in Appendix B. They agree with the results of Sec. III. The constant term b_N appearing in Eq. (5.9) comes from the "surface" regions (a) and (b), and contains all the mass dependence.

VI. CONCLUSIONS

Based on the cluster expansion in neutral pseudoscalar field theory, one can conclude that the Dirac form factor of the proton for rainbow diagrams has the structure[23]

$$F_1(q^2)=B\left(g^2,\frac{\mu^2}{m^2},\frac{M^2}{m^2}\right)\left(\frac{-q^2}{m^2}\right)^{-A(g^2)}, \qquad (6.1)$$

where g is the $\pi^0 p$ coupling constant, μ and M are the pion and proton mass, respectively, and m is an arbitrary scale factor. The functions A and B represent volume and surface effects, respectively, in a transverse momentum space of volume equal to $\ln(-q^2/m^2)$. They have the form

$$A(g^2)=\sum_{n=1}^{\infty}\left(\frac{g^2}{32\pi^2}\right)^n(-a_n),$$

$$B\left(g^2,\frac{\mu^2}{m^2},\frac{M^2}{m^2}\right)=\exp\left[\sum_{n=1}^{\infty}\left(\frac{g^2}{32\pi^2}\right)b_N\left(\frac{\mu^2}{m^2},\frac{M^2}{m^2}\right)\right]. \qquad (6.2)$$

In particular, $a_1=-1$, $b_1=0$, and $a_2=-\frac{5}{2}$.

According to experiments on electron-proton scattering, one should expect that A will be positive. If any of the a_n are positive in this model the sign of A may depend on the value of q^2 and g^2. However one is not led to expect this from the first two terms.

The following points should be emphasized regarding this calculation:

(1) It is particularly interesting that renormalization does not destroy the crucial factorization property. This indicates that the concept of cluster expansion may be useful for analyzing other processes involving spin. Its potential phenomenological usefulness for relating exclusive and inclusive production of spinless particles has recently been pointed out.[24] The basic ideas may also be valid for production of particles with spin.

(2) The rapid damping of the elastic form factor for large $-q^2$ is due to the increased probability that the pions will be emitted as bremmstrahlung. A previously studied model for the inelastic process,[25] similar in structure to the elastic one analyzed here, is shown in Fig. 12. When just the bare vertex is included the inelastic form factor W_2 in the deep-inelastic region factors in leading order into a part associated with the "outer rainbow" and one associated with the "inner rainbow." The outer rainbow depends on the fraction of longitudinal momentum on the proton immediately before the current insertion. If one does a Mellin transform on this parameter the outer rainbow factor is

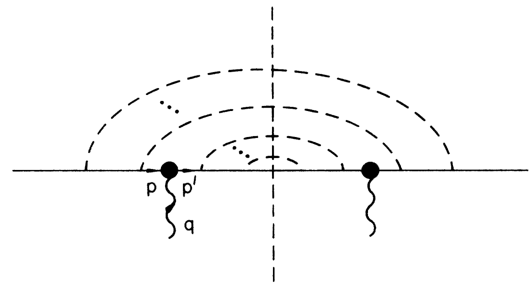

FIG. 12. A related model of the inelastic process. The rainbow form factor is generally not gauge invariant when $p'^2 \neq p^2$.

$$\exp\left[\frac{g^2}{16\pi^2}\frac{\ln(-q^2/m^2)}{\lambda(\lambda+1)}\right]-1, \qquad (6.3)$$

where λ is a "longitudinal impact parameter." The inner rainbow factor is simply

$$\exp\left[\frac{g^2}{32\pi^2}\ln\left(\frac{-q^2}{m^2}\right)\right]. \qquad (6.4)$$

The cluster expansion may provide a mechanism for inclusion of the nonleading terms. Secondly, comparison with the form factor result suggests that an appropriate vertex function in Fig. 12 may help restore scaling to νW_2. However, one would have to deal with the problem of gauge invariance.

(3) It is important to realize that the sum of leading terms can only suggest the general form of an amplitude. It is not by itself significant except in the weak coupling limit, as one can see from the first two terms in Eq. (6.2).

Grouping terms of the same order in g^2 and the same power in $\ln(-q^2/m^2)$ can be extremely misleading, as the sum of "next-to-leading" terms indicates. [See (5.13).] One must first identify the physical mechanism for the sum of leading terms, and include the nonleading terms as a modification on this fundamental unit. In the present model this mechanism is the association of $\ln(\vec{q}^2/m^2)$ with independent regions in the transverse phase space. In general all possible correlations are important. This strongly suggests that a nonperturbative approach is essential to a quantitative understanding of high-energy processes involving strong interactions.

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APPENDIX A

This appendix examines the two-pion on-shell rainbow, showing how a symmetry in Feynman parameters gives a conserved current. The argument is a straightforward generalization of the one-pion case.

In the notation of Fig. 1 the denominator factors can be combined as usual to give

$$\begin{aligned} D_2 &= \{\alpha_1[(p-w_1)^2-M^2]+\alpha_2[(p'-w_1)^2-M^2]+\alpha_3(w_1{}^2-\mu^2) \\ &\quad +\alpha_4[(p-w_1-w_2)^2-M^2]+\alpha_5[(p'-w_1-w_2)^2-M^2]+\alpha_6(w_2{}^2-\mu^2)\}^6 \\ &= \{(1-\alpha_6)w_1{}^2+(\alpha_4+\alpha_5+\alpha_6)w_2{}^2+(\alpha_4+\alpha_5)2w_1\cdot w_2 \\ &\quad -2w_1\cdot[(\alpha_1+\alpha_4)p+(\alpha_2+\alpha_5)p']-2w_2\cdot[\alpha_4 p+\alpha_5 p']-(\alpha_3+\alpha_6)\mu^2\}^6 . \end{aligned} \tag{A1}$$

The appropriate translation to cancel the cross terms is of the form

$$w_1 = w_1' + ap + a'p', \quad w_2 = w_2' + bp + b'p', \tag{A2}$$

where

$$\begin{aligned} a &= (\det A)^{-1}[\alpha_1(\alpha_4+\alpha_5)+\alpha_6(\alpha_1+\alpha_4)], \\ a' &= (\det A)^{-1}[\alpha_2(\alpha_4+\alpha_5)+\alpha_6(\alpha_2+\alpha_5)], \\ b &= (\det A)^{-1}[-(\alpha_1+\alpha_4)(\alpha_4+\alpha_5)+(1-\alpha_6)\alpha_4], \\ b' &= (\det A)^{-1}[-(\alpha_2+\alpha_5)(\alpha_4+\alpha_5)+(1-\alpha_6)\alpha_5], \end{aligned} \tag{A3}$$

and

$$A = \begin{pmatrix} 1-\alpha_6 & \alpha_4+\alpha_5 \\ \alpha_4+\alpha_5 & \alpha_4+\alpha_5+\alpha_6 \end{pmatrix}. \tag{A4}$$

In the primed variables one finds

$$\begin{aligned} D_2 &= \{(1-\alpha_6)w_1'^2+(\alpha_4+\alpha_5+\alpha_6)w_2'^2-p\cdot p'[(\alpha_1+\alpha_4)a'+(\alpha_2+\alpha_5)a+\alpha_4 b'+\alpha_5 b] \\ &\quad -M^2[\alpha_1+\alpha_4)a+(\alpha_2+\alpha_5)a'+\alpha_4 b+\alpha_5 b']-(\alpha_3+\alpha_6)\mu^2\}^6 . \end{aligned} \tag{A5}$$

It is easy to check that this is invariant under the permutation $\mathcal{P}=(\alpha_3)(\alpha_6)(\alpha_1\alpha_2)(\alpha_4\alpha_5)$ in which one interchanges α_1 and α_2, and α_4 and α_5. (Note that this interchanges a, a' and b, b'.) Since the α-space integration is completely symmetric only the part of the numerator invariant under $\mathcal{P}$ contributes. Thus one can make the following replacement in the numerator:

$$N_2(a', a; b', b) \to \tfrac{1}{2}[N_2(a', a; b', b)+N_2(a, a'; b, b')]. \tag{A6}$$

Terms linear in w_1 or w_2 integrate to zero. So do the antisymmetric parts of $w_{1\mu}'w_{1\nu}'$ and $w_{2\mu}'w_{2\nu}'$, so that one can also make the following replacements in the numerator:

$$w_{1\mu}'w_{1\nu}' \to \tfrac{1}{4}g_{\mu\nu}w_1'^2, \quad w_{2\mu}'w_{2\nu}' \to \tfrac{1}{4}g_{\mu\nu}w_2'^2. \tag{A7}$$

In the original variables the numerator can be written

$$\bar{u}(p')[(2w_1\cdot p-w_1{}^2)(2w_1\cdot p'-w_1{}^2)\gamma_\mu-(2w_1\cdot p-w_1{}^2)\not{w}_1\not{w}_2\gamma_\mu-(2w_1\cdot p'-w_1{}^2)\gamma_\mu\not{w}_2\not{w}_1+\not{w}_1\not{w}_2\gamma_\mu\not{w}_2\not{w}_1]u(p). \tag{A8}$$

The first term is clearly conserved. If one makes the substitution of Eq. (A2), dropping any terms linear in w_1' or w_2', and using (A7), both the sum of the cross terms and the last term in (A8) can be cast into the form

$$A(a, a'; b, b')\gamma_\mu + B(a, a'; b, b')p_\mu' + B(a', a; b', b)p_\mu . \tag{A9}$$

Thus the use of (A6) gives

$$\tfrac{1}{2}[A(a, a'; b, b')+A(a', a; b, b')]\gamma_\mu+\tfrac{1}{2}[B(a, a'; b, b')+B(a', a; b', b)](p'+p)_\mu, \tag{A10}$$

which is indeed conserved.

APPENDIX B

This appendix shows how the numerators to third order in g^2 can be cast into the form of a "transverse" trace. One begins with the relation

$$F_1^{(N)} = \tfrac{1}{8}\,\mathrm{Tr}[\gamma_+(\not p' + M)\Gamma_+^{(N)}(\not p + M)]. \tag{B1}$$

After the γ_5's are eliminated, the numerator takes the form

$$\tfrac{1}{8}\,\mathrm{Tr}[\gamma_+(\not A_1 + M)(\not A_2 - M)\cdots(\not A_N \pm M)\gamma_+(\not B_N \pm M)\cdots(\not B_2 - M)(\not B_1 + M)], \tag{B2}$$

where the A_i and B_i are 4-vectors and the sign of M alternates due to commuting the γ_5's. One can define $A_+^{(N)}$ by commuting the left-hand γ_+, so that

$$\gamma_+(\not A_1 + M)(\not A_2 - M)\cdots(\not A_N \pm M) \equiv 2A_+^{(N)} + (-)^N(\not A_1 - M)(\not A_2 + M)\cdots(\not A_N \mp M)\gamma_+ \tag{B3}$$

and similarly for $B_+^{(N)}$. Then one has

$$\begin{aligned}\mathrm{Tr}[2A_+^{(N)}2B_+^{(N)}] = {} & \mathrm{Tr}[\gamma_+(\not A_1 + M)(\not A_2 - M)\cdots(\not A_N \pm M)\gamma_+(\not B_N \pm M)\cdots(\not B_2 - M)(\not B_1 + M)]\\ & + \mathrm{Tr}[\gamma_+(\not A_1 - M)(\not A_2 + M)\cdots(\not A_N \mp M)\gamma_+(\not B_N \mp M)\cdots(\not B_2 + M)(\not B_1 - M)]\end{aligned} \tag{B4}$$

which follows from the cyclic property of the trace and the fact that $\gamma_+{}^2 = 0$. The second term on the right differs from the first only in the sign of M. However, any odd power of M is multiplied by the trace of an odd number of Dirac matrices, which vanishes. So the two terms in Eq. (B4) are equal. Thus one finds

$$\tfrac{1}{8}\,\mathrm{Tr}[\gamma_+(\not A_1 + M)\cdots(\not A_N \pm M)\gamma_+(\not B_N \pm M)\cdots(\not B_1 + M)] = \tfrac{1}{4}\,\mathrm{Tr}[A_+^{(N)}B_+^{(N)}]. \tag{B5}$$

In first order in g^2 one has

$$\gamma_+(\not p' + M)(\not p' - \not w_1 - M) = 2[(\not p' - \not w_1 - M) - (1 - x_1)(\not p' - M)] + (\not p' - M)(\not p' - \not w_1 + M)\gamma_+$$

and similarly for the other γ_+. So one obtains

$$A_+^{(1)} = x_1\not p' - \not w_1 - x_1 M, \quad B_+^{(1)} = \not w_1 - x_1\not p - x_1 M. \tag{B6}$$

Note that the plus components of $x_1 p' - w_1$ and $w_1 - x_1 p$ vanish. Thus their dot product is just $\vec{w}_1' \cdot \vec{w}_1$, where $\vec{w}_1' \equiv \vec{w}_1 - x_1\vec{q}$. In higher orders it is more convenient to leave the result in trace form. One can define a "transverse trace" by

$$\mathrm{Tr}(\vec{\not a}\vec{\not b}) = -\mathrm{Tr}(\vec{a}\cdot\vec{\gamma}\vec{b}\cdot\vec{\gamma}) = 4\vec{a}\cdot\vec{b} \tag{B7}$$

along with the usual rules for reducing larger traces. Then the one-pion numerator is

$$-\tfrac{1}{4}\,\mathrm{Tr}[(\vec{\not w}_1' - x_1 M)(\vec{\not w}_1 - x_1 M)] \tag{B8}$$

where the over-all sign is from $\gamma_5\gamma_+\gamma_5 = -\gamma_+$.

In $O(g^4)$ one has

$$\begin{aligned}\gamma_+(\not p' + M)(\not p' - \not w_1 - M)(\not p' - \not w_1 - \not w_2 + M)\\ = 2[(x_1\not p' - \not w_1 - x_1 M)(\not p' - \not w_1 - \not w_2 + M) + (1 - x_1 - x_2)(\not p' + M)(\not p' - \not w_1 + M)]\\ - (\not p' - M)(\not p' - \not w_1 + M)(\not p' - \not w_1 - \not w_2 - M)\gamma_+ .\end{aligned}$$

In order to cast the first term of $A_+^{(2)}$ as a product of factors with vanishing plus component, one can write

$$\begin{aligned}A_+^{(2)} = {} & (x_1\not p' - \not w_1 - x_1 M)[(x_1 + x_2)\not p' - \not w_1 - \not w_2 + M]\\ & + (1 - x_1 - x_2)[(x_1\not p' - \not w_1 - x_1 M)\not p' + (\not p' - M)(\not p' - \not w_1 + M)].\end{aligned} \tag{B9}$$

After the minus integrations one has $w_1{}^2 = \mu^2$. Then the term in square brackets becomes

$$(x_1\not p' - \not w_1 - x_1 M)\not p' - (\not p' - M)\not w_1 = \frac{1}{x_1}[(x_1\not p' - \not w_1 - x_1 M)(x_1\not p' - \not w_1) - \mu^2].$$

So the two terms in Eq. (B9) can be combined. $B_+^{(2)}$ has the same form with $x_1\not p' - \not w_1 \to \not w_1 - x_1\not p$. Thus the two-pion numerator after minus integrations can be written in the notation of Eq. (B8) as

$$\tfrac{1}{4}\,\mathrm{Tr}\left\{\left[(\vec{\not w}_1' - x_1 M)\left(\frac{1-x_2}{x_1}\vec{\not w}_1' + \vec{\not w}_2 + M\right) - \frac{1-x_1-x_2}{x_1}\mu^2\right]\left[\left(\frac{1-x_2}{x_1}\vec{\not w}_1 + \vec{\not w}_2 + M\right)(\vec{\not w}_1 - x_1 M) - \frac{1-x_1-x_2}{x_1}\mu^2\right]\right\}. \tag{B10}$$

Just as in Eq. (B9) one can write $A_+^{(3)}$ as $A_+^{(2)}[(x_1+x_2+x_3)\not p - \not w_1 - \not w_2 - \not w_3 - M]$ plus $(1-x_1-x_2-x_3)$ times the term

$$\left[(x_1\not p' - \not w_1 - x_1 M)\not p' - \frac{1-x_2}{x_1}\not w_1 - \not w_2 + M - \frac{1-x_1-x_2}{x_1}\mu^2\right]\not p' - (\not p' - M)(\not p' - \not w_1 + M)(\not p' - \not w_1 - \not w_2 - M). \tag{B11}$$

When $w_1^2 = \mu^2 = w_2^2$ the last term in this expression can, after some algebra, be written

$$-\frac{1}{x_1}\left\{[x_1\not p' - \not w_1 - x_1 M]\left(\not p' - \frac{1-x_2}{x_1}\not w_1 - \not w_2 + M\right) - \frac{1-x_1-x_2}{x_1}\mu^2\right\}\not w_1$$
$$+ (x_1\not p' - \not w_1 - x_1 M)\left[x_2\left(\frac{\not w_2'}{x_2} - \frac{\not w_1'}{x_1}\right)^2 - \frac{1}{x_1}(x_1\not p' - \not w_1)^2\right]$$
$$+ (x_1\not p' - \not w_1 - x_1 M)\left[M^2 - \frac{\mu^2}{x_1}\left(\frac{1-x_2}{x_1} + \frac{x_1+x_2}{x_2}\right)\right] + \frac{\mu^2}{x_1}(x_2\not p' - \not w_2 - x_1 M).$$

Then the three-pion numerator can be written:

$$-\tfrac{1}{4}\,\mathrm{Tr}\left(\left\{\left[(\vec{\not w}_1' - x_1 M)\left(\frac{1-x_2}{x_1}\vec{\not w}_1' + \vec{\not w}_2' + M\right) - \frac{1-x_1-x_2}{x_1}\mu^2\right]\left(\frac{1-x_2-x_3}{x_1}\vec{\not w}_1' + \vec{\not w}_2' + \vec{\not w}_3' - M\right)\right.\right.$$
$$\left.\left. + (\not w_1' - x_1 M)\left[x_2\left(\frac{\not w_2'}{x_2} - \frac{\not w_1'}{x_1}\right)^2 - \frac{w_1'^2}{x_1} + M^2 - \frac{\mu^2}{x_1}\left(\frac{1-x_2}{x_1} + \frac{x_1+x_2}{x_2}\right)\right] + \frac{\mu^2}{x_1}(\not w_2' - x_2 M)\right\}\left\{\vec{w}_{1,2}' \to \vec{w}_{1,2}\right\}\right), \tag{B12}$$

where it is understood in $\{\vec{w}_{1,2}' \to \vec{w}_{1,2}\}$ that the order of Dirac matrices is reversed, as in Eq. (B10).

APPENDIX C

Here the constants a_1 and a_2, associated with one and two-pion correlations in the interior, are calculated.

The one-pion correlation function in the interior region is easily found from Eq. (3.13) to be

$$c_1(\vec{0}, \vec{w}_1, 0) \equiv g_1^s(\vec{0}, \vec{w}_1, 0) = -\theta(1-x_1)x_1\,. \tag{C1}$$

Thus Eq. (5.15) gives

$$a_1 = \pi \int \frac{2dx_1}{\pi}[-\theta(1-x_1)x_1] = -1 \tag{C2}$$

which is just the result found in Eq. (3.17).

The two-pion correlation function in the interior is

$$\begin{aligned} c_2(\vec{0}, \vec{w}_1, \vec{w}_2, 0) &\equiv g_2^s(\vec{0}, \vec{w}_1, \vec{w}_2, 0) - g_1^s(\vec{0}, \vec{w}_1, 0)g_1^s(\vec{0}, \vec{w}_2, 0) \\ &\equiv [-\theta(1-x_1)\theta(1-x_1-x_2)f_2(\vec{0}, \vec{w}_1, \vec{w}_2, 0) + \theta(1-x_1)\theta(1-x_2)f_1(\vec{0}, \vec{w}_1, 0)f_1(\vec{0}, \vec{w}_2, 0)] \\ &\quad + [-\theta(1-x_2)\theta(1-x_1-x_2)f_2(\vec{0}, \vec{w}_2, \vec{w}_1, 0) + \theta(1-x_1)\theta(1-x_2)f_1(\vec{0}, \vec{w}_1, 0)f_1(\vec{0}, \vec{w}_2, 0)] \\ &\quad - \theta(1-x_1)\theta(1-x_2)f_1(\vec{0}, \vec{w}_1, 0)f_1(\vec{0}, \vec{w}_1, 0) \\ &= -\theta(1-x_1)\theta(1-x_1-x_2)f_2(\vec{0}, \vec{w}_1, \vec{w}_2, 0) - \theta(1-x_2)\theta(1-x_1-x_2)f_2(\vec{0}, \vec{w}_2, \vec{w}_1, 0) \\ &\quad + \theta(1-x_1)\theta(1-x_2)f_1(\vec{0}, \vec{w}_1, 0)f_1(\vec{0}, \vec{w}_2, 0)\,. \end{aligned} \tag{C3}$$

The function f_2 given by Eq. (3.21) becomes relatively simple in this region. It appears in Eq. (3.28). After changing variables to $\vec{v}_2 = \vec{w}_2/|\vec{w}_1|$ one finds

$$a_2=\frac{\pi}{2!}\int\frac{2dx_1}{\pi}\int\frac{2dx_2}{\pi}\int\frac{d^2v_2}{\vec{v}_2^{\,2}}$$

$$\times\left(\left\{-\frac{\frac{x_1x_2}{(1-x_1)^2}\theta(1-x_1)\theta(1-x_1-x_2)\vec{v}_2^{\,2}\left(\vec{v}_2+\frac{1-x_2}{x_1}\hat{w}_1\right)^2-\frac{x_2}{x_1}\theta(1-x_2)\theta(1-x_1-x_2)\left(\vec{v}_2+\frac{x_2}{1-x_1}\hat{w}_1\right)^2}{\left[\vec{v}_2^{\,2}+2\frac{x_2}{1-x_1}\hat{w}_1\cdot\vec{v}_2+\frac{x_2(1-x_2)}{x_1(1-x_1)}\right]^2}\right\}+x_1x_2\theta(1-x_1)\theta(1-x_2)\right), \tag{C4}$$

where $\hat{w}_1$ is a unit transverse vector, and $x_{1,2}>0$. Letting $x_2\to(1-x_1)x_2$ in the first term and $x_1\to(1-x_2)x_1$ in the second, this becomes

$$a_2=\frac{\pi}{2!}\int_0^1\frac{2dx_1}{\pi}\int_0^1\frac{2dx_2}{\pi}(x_1x_2)\int\frac{d^2v_2}{\vec{v}_2^{\,2}}$$

$$\times\left(\frac{-\vec{v}_2^{\,2}\left[\vec{v}_2+\frac{1-x_2(1-x_1)}{x_1}\hat{w}_1\right]^2}{\left\{\vec{v}_2^{\,2}+2x_2\hat{w}_1\cdot\vec{v}_2+\frac{x_2[1-x_2(1-x_1)]}{x_1}\right\}^2}+\frac{-\frac{1}{x_1^2}\left[\vec{v}_2+\frac{x_2}{1-x_1(1-x_2)}\hat{w}_1\right]^2}{\left\{\vec{v}_2^{\,2}+2\frac{x_2}{1-x_1(1-x_2)}\hat{w}_1\cdot\vec{v}_2+\frac{x_2}{x_1[1-x_1(1-x_2)]}\right\}^2}+1\right). \tag{C5}$$

Note that as $\vec{v}_2^{\,2}\to\infty$ the first and third terms cancel and the second converges. As $\vec{v}_2^{\,2}\to 0$ the first term converges and the second and third cancel. Thus if the terms are integrated individually from ϵ to $1/\epsilon'$ where $\epsilon,\epsilon'\ll 1$, the divergent parts will cancel. Since the first term is only logarithmically divergent at the upper limit, a shift will not affect it. The second and third terms are divergent at the origin, however, and must be shifted together.

In the first term one can put

$$\vec{v}_2=\vec{v}_2'-x_2\hat{w}_1 \tag{C6}$$

to eliminate the angular dependence in the denominator. The cross term in the numerator then integrates to zero, and the remaining transverse integral is

$$-\pi\int_\epsilon^{1/\epsilon'}dv_2'^2\,\frac{\vec{v}_2'^2+\left(\frac{1-x_2}{x_1}\right)^2}{\left(\vec{v}_2'^2+x_2\frac{1-x_2}{x_1}\right)^2}=-\pi\left[\ln\left(\frac{1}{\epsilon'}\right)-\ln\left|\frac{x_2(1-x_2)}{x_1}\right|-1+\frac{1-x_2}{x_1x_2}\right]. \tag{C7}$$

This has the same form as the second- and third-from-the-last terms in Eq. (3.29), where no shift was performed.

In the last two terms of Eq. (C5) one can perform the shift:

$$\vec{v}_2=\vec{v}_2'-\frac{x_2}{1-x_1(1-x_2)}\hat{w}_1\,. \tag{C8}$$

Then the transverse integral becomes

$$-\int\frac{d^2v_2'}{\left(\vec{v}_2'-\frac{x_2}{1-x_1(1-x_2)}\hat{w}_1\right)^2}\left(\frac{\frac{1}{x_1^2}\vec{v}_2'^2}{\left\{\vec{v}_2'^2+\frac{x_2(1-x_1)}{x_1[1-x_1(1-x_2)]^2}\right\}^2}-1\right)$$

$$= -\pi \int_0^{1/\epsilon'} \frac{dv_2'^2}{\left|\vec{v}_2'^2 - \left(\frac{x_2}{1-x_1(1-x_2)}\right)^2\right|} \left(\frac{\frac{1}{x_1^2}\vec{v}_2'^2}{\left\{\vec{v}_2'^2 + \frac{x_2(1-x_1)}{x_1[1-x_1(1-x_2)]^2}\right\}^2} - 1 \right). \tag{C9}$$

The divergence at the origin has been shifted to $\vec{v}_2'^2 = \{x_2/[1-x_1(1-x_2)]\}^2$. The integrand vanishes at that point, so only the principal part contributes. Because of the absolute value, the integral takes the form $F(\infty) - F(a+\epsilon) - [F(a-\epsilon) - F(0)]$ where a is the singular point. Then (C9) becomes

$$-\pi \left\{ -\ln\left(\frac{1}{\epsilon'}\right) - \ln\left[\frac{x_1(1-x_1)}{x_2}\right] - 1 + \frac{1-x_1}{x_1 x_2} \right\}. \tag{C10}$$

When this is added to (C7) the ϵ' cancels. Inserting the result in place of the transverse integral in Eq. (C5) one obtains:

$$a_2 = -\left(\tfrac{3}{2} - 1 + 2\right) = -\tfrac{5}{2} \tag{C11}$$

which agrees with the result found in Eq. (3.30).

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[1]For an excellent summary and references, see R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966), especially Sections (3.3) and (3.6).

[2]S.-J. Chang, T.-M. Yan, and Y.-P. Yao, Phys. Rev. D 1, 3012 (1971).

[3]D. K. Campbell and S.-J. Chang, Phys. Rev. D 4, 1151 (1971); see also A. H. Mueller, *ibid.* 4, 150 (1971).

[4]D. K. Campbell and S.-J. Chang, Phys. Rev. D 4, 3658 (1971).

[5]R. P. Feynman, Phys. Rev. Letters 23, 1415 (1969); K. Wilson, Cornell University Report No. CLNS-131, 1970 (unpublished).

[6]A metric is used in which $q^2 < 0$.

[7]T. Appelquist and J. R. Primack, Phys. Rev. D 1, 1144 (1970).

[8]See, for example, J. D. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 198ff.

[9]P. M. Fishbane and J. D. Sullivan, Phys. Rev. D 4, 458 (1971); see also R. Jackiw, Ann. Phys. (N.Y.) 48, 292 (1968); T. Appelquist and J. R. Primack, Phys. Rev. D 4, 2454 (1971).

[10]For a good introduction to the properties of this representation, see, for example, S.-J. Chang and S. K. Ma, Phys. Rev. 180, 1506 (1969), and also L. Susskind and G. Frye, *ibid.* 165, 1535 (1968).

[11]R. P. Feynman in, *High Energy Collisions*, Third International Conference held at the State University of New York, Stony Brook, 1969, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969), p. 237; J. D. Bjorken and E. Paschos, Phys. Rev. 185, 1975 (1969); S. D. Drell, D. J. Levy, and T.-M. Yan, Phys. Rev. D 1, 1035 (1970).

[12]Actually, the unrenormalized coupling constant g_0 should appear in Eq. (3.1). It is replaced by g after appropriate renormalization.

[13]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[14]The quantity subtracted out is absorbed into the renormalized charge e.

[15]In writing $f_1(\vec{q}, \vec{w}_1, m)$, for example, one must recall that there is also x_1 dependence. However, since pseudoscalar theory has no infrared divergence, $f_1 \to 0$ as $x_1 \to 0$. Similarly all the x_i are $O(1)$ and introduce no q^2 dependence. This is not true in quantum electrodynamics where regions $x_i \to 0$ introduce an extra $\ln(-q^2/m^2)$. (See Ref. 9.)

[16]Regions in which $f_1 \ll 1$ can only contribute to $O(m^2/-q^2)$ and are neglected.

[17]Recall that μ^2 was neglected here. However, it will be shown in Sec. V that only b_1 and b_2 depend on the masses, but a_1 and a_2 do not.

[18]One way to see this is to scale out factors like $(|\vec{q}| + |\vec{w}_1| + \cdots + |\vec{w}_i|)$ by Lorentz transforming each propagator factor. One can then show that the remaining trace is $O(1)$.

[19]One way to see this is to transform these factors as in Eq. (2.6) with $e^{-\xi} = |\vec{w}| \approx (|\vec{w}_{i+1}|, \ldots, |\vec{w}_N|)$. After the inverse transformation, only the coefficient of γ_+ behaves as $(\vec{w}^2)^{N-i}$.

[20]See for example Kerson Huang, *Statistical Mechanics* (Wiley, New York, 1963), p. 303 ff.

[21]This result can be compared with the corresponding quantity in Ref. 7, given there by Eq. (6.6). Nonleading terms given there are numerically different because the internal subtractions have been performed at $p = 0 = q$. However, the result is still larger by $\ln(-q^2/M^2)$ than the "leading" sum. It is felt that the simplicity of the present result argues for doing all subtractions on-shell in studying nonleading terms.

[22]There is no q^2 dependence other than that appearing in the $\vec{w}_i'$. This is clear for D_N defined in Eq. (4.5), and can be shown for $\mathfrak{N}_N$ by scaling the propagators as mentioned in footnote 18.

[23]During the writing of this paper, a study of the rainbow diagrams by Y. Shimizu, Phys. Rev. D (to be published), was received. By Mellin transform techniques he obtains results consistent with Eq. (6.1) but including additional terms which are here seen to be vanishing by the cluster decomposition method.

[24]D. K. Campbell and S.-J. Chang, University of Illinois Report No. Th-71/11, 1971 (unpublished).

[25]S.-J. Chang and P. M. Fishbane, Phys. Rev. D 2, 1084 (1970).

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Short-Distance Behavior of Quantum Electrodynamics and an Eigenvalue Condition for α

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We review and extend earlier work dealing with the short-distance behavior of quantum electrodynamics. We show that if the renormalized photon propagator is asymptotically finite, then in the limit of zero fermion mass all of the single-fermion-loop $2n$-point functions, regarded as functions of the coupling constant, must have a common infinite-order zero. In the usual class of asymptotically finite solutions introduced by Gell-Mann and Low, the asymptotic coupling α_0 is fixed to be this infinite-order zero and the physical coupling $\alpha < \alpha_0$ is a free parameter. We show that if the single-fermion-loop diagrams actually possess the required infinite-order zero, there is a unique, additional solution in which the physical coupling α is fixed to be the infinite-order zero. We conjecture that this is the solution chosen by nature. According to our conjecture, the fine-structure constant is determined by the eigenvalue condition $F^{[1]}(\alpha) = 0$, where $F^{[1]}$ is a function related to the single-fermion-loop vacuum-polarization diagrams. The eigenvalue condition is independent of the number of fundamental fermion species which are assumed to be present.

I. INTRODUCTION AND SUMMARY

The fundamental constant regulating all microscopic electronic phenomena, from atomic physics to quantum electrodynamics, is the fine-structure constant α. Experimentally, the current value[1] $\alpha = 1/(137.03602 \pm 0.00021)$ is one of the best determined numbers in physics. Theoretically, the reason why nature selects this particular numerical value has remained a mystery, and has provoked much interesting speculation. The speculations may be divided roughly into three general types: (a) those in which α is cosmologically determined, either as a cosmological boundary condition (which makes α undeterminable) or as a function of time-varying cosmological parameters (which makes α a function of time)[2]; (b) theories in which α is a constant which is determined microscopically through the interplay of the electromagnetic interaction with interactions of other types, either strong, weak, or gravitational.[3] Since these interactions are currently even less well understood than is the electromagnetic interaction, such theories seem at present to offer little promise of an actual computation of α; (c) finally, theories in which α is microscopically determined through properties of the electromagnetic interaction alone, considered in isolation from other interactions. It is this restricted class of theories to which we will address ourselves in the present paper.

The idea that α may be determined electromagnetically is an old one. In the early days of renormalization theory there were hopes that α could be fixed by requiring the logarithmic divergences appearing in higher orders of perturbation theory to cancel or "compensate" the second-order divergence in the photon wave-function renormalization Z_3,[4] so that the renormalized photon propagator would be asymptotically finite. These hopes received a setback, however, when Jost and Luttinger[5] calculated the order-α^2 logarithmically divergent contribution to Z_3 and found that it has the same sign as the order-α divergence. Of course, it was obvious that the question could not be settled by calculations to any finite order of perturbation theory. A systematic nonperturbative attack on the problem was made by Gell-Mann and Low[6] in their classic 1954 paper on renormalization-group methods. They showed that there is