Thermal Noise in Lossy Waveguides

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Abstract- **This work rigorously treats thermal electromagnetic noise in lossy waveguides and develops explicit modal equivalent-circuit representations for the noise generated by arbitrary passive networks embedded in them. The results show that the formulations in common use are limited to lossless transmission media.**

INTRODUCTION

Here we will place the theory of electrical noise in electromagnetic waveguides on firm theoretical ground, developing explicit expressions for the spectral densities and the correlations of modal Thevenin-equivalent voltage sources describing the electrical noise generated by arbitrary passive circuits embedded in lossy guides.

In 1928 Nyquist [1] explained Johnson's measurements of the electrical noise voltage of a resistor [2] by examining the interaction between the resistor and a lossless transmission line supporting a single dominant mode of propagation. Nyquist's arguments were based on the assumption that the modes of an electromagnetic resonator form a closed system to which the second law of thermodynamics may be applied; maximizing the entropy of this system shows that the average energy per unit bandwidth of each mode of the resonator is $hf/(e^{hf/kT}-1)$, where f is the frequency, k is the Boltzmann constant, h is the Planck constant, and T is the absolute temperature of the system. By applying this result to resonators formed from increasingly long sections of lossless transmission line, Nyquist was able to determine the power spectral density of the electromagnetic energy of a single lossless mode

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in an infinite transmission line in thermal equilibrium with its environment. He then examined the interaction between a resistor and the line; the requirement that the average power flow between them be balanced in thermal equilibrium fixed the electromagnetic energy radiated by the resistor into the lossless mode of the line. This determined the spectral density of the resistor's Thevenin-equivalent voltage describing its electromagnetic noise.

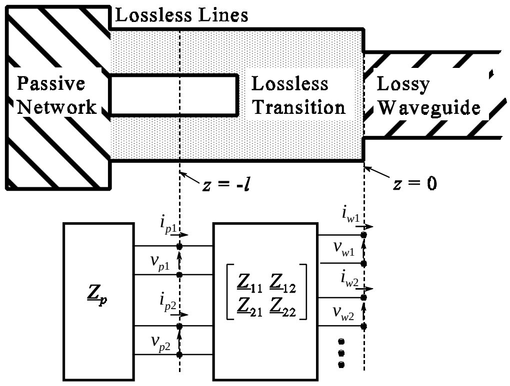

Figure 1. A passive but otherwise arbitrary multiport network embedded in the lossless transmission lines connected to a lossy waveguide and its equivalent circuit.

Strictly speaking, Nyquist's Thevenin-equivalent voltage was not actually a property of the resistor itself, but of the electromagnetic radiation emitted by the resistor into a mode of a lossless transmission line. Discussions of Nyquist's results are found in [3] and [4], and in numerous modern quantum-mechanical treatments of thermal electrical noise.

From Nyquist's results Schremp [5] developed Thevenin-equivalent representations for the electromagnetic noise generated by reciprocal and passive but otherwise arbitrary multiport networks embedded in lossless transmission lines. Twiss [6] extended these results to arbitrary passive multiport networks embedded in those lines. Bosma [7] discusses their wave representations.

Nyquist's arguments cannot be extended directly to lossy waveguides because, when he applies the second law of thermodynamics to a waveguide mode, he assumes that it forms a closed system. In fact, a mode of a lossy waveguide does not form a closed system, as it is coupled to and dissipated by the materials composing the guide. Here we determine the thermal noise generated by arbitrary passive networks by considering how thermal energy is transferred from passive networks embedded in lossless transmission lines, the special case where the results of Nyquist and Twiss can be applied directly, to lossy waveguides.

Figure 1 illustrates the argument. It shows a passive multiport network at the left embedded in a set of lossless transmission lines (for clarity only two are shown in the figure). To simplify the arguments, we will assume that the lossless lines support only a single propagating mode and that all the other modes of the lines have decayed away at $z=-l$; this allows us to apply in a straightforward manner the results of Twiss [6] to characterize the noise there. The lossless transmission lines are connected to the lossy waveguide by a transition that is composed entirely of lossless materials, begins at $z=-l$, and abruptly terminates in the lossy waveguide at $z=0$. We will account for all of the modes in the lossy waveguide; by this full accounting we will eliminate sources of electromagnetic noise in the transition due to the excitation of high-order modes in the lossy guide. This and the restriction that the transition is constructed only of lossless materials will allow us to treat it in the context of the theory of [8] as truly lossless and sourceless. The simplest such transition is formed by continuing the lossless transmission lines to z=0 and abruptly connecting them to the lossy guide there.

In what follows we will use the general waveguide circuit theory of [8] to examine the flow of the noise from the passive network of Figure 1, which can be characterized using the results of [6], through the lossless and sourceless transition to the lossy waveguide. This will allow us to develop expressions for the noise generated by the network to the left of z=0 in the lossy guide, which we will express in terms of the spectral densities of modal Thevenin-equivalent voltage sources and their correlations. Since we place no restrictions on the passive network, we will conclude that the expression is general, valid for any passive network embedded in the lossy guide.

MODAL VOLTAGES AND CURRENTS

We require that the lossless transmission lines and lossy guide of Figure 1 be closed, uniform in z, and constructed entirely of materials with isotropic permittivity and permeability. These restrictions ensure that the electromagnetic eigenvalue problem is separable and that the lines and guide support discrete and complete sets of forward and backward modes [8,9]; the continuous spectrum of radiation modes supported by open structures are neglected here. As outlined in the introduction, we also assume only a single dominant mode in each of the lossless lines at z=-l. We can now apply the general waveguide theory of [8], and express the total transverse electric field $\boldsymbol{E}_{ptn}$ and total transverse magnetic field $\boldsymbol{H}_{ptn}$ in the nth lossless line with a single modal voltage v_{pn} and modal current i_{pn} defined by

$$\boldsymbol{E}_{ptn}(z=-l) = \frac{v_{pn}}{v_{p0n}}\,\boldsymbol{e}_{ptn}\ ; \quad \boldsymbol{H}_{ptn}(z=-l) = \frac{i_{pn}}{i_{p0n}}\,\boldsymbol{h}_{ptn}, \tag{1}$$

where $\boldsymbol{e}_{ptn}$ and $\boldsymbol{h}_{ptn}$ are the transverse fields of the forward propagating dominant mode and v_{p0n} and i_{p0n} are normalizing factors. We define the modal voltages and currents in the lossy waveguide in an analogous manner, writing the total transverse electric field $\boldsymbol{E}_{wt}$ and magnetic field $\boldsymbol{H}_{wt}$ there in terms of the modal voltages v_{wm} and modal currents i_{wm} as

$$\boldsymbol{E}_{wt}(z=0) = \sum_{m=1}^{\infty} \frac{v_{wm}}{v_{w0m}} \boldsymbol{e}_{wtm} \; ; \quad \boldsymbol{H}_{wt}(z=0) = \sum_{m=1}^{\infty} \frac{i_{wm}}{i_{w0m}} \boldsymbol{h}_{wtm}, \tag{2}$$

where $\boldsymbol{e}_{wtm}$ and $\boldsymbol{h}_{wtm}$ are the transverse fields of the mth forward propagating mode of the lossy waveguide, v_{w0m} and i_{w0m} are normalizing factors, and the sums over m span the set of all modes in the guide, typically infinite in number. This assignment of discrete modal voltages and currents to each mode cannot be made in open guides, which support in addition a continuous spectrum of radiation modes, necessitating the restriction here to closed guides.

In accordance with [8] and [10], we place the restrictions $v_{p0n} i_{p0n}{}^{*} = \int \boldsymbol{e}_{ptn} \times \boldsymbol{h}_{ptn}{}^{*} \cdot \boldsymbol{z} \, \mathrm{d}S$ and $v_{w0m} i_{w0m}{}^{*} = \int \boldsymbol{e}_{wtm} \times \boldsymbol{h}_{wtm}{}^{*} \cdot \boldsymbol{z} \, \mathrm{d}S$, where "*" indicates the complex conjugate, on the normalizing factors v_{p0n}, i_{p0n}, v_{w0m}, and i_{w0m}: this restriction assures that, when only one mode is present, the power transmitted across a reference plane by that mode alone is $\mathrm{Re}\,(v_{pn} i_{pn}{}^{*})$ or $\mathrm{Re}\,(v_{wm} i_{wm}{}^{*})$, as appropriate. If we choose v_{w0m} to be the integral of $\boldsymbol{e}_{wtm}$ over a given path in the transverse plane of the lossy guide, then v_{wm} will correspond to the integral of $\boldsymbol{E}_{wt}$ over that same path when only the mth mode is present. Likewise, if we choose i_{w0m} to be the integral of $\boldsymbol{h}_{wtm}$ around a given closed path in the transverse plane of the lossy guide, then i_{wm} will correspond to the integral of $\boldsymbol{H}_{wt}$ around that same path when only the mth mode is present. However, choosing either v_{w0m} or i_{w0m} fixes the other. These considerations also apply in the lossless transmission lines.

Denoting the vectors of voltages v_{pn} and currents i_{pn} by $\underline{v}_p$ and $\underline{i}_p$, respectively, the total real power crossing the reference plane at z=-l is $P(z=-l) = \mathrm{Re}(\underline{i}_p{}^{\dagger} \, \underline{v}_p)$, where the superscript "†" indicates the Hermitian adjoint (conjugate transpose). The total real power transferred across the reference plane at z=0 in the lossy guide is

$$P = \mathrm{Re}\left(\int_{z=0} \boldsymbol{E}_{wt} \times \boldsymbol{H}_{wt}{}^{*} \cdot \boldsymbol{z} \, \mathrm{d}S \right) = \mathrm{Re}\left(\int \left(\sum_{n=1}^{\infty} \frac{v_{wn}}{v_{w0n}} \boldsymbol{e}_{wtn} \right) \times \left(\sum_{m=1}^{\infty} \frac{i_{wm}}{i_{w0m}} \boldsymbol{h}_{wtm} \right)^{*} \cdot \boldsymbol{z} \, \mathrm{d}S \right). \tag{3}$$

Defining the elements of the cross-power matrix $\underline{X}$ to be

$$X_{mn} \equiv \frac{1}{v_{w0n} \, i_{w0m}{}^{*}} \int \boldsymbol{e}_{wtn} \times \boldsymbol{h}_{wtm}^{*} \cdot \boldsymbol{z} \, \mathrm{d}S, \tag{4}$$

can compactly express equation **(3)** as

$$P = \mathrm{Re}(\, \underline{i}_w^{\dagger}\, \underline{X}\, \underline{v}_w \,)\,. \qquad \textbf{(5)}$$

The diagonal elements of $\underline{X}$ are equal to 1; in the presence of loss, the off-diagonal terms of $\underline{X}$ will generally differ from 0.

IMPEDANCE MATRICES AND THEVENIN-EQUIVALENT SOURCES

We will represent the electromagnetic noise of thermal origin generated in the passive network of Figure 1 at z=-l by the vector $\underline{\hat{v}}_p$ of modal Thevenin-equivalent voltage sources. The vector $\underline{\hat{v}}_p$ is defined by

$$\underline{v}_p = -\underline{Z}_p\, \underline{i}_p + \underline{\hat{v}}_p\,, \qquad \textbf{(6)}$$

where $\underline{Z}_p$ is the impedance matrix of the passive network embedded in the lossless transmission lines. The negative sign in **(6)** accounts for the fact that $\underline{Z}_p$ is defined with respect to currents which enter the passive network, whereas the modal currents $\underline{i}_p$ are associated with the forward modes in the transmission lines.

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$$\underline{v}_w = -\underline{Z}_w\, \underline{i}_w + \underline{\hat{v}}_w\,, \qquad \textbf{(7)}$$

where $\underline{Z}_w$ is the impedance matrix of the passive network *and* the lossless transition embedded in the lossy guide. Again, the negative sign in **(7)** accounts for the fact that $\underline{Z}_w$ is defined with respect to currents which enter the lossless transition, whereas the modal currents $\underline{i}_w$ are associated with the forward modes in the lossy guide.

We define the impedance matrix of the lossless transition connecting the embedded network to the lossy guide by

$$\begin{bmatrix} \underline{v}_p \\ \underline{v}_w \end{bmatrix} = \begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12} \\ \underline{Z}_{21} & \underline{Z}_{22} \end{bmatrix} \begin{bmatrix} \underline{i}_p \\ -\underline{i}_w \end{bmatrix}. \qquad \textbf{(8)}$$

Because we have accounted for all modes in the problem, we can speak of this transition as lossless, which explains the absence of source terms in **(8)**. The negative sign in **(8)** accounts for the fact that the modal currents $\underline{i}_w$ are associated with the forward modes in the lossy guide, which leave, rather than enter, the transition.

NOISE CORRELATION MATRIX

The noise properties of the passive network at z=-l are conveniently expressed in the frequency domain by the matrix $\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}}$, where the overbar indicates the spectral density of the quantity below it [11]. The nth diagonal element of $\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}}$ is the spectral density $\overline{|\hat{v}_{pn}|^2}$ of $|\hat{v}_{pn}|^2$. The nmth element of $\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}}$ is the spectral density of $\hat{v}_{pn}\ \hat{v}_{pm}{}^*$. These off-diagonal elements of $\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}}$ contain the correlations between the elements of $\underline{\hat{v}}_p$.

Twiss [6] shows that when the passive circuit is in thermal equilibrium, $\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}}$ is given by

$$\overline{\underline{\hat{v}}_p\ \underline{\hat{v}}_p^{\dagger}} = 2\frac{hf}{e^{hf/kT}-1}\left[\underline{Z}_p + \underline{Z}_p^{\dagger}\right]. \qquad \textbf{(9)}$$

This result is obtained directly from Nyquist's expression for the spectral density of the Thevenin-equivalent voltage source that describes the noise of a resistor in a lossless line and arguments of thermal equilibrium. In what follows, we will try to develop an expression comparable to **(9)** for the noise behavior of the network in the lossy guide at the reference plane z=0.

We can determine the Thevenin-equivalent voltage sources $\underline{\hat{v}}_w$ in terms of their counterparts $\underline{\hat{v}}_p$ by applying the boundary condition $\underline{i}_w = 0$ in **(7)** and **(8)**, in which case $\underline{\hat{v}}_w = \underline{v}_w = \underline{Z}_{21}\underline{i}_p$. Substituting **(6)** into **(8)** to eliminate $\underline{v}_p$ and $\underline{i}_p$ gives the desired result

$$\underline{\hat{v}}_w = \underline{Z}_{21}\left(\underline{Z}_p + \underline{Z}_{11}\right)^{-1}\underline{\hat{v}}_p. \qquad \textbf{(10)}$$

Thus the matrix $\overline{\underline{\hat{v}}_w\ \underline{\hat{v}}_w^{\dagger}}$ is

$$\overline{\underline{\hat{v}}_w\ \underline{\hat{v}}_w^{\dagger}} = \overline{\underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\underline{\hat{v}}_p\left(\underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\underline{\hat{v}}_p\right)^{\dagger}} = \underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\ \overline{\underline{\hat{v}}_p\underline{\hat{v}}_p^{\dagger}}\left((\underline{Z}_p+\underline{Z}_{11})^{-1}\right)^{\dagger}\underline{Z}_{21}^{\dagger}. \qquad \textbf{(11)}$$

Substitution of **(9)** into **(11)** results in

$$\overline{\underline{\hat{v}}_w\, \underline{\hat{v}}_w^{\dagger}} = 2\frac{hf}{e^{hf/kT}-1}\, \underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\left[\underline{Z}_p + \underline{Z}_p^{\dagger}\right]\left((\underline{Z}_p+\underline{Z}_{11})^{-1}\right)^{\dagger}\underline{Z}_{21}^{\dagger}. \qquad \textbf{(12)}$$

PROPERTIES OF THE LOSSLESS TRANSITION

We will now use the lossless property of the transition to simplify **(12)**, eventually expressing the factors on the right involving $\underline{Z}_p$, $\underline{Z}_{11}$, and $\underline{Z}_{21}$ in terms of $\underline{Z}_w$ and the cross-power matrix $\underline{X}$. Appendix 1 shows that the impedance matrix $\underline{Z}_L$ of a passive and lossless circuit satisfies

$$\underline{X}_L\, \underline{Z}_L + \underline{Z}_L^{\dagger}\, \underline{X}_L^{\dagger} = 0, \qquad \textbf{(13)}$$

where $\underline{X}_L$ is defined in accordance with **(4)**. When we apply **(13)** to the impedance matrix of our transition, we obtain the condition

$$\begin{bmatrix} 1 & 0 \\ 0 & \underline{X} \end{bmatrix}\begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12} \\ \underline{Z}_{21} & \underline{Z}_{22} \end{bmatrix} + \begin{bmatrix} \underline{Z}_{11}^{\dagger} & \underline{Z}_{21}^{\dagger} \\ \underline{Z}_{12}^{\dagger} & \underline{Z}_{22}^{\dagger} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \underline{X}^{\dagger} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \textbf{(14)}$$

which is really the four conditions

$$\begin{bmatrix} \underline{Z}_{11}+\underline{Z}_{11}^{\dagger} & \underline{Z}_{12}+\underline{Z}_{21}^{\dagger}\underline{X}^{\dagger} \\ \underline{X}\underline{Z}_{21}+\underline{Z}_{12}^{\dagger} & \underline{X}\underline{Z}_{22}+\underline{Z}_{22}^{\dagger}\underline{X}^{\dagger} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \textbf{(15)}$$

Using the condition $\underline{Z}_{11}+\underline{Z}_{11}^{\dagger}=0$ from **(15)**, we can now write the term $\underline{Z}_p+\underline{Z}_p^{\dagger}$ in **(12)** as $\underline{Z}_p+\underline{Z}_p^{\dagger}=(\underline{Z}_p+\underline{Z}_{11})+(\underline{Z}_p^{\dagger}+\underline{Z}_{11}^{\dagger})$. Substitution into **(12)** gives

$$\overline{\underline{\hat{v}}_w\, \underline{\hat{v}}_w^{\dagger}} = 2\frac{hf}{e^{hf/kT}-1}\left[\underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\,\underline{Z}_{21}^{\dagger} + \left(\underline{Z}_{21}(\underline{Z}_p+\underline{Z}_{11})^{-1}\,\underline{Z}_{21}^{\dagger}\right)^{\dagger}\right]. \qquad \textbf{(16)}$$

Using the condition $\underline{Z}_{12} + \underline{Z}_{21}{}^{\dagger}\underline{X}^{\dagger} = 0$ from **(15)**, we can now write **(16)** as

$$\overline{\underline{\hat{v}}_w \, \underline{\hat{v}}_w{}^{\dagger}} = -2 \frac{hf}{e^{hf/kT} - 1} \left[\underline{Z}_{21} (\underline{Z}_p + \underline{Z}_{11})^{-1} \, \underline{Z}_{12} (\underline{X}^{\dagger})^{-1} + \left(\underline{Z}_{21} (\underline{Z}_p + \underline{Z}_{11})^{-1} \, \underline{Z}_{12} (\underline{X}^{\dagger})^{-1} \right)^{\dagger} \right]. \qquad \textbf{(17)}$$

We can write $\underline{Z}_p$ in terms of $\underline{Z}_w$ by using the fact that $\underline{Z}_w$ is defined by the relation $\underline{v}_w = \underline{Z}_w\,(-\underline{i}_w)$ when $\underline{\hat{v}}_w$ and $\underline{\hat{v}}_p$ are set to 0. Then equation **(6)** gives $\underline{v}_p = -\underline{Z}_p\,\underline{i}_p$, while **(8)** gives $\underline{v}_p = \underline{Z}_{11}\,\underline{i}_p - \underline{Z}_{12}\,\underline{i}_w$. Combining the two gives $\underline{i}_p = (\underline{Z}_p + \underline{Z}_{11})^{-1}\,\underline{Z}_{12}\,\underline{i}_w$. Substitution into **(8)** yields $\underline{v}_w = \underline{Z}_{21}\,(\underline{Z}_p + \underline{Z}_{11})^{-1}\,\underline{Z}_{12}\,\underline{i}_w - \underline{Z}_{22}\,\underline{i}_w$, which leads us to deduce the expression for $\underline{Z}_w$ in terms of $\underline{Z}_p$:

$$\underline{Z}_w = \underline{Z}_{22} - \underline{Z}_{21}\,(\underline{Z}_p + \underline{Z}_{11})^{-1}\,\underline{Z}_{12}. \qquad \textbf{(18)}$$

This expression can be used to replace the terms $\underline{Z}_{21}\,(\underline{Z}_p + \underline{Z}_{11})^{-1}\,\underline{Z}_{12}$ in **(17)** with $-(\underline{Z}_w - \underline{Z}_{22})$:

$$\overline{\underline{\hat{v}}_w \, \underline{\hat{v}}_w{}^{\dagger}} = 2 \frac{hf}{e^{hf/kT} - 1} \left[(\underline{Z}_w - \underline{Z}_{22})(\underline{X}^{\dagger})^{-1} + \left((\underline{Z}_w - \underline{Z}_{22})(\underline{X}^{\dagger})^{-1} \right)^{\dagger} \right]. \qquad \textbf{(19)}$$

Now the condition $\underline{X}\underline{Z}_{22} + \underline{Z}_{22}{}^{\dagger}\underline{X}^{\dagger} = 0$ from **(15)** eliminates the terms involving $\underline{Z}_{22}$ in **(19)**, giving the desired result

$$\overline{\underline{\hat{v}}_w \, \underline{\hat{v}}_w{}^{\dagger}} = 2 \frac{hf}{e^{hf/kT} - 1} \left[\underline{Z}_w (\underline{X}^{\dagger})^{-1} + \left(\underline{Z}_w (\underline{X}^{\dagger})^{-1} \right)^{\dagger} \right]. \qquad \textbf{(20)}$$

This is a concise expression for the modal Thevenin-equivalent voltage sources describing the noise of a passive circuit embedded in the lossy waveguide. Appendix 2 gives the Norton-equivalent current and scattering-parameter forms of **(20)**.

DISCUSSION

Since we placed no restrictions on the embedded network other than it be passive and no restriction on the intervening transmission lines and transition other than that they are lossless, relation **(20)** is a very general result that must be satisfied by any passive network embedded in the lossy waveguide. There are a number of interesting applications and special cases.

Lossless Networks: Substitution of the lossless condition **(13)**, derived in Appendix 1, into **(20)** shows that $\overline{\underline{\hat{v}}_w \underline{\hat{v}}_w^{\dagger}} = 0$ for lossless networks.

Multi-port Networks: Since no restriction was placed on the lossy waveguide except that it be closed and be constructed only of isotropic materials, we can replace the single lossy waveguide with multiple lossy guides, as illustrated in Figure 2. Equation **(20)** is still applicable, except that $\underline{X}$ is given by the block-diagonal matrix

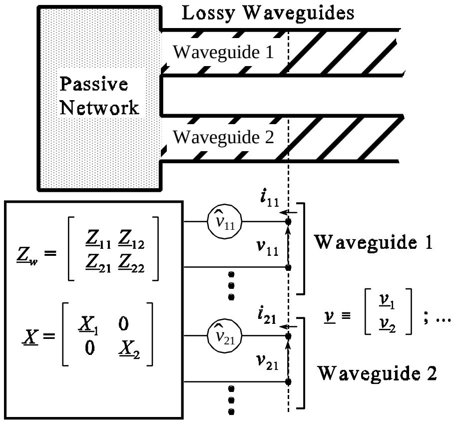

Figure 2. Extension to multiport network. Only two guides are shown for clarity.

$$\underline{X} \equiv \begin{bmatrix} \underline{X}_1 & 0 & \cdots \\ 0 & \underline{X}_2 & \\ \vdots & & \ddots \end{bmatrix} \qquad \textbf{(21)}$$

and $\underline{\hat{v}}_w$ by the vector

$$\underline{\hat{v}}_w \equiv \begin{bmatrix} \underline{\hat{v}}_{w1} \\ \underline{\hat{v}}_{w2} \\ \vdots \end{bmatrix}, \qquad \textbf{(22)}$$

where $\underline{\hat{v}}_{wn}$ and $\underline{X}_n$ refer to the source vector and cross-power matrix of the nth guide, respectively.

Alternate Form: The general waveguide theory of [8] determines the symmetry of the impedance matrix of a waveguide junction composed only of passive reciprocal materials. The result is that the impedance matrix of these waveguide junctions satisfy $\underline{Z}^t = \underline{W}\,\underline{Z}\,\underline{W}^{-1}$, where superscript "$t$" indicates the transpose, $\underline{W} \equiv \mathrm{diag}\left(\int \boldsymbol{e}_{tn} \times \boldsymbol{h}_{tn} \cdot \boldsymbol{z}\,\mathrm{d}S \;/\; v_{0n} i_{0n}\right) \equiv \mathrm{diag}\left((v_{0n}^{*}/v_{0n})K_n\right)$ is the diagonal reciprocity matrix, and the K_n are the reciprocity factors of [12] for each mode. Appendix 3 shows that these reciprocity factors are related to the cross-power matrix $\underline{X}$ through $(\underline{X}^{\dagger})^{-1} = \underline{W}^{-1}\,\underline{X}^{t}\,(\underline{W}^{-1})^{\dagger}$. Thus **(20)** may be written as

$$\overline{\underline{\hat{v}}_w \, \underline{\hat{v}}_w^{\dagger}} = 2 \frac{hf}{e^{hf/kT}-1} \left[\underline{Z}_w \, \underline{W}^{-1} \, \underline{X}^t \, (\underline{W}^{-1})^{\dagger} + \left(\underline{Z}_w \, \underline{W}^{-1} \, \underline{X}^t \, (\underline{W}^{-1})^{\dagger} \right)^{\dagger} \right] . \quad \textbf{(23)}$$

Dominant Modes: When the first N modes contain at least all of the dominant ones and the circuit is embedded in a length of waveguide sufficiently long to damp out all of the modes except the dominant ones, then $\underline{Z}_w$ takes the form

$$\underline{Z}_w = \begin{bmatrix} \underline{Z}_{w\mathrm{d}} & 0 \\ 0 & \underline{Z}_{w0} \end{bmatrix} , \quad \textbf{(24)}$$

where $\underline{Z}_{w\mathrm{d}}$ is an N by N matrix and $\underline{Z}_{w0}$ is a diagonal matrix containing the characteristic impedances of all but the first N modes. Now **(20)** gives

$$\overline{\underline{\hat{v}}_{w\mathrm{d}} \, \underline{\hat{v}}_{w\mathrm{d}}^{\dagger}} = 2 \frac{hf}{e^{hf/kT}-1} \left[\underline{Z}_{w\mathrm{d}} Q + \left(\underline{Z}_{w\mathrm{d}} Q \right)^{\dagger} \right] , \quad \textbf{(25)}$$

where $\underline{\hat{v}}_{w\mathrm{d}}$ is the subvector of $\underline{\hat{v}}_w$ containing its first N elements and Q is the upper left-hand N by N submatrix of $(\underline{X}^{\dagger})^{-1}$. Equation **(23)** and appendix 3 show that $Q = \underline{W}_{\mathrm{d}}^{-1} \, \underline{X}_{\mathrm{d}}^{t} \, (\underline{W}_{\mathrm{d}}^{-1})^{\dagger}$, where $\underline{X}_{\mathrm{d}}$ and $\underline{W}_{\mathrm{d}}$ are the upper left-hand N by N submatrices of $\underline{X}$ and $\underline{W}$, respectively. This last relation is useful when not all of the elements of $\underline{X}$, which is generally infinite in dimension, are known.

Power-Normalized Conductor Representation: The “conductor” voltages and currents of [13] are linear transformations of the modal voltages and currents $\underline{v}_w$ and $\underline{i}_w$. By analogy with [13] we define the “power-normalized” conductor voltages $\underline{v}_c$ and currents $\underline{i}_c$ by

$$\underline{v}_c \equiv \underline{M}_v \, \underline{v}_w \, ; \quad \underline{i}_c \equiv \underline{M}_i \, \underline{i}_w \, , \quad \textbf{(26)}$$

where $\underline{v}_c$ and $\underline{i}_c$ are generally infinite in dimension and $\underline{M}_v$ and $\underline{M}_i$ are invertible and satisfy $\underline{M}_i^{\dagger} \, \underline{M}_v = \underline{X}$: this latter restriction ensures that the total power is given by $p = \underline{i}_c^{\dagger} \, \underline{v}_c$. Equation **(20)** becomes [6]

$$\overline{\underline{\hat{v}}_c \, \underline{\hat{v}}_c^{\dagger}} = 2 \frac{hf}{e^{hf/kT}-1} \left[\underline{Z}_c + \underline{Z}_c^{\dagger} \right] \quad \textbf{(27)}$$

in this representation, where $\underline{Z}_c$, the impedance matrix in the conductor representation, is defined by

$$\underline{Z}_c \equiv \underline{M}_v \underline{Z}_w \underline{M}_i^{-1}.$$

ILLUSTRATION

In low-loss circular, rectangular, and coaxial waveguides the off-diagonal elements of the cross-power matrix $\underline{X}$ linking the dominant waveguide mode to other modes in the guide are generally small except at frequencies where the modes are nearly degenerate (i.e. when their propagation constants are nearly equal). At these frequencies the modes couple and the field patterns of each of the lossy coupled modes can be represented to first order as linear combinations of the field patterns of lossless uncoupled modal solutions [9], [14]. While this results in large off-diagonal elements of $\underline{X}$, this coupling phenomena is limited to narrow bands of frequencies at or above the upper frequency limit of the guide, and so may usually be ignored in practice.

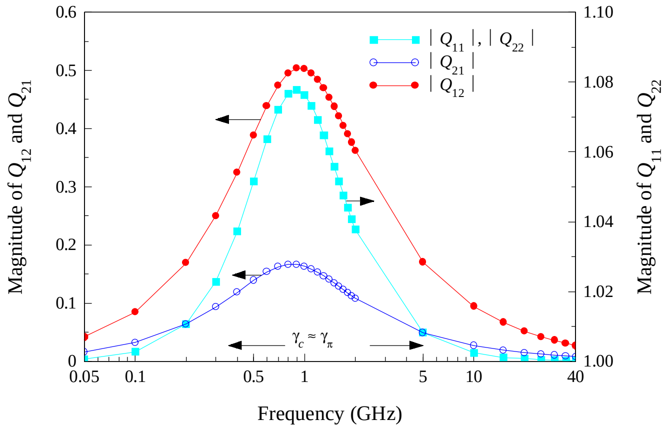

Figure 4. The magnitudes of the elements of the matrix Q for the coupled lines of Figure 3. The frequencies where the imaginary parts of γ_c and γ_π cross and the quantity $|\gamma_c\text{-}\gamma_\pi|/\beta_0$ reaches a broad minimum define the frequency range labeled $\gamma_c \approx \gamma_\pi$ in the figure.

However large off-diagonal elements of $\underline{X}$ linking dominant modes often do occur in multiconductor transmission lines over broad ranges of useful frequencies. The lossy asymmetric coupled microstrip lines of Figure 3 illustrate this phenomena. This transmission line structure supports two quasi-TEM dominant modes, commonly referred to as the c and π modes, which correspond to the even and the odd mode of the symmetric case, respectively. The propagation constants of the c and π modes of the structure of Figure 3 become close in the frequency range 300 MHz - 5 GHz. While the low-loss assumptions of [9] and [14] are not met by this high-loss guide, our calculations based on the full-wave method of Heinrich [15] show that this near degeneracy is accompanied by large off-diagonal elements of $\underline{X}$.

Since the c and π modes are the dominant ones the impedance of a termination embedded in a sufficiently

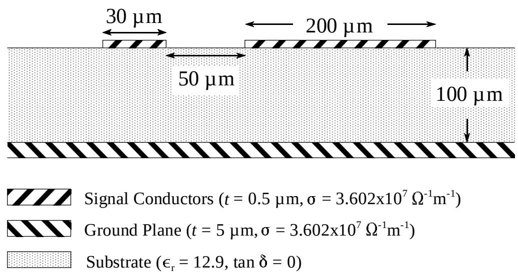

Figure 3. Asymmetric coupled microstrip lines on a lossless substrate.

long length of line will take the form **(24)**, where $\underline{Z}_{wd}$ is the two-by-two c-mode/π-mode impedance matrix. We can calculate the two-by-two matrix $\overline{\underline{\hat{v}}_{wd}\ \underline{\hat{v}}_{wd}^{\dagger}}$ of c-mode and π-mode Thevenin-equivalent sources from $\underline{Z}_{wd}$ and Q using equation **(25)**.

Figure 4 plots magnitudes of the elements of the matrix Q calculated with the method of [15]. It shows that Q differs significantly from the identity matrix in the region where the modes couple; the conventional formulation, in which Q is absent in the expression relating $\overline{\underline{\hat{v}}_{wd}\ \underline{\hat{v}}_{wd}^{\dagger}}$ to $\underline{Z}_{wd}$, will fail there.

While beyond the scope of this work, [13] shows for the case of Figure 3 that the expression for $\overline{\underline{\hat{v}}_{cd}\ \underline{\hat{v}}_{cd}^{\dagger}}$, the Thevenin-equivalent voltage sources in the power-normalized dominant-mode conductor representation, assumes the conventional form $2\ hf/(e^{hf/kT}-1)\ [\underline{Z}_{cd}+\underline{Z}_{cd}^{\dagger}]$, where $\underline{Z}_{cd}$ is the impedance matrix in that representation.

CONCLUSION

We have developed a rigorous representation for the thermal electromagnetic noise of circuits embedded in lossy waveguides based on modal Thevenin-equivalent voltage sources and derived explicit expressions describing the noise generated by passive networks. The results form a firm foundation for the theory of electrical noise in lossy waveguides and show that the spectral densities of the modal Thevenin-equivalent voltage sources depend on the cross-power matrix $\underline{X}$, a result that cannot be predicted directly from Nyquist's theory. We illustrated the results with a practical example in which the off-diagonal elements of $\underline{X}$ are large and the conventional formulation fails.

APPENDIX 1

LOSSLESS CONDITION

The net power P entering a lossless circuit with impedance matrix $\underline{Z}$ is

$$P = \mathrm{Re}(\underline{i}^{\dagger}\underline{X}\ \underline{v}) = \mathrm{Re}(\underline{i}^{\dagger}\underline{X}\ \underline{Z}\ \underline{i}) = \frac{1}{2}(\underline{i}^{\dagger}\underline{X}\ \underline{Z}\ \underline{i}+[\underline{i}^{\dagger}\underline{X}\ \underline{Z}\ \underline{i}]^{*}). \qquad \textbf{(28)}$$

The quantity $\underline{i}^{\dagger}\underline{X}\ \underline{Z}\ \underline{i}$ is a scalar and so is equal to its transpose. Therefore **(28)** is

$$P = \frac{1}{2}\ (\underline{i}^{\dagger}\underline{X}\ \underline{Z}\ \underline{i}+\underline{i}^{\dagger}\underline{Z}^{\dagger}\underline{X}^{\dagger}\underline{i}) = \frac{1}{2}\ \underline{i}^{\dagger}(\underline{X}\ \underline{Z}+\underline{Z}^{\dagger}\underline{X}^{\dagger})\underline{i}. \qquad \textbf{(29)}$$

Since the circuit is lossless, P must equal 0 for all current vectors $\underline{i}$, which implies that lossless networks satisfy the relation $\underline{X}\,\underline{Z} + \underline{Z}^\dagger \underline{X}^\dagger = 0$.

APPENDIX 2

OTHER REPRESENTATIONS OF THERMAL NOISE

The Norton-equivalent current sources $\hat{\underline{i}}_w$, defined analogously to the Thevenin-equivalent voltage sources, satisfy

$$-\underline{i}_w = \underline{Y}_w \underline{v}_w - \hat{\underline{i}}_w, \tag{30}$$

where $\underline{Y}_w = \underline{Z}_w^{-1}$ is the admittance matrix of the circuit. We can relate $\hat{\underline{i}}_w$ to $\hat{\underline{v}}_w$ by

$$\hat{\underline{i}}_w \equiv \underline{i}_w\big|_{\underline{v}_w=0} = \underline{Z}_w^{-1}\, \hat{\underline{v}}_w, \tag{31}$$

so, using **(20)**,

$$\overline{\hat{\underline{i}}_w \hat{\underline{i}}_w^{\dagger}} = \underline{Z}_w^{-1}\, \overline{\hat{\underline{v}}_w \hat{\underline{v}}_w^{\dagger}}\, (\underline{Z}_w^{-1})^\dagger = 2\frac{hf}{e^{hf/kT}-1}\left[\underline{Y}_w \underline{X}^{-1} + \left(\underline{Y}_w \underline{X}^{-1}\right)^\dagger\right]. \tag{32}$$

We can also express **(20)** in terms of the pseudo-wave parameters of [8] in the lossy guide. The pseudo-waves correspond to traveling waves when their reference impedance is set equal to the characteristic impedance of the mode. They correspond to the waves conventionally used in microwave design when their reference impedance is set real. The vectors of forward pseudo-waves $\underline{a}_w$ and backward pseudo-waves $\underline{b}_w$ with reference impedances Z_{ref}^n are related to the voltages and currents by [8]

$$\underline{a}_w = \frac{1}{2}\,\underline{U}\,(\underline{v}_w + \underline{Z}_{\mathrm{ref}}\,\underline{i}_w) \tag{33}$$

and

$$\underline{b}_w = \frac{1}{2}\,\underline{U}\,(\underline{v}_w - \underline{Z}_{\mathrm{ref}}\,\underline{i}_w), \tag{34}$$

where the diagonal matrices $\underline{Z}_{\rm ref}$ and $\underline{U}$ are defined by $\underline{Z}_{\rm ref} = \mathrm{diag}(Z_{\rm ref}{}^{n})$ and

$$\underline{U} \;=\; \mathrm{diag}\left(\frac{|v_{0n}|}{v_{0n}} \, \frac{\sqrt{\mathrm{Re}(Z_{\rm ref}{}^{n})}}{|Z_{\rm ref}{}^{n}|} \right) . \qquad \textbf{(35)}$$

The Thevenin-equivalent voltage is

$$\hat{\underline{v}}_{w} \;=\; \underline{v}_{w}\big|_{\underline{i}_{w}=0} \;=\; \underline{U}^{-1}(\underline{a}_{w}+\underline{b}_{w})\big|_{\underline{a}_{w}=\underline{b}_{w}} \;=\; 2\;\underline{U}^{-1}\;\underline{a}_{w}\big|_{\underline{a}_{w}=\underline{b}_{w}} . \qquad \textbf{(36)}$$

forward wave sources $\hat{\underline{a}}_{w}$ and pseudo-wave reflection coefficient matrix $\underline{\Gamma}_{w}$ are defined by [8]

$$\underline{a}_{w} \;=\; \underline{\Gamma}_{w}\;\underline{b}_{w} + \hat{\underline{a}}_{w} , \qquad \textbf{(37)}$$

which implies that

$$\hat{\underline{a}}_{w} \;=\; \frac{1}{2}\;(1-\underline{\Gamma}_{w})\;\underline{U}\;\hat{\underline{v}}_{w} . \qquad \textbf{(38)}$$

Now we can express $\overline{\hat{\underline{a}}_{w}\,\hat{\underline{a}}_{w}{}^{\dagger}}$ in terms of $\overline{\hat{\underline{v}}_{w}\,\hat{\underline{v}}_{w}{}^{\dagger}}$:

$$\overline{\hat{\underline{a}}_{w}\,\hat{\underline{a}}_{w}{}^{\dagger}} \;=\; \frac{1}{4}\;(1-\underline{\Gamma}_{w})\;\underline{U}\;\overline{\hat{\underline{v}}_{w}\,\hat{\underline{v}}_{w}{}^{\dagger}}\;\underline{U}^{\dagger}\;(1-\underline{\Gamma}_{w})^{\dagger} . \qquad \textbf{(39)}$$

Substituting **(20)** into **(39)** gives

$$\overline{\hat{\underline{a}}_{w}\,\hat{\underline{a}}_{w}{}^{\dagger}} \;=\; \frac{1}{2}\;\frac{hf}{e^{hf/kT}-1}\;(1-\underline{\Gamma}_{w})\underline{U}\left[\;\underline{Z}_{w}\;(\underline{X}^{\dagger})^{-1} \;+\; \left(\underline{Z}_{w}\;(\underline{X}^{\dagger})^{-1}\right)^{\dagger}\;\right]\underline{U}^{\dagger}(1-\underline{\Gamma}_{w})^{\dagger} . \qquad \textbf{(40)}$$

The relation $\underline{Z}_{w} = (1-\underline{U}^{-1}\underline{\Gamma}_{w}\;\underline{U})^{-1}(1+\underline{U}^{-1}\underline{\Gamma}_{w}\;\underline{U})\underline{Z}_{\rm ref}$ from Appendix E of [8] shows that **(40)** is

$$\overline{\hat{\underline{a}}_{w}\,\hat{\underline{a}}_{w}{}^{\dagger}} \;=\; \frac{1}{2}\;\frac{hf}{e^{hf/kT}-1}\left[\;(1-\underline{\Gamma}_{w})\;\underline{A}^{\dagger}(1+\underline{\Gamma}_{w})^{\dagger} \;+\; (1+\underline{\Gamma}_{w})\;\underline{A}(1-\underline{\Gamma}_{w})^{\dagger}\;\right] , \qquad \textbf{(41)}$$

where

$$\underline{A} \equiv \underline{Z}_{\text{ref}}\, \underline{U}\, (\underline{X}^{\dagger})^{-1}\, \underline{U}^{\dagger}. \tag{42}$$

As explained in the text, these results are also applicable to multiport networks. In that case, the pseudo-wave scattering-parameter matrix replaces $\underline{\Gamma}_w$ in **(41)**.

APPENDIX 3

RELATIONS BETWEEN $\underline{X}$ AND $\underline{W}$

Figure 5 shows the abrupt connection of a lossless line to a lossy waveguide. The transition is again defined to begin at z=-l and to terminate at z=0 and contains only lossless and reciprocal materials. If we account for all of the modes at the two reference planes at z=-l and z=0, we can say, from the preceding arguments, that its impedance matrix satisfies **(15)**. Since the materials comprising the transition are reciprocal, we can also apply the condition $\underline{Z}^t = \underline{W}\,\underline{Z}\,\underline{W}^{-1}$ of [8] and [12], where $\underline{W} \equiv \mathrm{diag}\left(\int \boldsymbol{e}_{wtm} \times \boldsymbol{h}_{wtm} \cdot \boldsymbol{z}\, \mathrm{d}S \;/\; v_{w0m} i_{w0m}\right)$, to its impedance matrix **(8)**. The result is

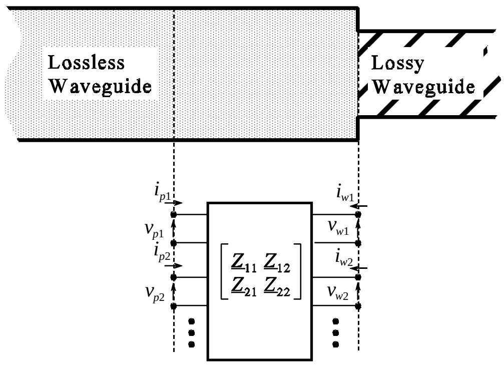

Figure 5. Abrupt connection of lossless transmission line to a lossy waveguide.

$$\begin{bmatrix} \underline{Z}_{11}^{\;t} & \underline{Z}_{21}^{\;t} \\ \underline{Z}_{12}^{\;t} & \underline{Z}_{22}^{\;t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \underline{W} \end{bmatrix} \begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12} \\ \underline{Z}_{21} & \underline{Z}_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \underline{W}^{-1} \end{bmatrix} = \begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12}\underline{W}^{-1} \\ \underline{W}\,\underline{Z}_{21} & \underline{W}\,\underline{Z}_{22}\underline{W}^{-1} \end{bmatrix}, \tag{43}$$

where $\underline{W}$ is the reciprocity matrix for the lossy waveguide (the reciprocity matrix for the lossless guide is the identity matrix [12]).

Now, combing the lower-left conditions of **(15)** and **(43)** gives

$$\underline{Z}_{12}^{\;\dagger} = -\underline{X}\,\underline{Z}_{21} = -\underline{X}\,(\underline{W}^{-1}\underline{Z}_{12}^{\;t}). \tag{44}$$

The upper-right condition of **(15)** gives

$$\underline{Z}_{12}{}^{t} = -\underline{X}^{*}\,\underline{Z}_{21}{}^{*}, \tag{45}$$

while the upper-right condition of **(43)** gives

$$\underline{Z}_{12}{}^{\dagger} = \underline{W}^{\dagger}\,\underline{Z}_{21}{}^{*}. \tag{46}$$

Substitution of these two results into **(44)** gives

$$\underline{W}^{\dagger}\,\underline{Z}_{21}{}^{*} = -\underline{X}\,\underline{W}^{-1}\,\underline{X}^{*}\,\underline{Z}_{21}{}^{*}. \tag{47}$$

Since equation **(47)** is true for the connection of *any* lossless guide to the lossy guide, we must in general have

$$\underline{W}^{\dagger} = -\underline{X}\,\underline{W}^{-1}\,\underline{X}^{*}, \tag{48}$$

which in turn implies

$$(\underline{X}^{\dagger})^{-1} = -\underline{W}^{-1}\,\underline{X}^{t}\,(\underline{W}^{\dagger})^{-1}. \tag{49}$$

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REFERENCES

[1] H. Nyquist, “Thermal agitation of electric charge in conductors,” *Physical Review*, vol. 32, pp. 110-113, July 1928.

[2] J. B. Johnson, “Thermal agitation of electricity in conductors,” *Physical Review*, vol. 32, pp. 97-109, July 1928.

[3] F. Reif, *Fundamentals of Statistics and Thermal Physics*, McGraw-Hill, New York, 1965.

[4] B. M. Oliver, “Thermal and quantum noise,” *Proc. IEEE*, vol. 53, pp. 436-454, May 1965.

[5] E. J. Schremp, MIT Rad. Lab. Series 18, Valley and Wallman, *Vacuum Tube Amplifiers*, McGraw-Hill, Inc., NY, 1949, chapter 12.

[6] R. Q. Twiss, "Nyquist's and Thevenin's theorems generalized for nonreciprocal linear networks," *J. Applied Phys.*, vol. 26, no. 5, pp. 599-602, May 1955.

[7] H. Bosma, "On the theory of linear noisy systems," *Phillips Res. Repts. Suppl.*, No. 10, 1967.

[8] R. B. Marks and D. F. Williams, "A general waveguide circuit theory," *J. Res. Natl. Inst. Stand. Technol.*, vol. 97, pp. 533-561, Sept.-Oct. 1992.

[9] R. E. Collin, *Field Theory of Guided Waves*, McGraw-Hill, New York, 1960.

[10] J. R. Brews, "Transmission line models for lossy waveguide interconnections in VLSI," *IEEE Trans. Electron Dev.*, pp. 1356-1365, 1986.

[11] H. A. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks*, John Wiley and Sons, Inc., New York, 1959.

[12] D. F. Williams and R. B. Marks, "Reciprocity Relations in Waveguide Junctions," *IEEE Trans. Microwave Theory and Tech.*, vol. 41, pp. 1105-1110, July 1993.

[13] D. F. Williams, L. A. Hayden, and R. B. Marks, "A multimode equivalent-circuit theory for electrical design," submitted to *IEEE Trans. Microwave Theory and Tech.*

[14] V. M. Papadopoulos, "Propagation of electromagnetic waves in cylindrical wave-guides with imperfectly conducting walls," *Quart. Journ. Mech. and Applied Math.*, vol. VII, pp. 325-334, September 1954.

[15] W. Heinrich, "Full-wave analysis of conductor losses on MMIC transmission lines," *IEEE Trans. on Microwave Theory and Tech.*, pp. 1468-1472, Oct. 1990.