

Causality and Waveguide Circuit Theory

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***Abstract-* We develop a new causal power-normalized waveguide equivalent-circuit theory that, unlike its predecessors, results in network parameters usable in both the frequency and time domains in a broad class of waveguides. Enforcing simultaneity of the voltages, currents, and fields and a power normalization fixes all of the parameters of the new theory within a single normalization factor, including both the magnitude and phase of the characteristic impedance of the waveguide.**

INTRODUCTION

We develop a causal power-normalized waveguide equivalent-circuit theory. The theory determines voltages, currents, and network parameters suitable for use in both frequency- and time-domain circuit simulations from fields in a single-moded waveguide. The theory maintains the simultaneity of the voltages, currents, and fields inherent in classical waveguide circuit theory but is not restricted to TEM, TE, and TM guides.

Waveguide equivalent-circuit theories prescribe methods for constructing a waveguide voltage v and current i from the electromagnetic fields in uniform waveguides. The intent is to construct v and i so that the electromagnetic problem reduces to a simpler circuit problem that can be solved with conventional circuit simulators.

Classical waveguide equivalent-circuit theories, of which [1] is representative, are based on frequency-independent modal solutions with a constant wave impedance. While we will see that the network parameters of these classic theories satisfy the causality and power-normalization conditions we develop here, they are strictly limited to TEM, TE, and TM waveguides.

The theories of [2] and [3] attempt to eliminate the restriction to TEM, TE, and TM waveguides by adding a power normalization, an approach first suggested by Brews [4]. The power normalizations used in these theories ensure that the real part of the impedance of passive circuits is always positive, a basic requirement for stable circuit simulation.

Nevertheless the waveguide circuit theories of [2] and [3] do not fix all of their parameters uniquely: they require in addition a user-defined integration path to define either the voltage or current. Since they construct v and i independently at each frequency, they also do not explicitly relate the behavior of their parameters in the frequency domain to their behavior in the time domain, and so leave unspecified the temporal properties of v and i . In particular, the voltage and current may not start simultaneously with the electric and magnetic field.

Leaving the temporal properties of v and i unspecified can have serious consequences. For example, the network parameters of passive devices in the circuit theories of [2] and [3] are not constrained to be causal. That is, passive circuits may appear to respond to inputs *before*, rather than after, the input signal reaches the device. This complicates the

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interpretation of the circuits network parameters in the time domain, and renders them unsuitable for use with conventional time-based simulation tools.

This new waveguide circuit theory enforces simultaneity of its voltages and currents with the actual fields in the circuit while eliminating the TEM, TE, and TM restrictions of classical waveguide circuit theories. This simultaneity ensures that the network parameters of passive devices are causal, a necessary condition for stable time-domain simulations. The theory also employs the power-normalization of [2], so that the actual time-averaged power p in the circuit is equal to $\frac{1}{2}vi^*$.¹

The simultaneity and power constraints fix all of the parameters of this new causal circuit theory, including the characteristic impedance Z_0 , within a single positive frequency-independent multiplier that defines the overall impedance normalization. The implications are significant, and some have already been explored in [5]. For example, the use of $Z_0 = 1$ for the TE₁₀ mode of rectangular waveguide, a choice permitted in some waveguide circuit theories (see [2], [3], and chapter 4 of [6]), is not consistent with the causal theory developed here.

VOLTAGE AND CURRENT

We begin with a closed waveguide that is uniform in the axial direction. The waveguide must have only a single dominant mode and be long enough to support only that mode at a reference plane where v and i are defined. We also require that the dominant mode be unique and distinct from any other modes in the system: modes with degeneracies or modes that bifurcate violate this restriction.

¹ The factor of $\frac{1}{2}$ appears in the relation for time-averaged power because the complex magnitude of voltages, currents and fields are defined here as the peak values. The factor of $\frac{1}{2}$ does not appear in [2] because there complex magnitudes are defined in terms of the root-mean-square of the peak values.

In the following development we will refer only to the transverse electric and magnetic field components in the waveguide, for they capture all of the physics required to construct a waveguide circuit theory. However there is no implication that the axial components of the fields vanish; the analysis is not restricted to TEM, TE, or TM modes and the longitudinal fields can always be reconstructed from the transverse fields [2].

We write the transverse electric field $\hat{\mathbf{E}}_t(t, \mathbf{r}, z)$ and magnetic field $\hat{\mathbf{H}}_t(t, \mathbf{r}, z)$ at a given time t , transverse coordinate $\mathbf{r} = (x, y)$, and longitudinal position z in the guide in terms of their frequency-domain representations

$$\hat{\mathbf{E}}_t(t, \mathbf{r}, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}_t(\omega, \mathbf{r}, z) e^{j\omega t} d\omega \quad (1)$$

and

$$\hat{\mathbf{H}}_t(t, \mathbf{r}, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{H}_t(\omega, \mathbf{r}, z) e^{j\omega t} d\omega, \quad (2)$$

where ω is the angular frequency.

We introduce the voltage $v(\omega, z)$ and frequency-dependent normalization $v_0(\omega)$ with

$$\begin{aligned} \mathbf{E}_t(\omega, \mathbf{r}, z) &= [c_+(\omega)e^{-\gamma z} + c_-(\omega)e^{\gamma z}] \mathbf{e}_t(\omega, \mathbf{r}) \\ &\equiv \frac{v(\omega, z)}{v_0(\omega)} \mathbf{e}_t(\omega, \mathbf{r}) \end{aligned} \quad (3)$$

and the current $i(\omega, z)$ and normalization $i_0(\omega)$ with

$$\begin{aligned} \mathbf{H}_t(\omega, \mathbf{r}, z) &= [c_+(\omega)e^{-\gamma z} - c_-(\omega)e^{\gamma z}] \mathbf{h}_t(\omega, \mathbf{r}) \\ &\equiv \frac{i(\omega, z)}{i_0(\omega)} \mathbf{h}_t(\omega, \mathbf{r}), \end{aligned} \quad (4)$$

where \mathbf{e}_t and \mathbf{h}_t are the transverse modal electric and magnetic fields of the single propagating mode, γ is its modal propagation constant, and c_+ and c_- are the amplitudes of the mode in the forward and reverse

directions. The two normalizing factors v_0 and i_0 define v and i in terms of the modal field solution $\{\mathbf{e}_t, \mathbf{h}_t\}$, which has a fixed but unspecified normalization.

The normalized transverse modes are defined by the equations

$$\mathbf{E}_t(\omega, \mathbf{r}) \equiv \frac{\mathbf{e}_t(\omega, \mathbf{r})}{v_0(\omega)}; \quad \mathbf{H}_t(\omega, \mathbf{r}) \equiv \frac{\mathbf{h}_t(\omega, \mathbf{r})}{i_0(\omega)}, \quad (5)$$

which imply $\mathbf{E}_t = v\mathbf{E}_t$ and $\mathbf{H}_t = i\mathbf{H}_t$. The power normalization is achieved with the constraint

$$v_0(\omega)i_0^*(\omega) = p_0(\omega) \equiv \int \mathbf{e}_t(\omega, \mathbf{r}) \times \mathbf{h}_t^*(\omega, \mathbf{r}) \cdot \mathbf{z} \, d\mathbf{r}, \quad (6)$$

where the integral in (6) is over the entire guide cross section. This normalization implies that $\int \mathbf{E}_t(\omega, \mathbf{r}) \times \mathbf{H}_t^*(\omega, \mathbf{r}) \cdot \mathbf{z} \, d\mathbf{r} = 1$, and ensures that the total time-averaged power p is given by $p \equiv \frac{1}{2} \int \mathbf{E}_t(\omega, \mathbf{r}) \times \mathbf{H}_t^*(\omega, \mathbf{r}) \cdot \mathbf{z} \, d\mathbf{r} = \frac{1}{2} v i^*$.

We say that a function $\hat{F}(t, \mathbf{r})$ starts at a time t_0 if $\hat{F}(t, \mathbf{r}) = 0$ for $t < t_0$ and is nonvanishing at some \mathbf{r} starting at $t = t_0$. Equations (1)-(5) imply that

$$\begin{aligned} \hat{\mathbf{E}}_t(t, \mathbf{r}, z) &\equiv \hat{v}(t, z) \otimes \hat{\mathbf{E}}_t(t, \mathbf{r}) \\ \hat{\mathbf{H}}_t(t, \mathbf{r}, z) &\equiv \hat{i}(t, z) \otimes \hat{\mathbf{H}}_t(t, \mathbf{r}), \end{aligned} \quad (7)$$

where Fourier transformation gives the temporal voltage

$$\hat{v}(t, z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} v(\omega, z) e^{j\omega t} \, d\omega \quad (8)$$

and the temporal current

$$\hat{i}(t, z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} i(\omega, z) e^{j\omega t} \, d\omega, \quad (9)$$

and \otimes represents convolution with respect to the time t . We observe that if $\hat{\mathbf{E}}_t$ and $\hat{\mathbf{H}}_t$ start at $t = 0$, then the temporal voltage \hat{v} starts simultaneously with $\hat{\mathbf{E}}_t$, and the temporal current \hat{i} starts simultaneously with $\hat{\mathbf{H}}_t$.

In what follows we will present a prescription for determining v_0 and i_0 consistent with the power normalization (6) such that $\hat{\mathbf{E}}_t$ and $\hat{\mathbf{H}}_t$ start at $t = 0$. These latter constraints ensure simultaneity of \hat{v} and $\hat{\mathbf{E}}_t$ and of \hat{i} and $\hat{\mathbf{H}}_t$.

TEM, TE, AND TM GUIDES

Construction of causal v_0 and i_0 that satisfy the power normalization (6) is straightforward in TEM, TE, and TM guides. In those guides there exists a unique wave impedance $Z_w(\omega)$ and the modal fields can be written as

$$\mathbf{e}_t(\omega, \mathbf{r}) = \mathbf{f}(\mathbf{r}); \quad \mathbf{h}_t(\omega, \mathbf{r}) = \frac{\mathbf{z} \times \mathbf{f}(\mathbf{r})}{Z_w(\omega)}, \quad (10)$$

where $\mathbf{f}(\mathbf{r})$ is real. Without loss of generality, we can set $\int |\mathbf{f}(\mathbf{r})|^2 \, d\mathbf{r} = 1$.

With this normalization, $p_0 = 1/Z_w^*$, and $v_0 = \lambda$ and $i_0 = 1/(\lambda Z_w)$, where λ is a positive constant multiplier, satisfy the power normalization (6). We also have $\hat{\mathbf{E}}_t(t, \mathbf{r}) = \lambda^{-1} \delta(t) \mathbf{f}(\mathbf{r})$ and $\hat{\mathbf{H}}_t(t, \mathbf{r}) = \lambda \delta(t) \mathbf{z} \times \mathbf{f}(\mathbf{r})$, where δ is the Dirac delta function, so we see that \hat{v} starts simultaneously with $\hat{\mathbf{E}}_t$, and \hat{i} starts simultaneously with $\hat{\mathbf{H}}_t$.

If we choose $\lambda=1$, then $\mathbf{E}_t = v \mathbf{e}_t$ and $\mathbf{H}_t = i Z_w \mathbf{h}_t = i \mathbf{z} \times \mathbf{e}_t$, and we see that v and i correspond to the voltages and currents of the classic theory. Thus in both the classic waveguide circuit theory and our causal generalization of that theory, the voltage and current start simultaneously with the electric and magnetic field, and the characteristic impedance Z_0 is proportional to the wave impedance of TEM, TM, and TE modes. Other choices, such as setting $|Z_0| = 1$ in lossless rectangular waveguide, which is allowed in [2], [3], and [6], will not be consistent with the classic waveguide circuit theory or with our causal generalization.

Appendix 1 constructs a normalizing voltage v_0' such that the time-domain voltage \hat{v}' associated with it and the electric field start at the same time when the modal electric field \mathbf{e}_t is separable and can be expanded as a finite sum

$$\mathbf{e}_t(\omega, \mathbf{r}) = \sum_{m=1}^n c_m(\omega) \mathbf{f}_m(\mathbf{r}), \quad (11)$$

where the $c_m(\omega)$ are rational functions of ω and the $\mathbf{f}_m(\mathbf{r})$ are real vector functions that satisfy the orthogonality condition

$$\int \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{f}_j(\mathbf{r}) \, d\mathbf{r} \equiv \begin{cases} 1 & \text{if } m=j \\ 0 & \text{if } m \neq j \end{cases}. \quad (12)$$

A similar argument shows that when \mathbf{h}_t separates in this way we can construct a normalizing current i_0' such that the time-domain current \hat{i}' associated with it and the magnetic field start at the same time.

We now apply a Wiener-Hopf decomposition [7] to the function $f \equiv p_0/v_0' i_0'^*$. Define the auxiliary function G from $\arg(G) = \arg(f)$ and $\mathcal{H}(\ln|G|) = \arg(f)$, where \mathcal{H} is the Hilbert transform. (See Appendix 2 for a discussion of minimum-phase functions and the Hilbert transform \mathcal{H} .) Then we have $\mathcal{H}(\ln|G|) = \arg(G) = \arg(f)$, with G minimum phase by construction.

Then define K_v and K_i from $\ln|K_v| \equiv \frac{1}{2}(\ln|f| + \ln|G|)$, $\ln|K_i| \equiv \frac{1}{2}(\ln|f| - \ln|G|)$, $\arg(K_v) \equiv \mathcal{H}(\ln|K_v|)$, and $\arg(K_i) \equiv \mathcal{H}(\ln|K_i|)$, and v_0 and i_0 from $v_0 \equiv \lambda K_v v_0'$ and $i_0 \equiv K_i i_0'/\lambda$, where λ is a positive constant multiplier.

K_v and K_i are minimum phase by construction, so \hat{v} starts simultaneously with \hat{v}' , and therefore with $\hat{\mathbf{E}}_t$, and \hat{i} starts simultaneously with \hat{i}' , and therefore with $\hat{\mathbf{H}}_t$. Furthermore, $v_0 i_0^* = K_v K_i^* p_0/f$, $|v_0 i_0^*| = |p_0|$, and $\arg(v_0 i_0^*) = \arg(K_v) - \arg(K_i) + \arg(p_0) - \arg(f) = \arg(p_0)$, so v_0 and i_0 satisfy the power normalization constraint (6).

For TEM, TM, and TE modes our causal theory reduces to the classic circuit theory. These modes include a number of useful idealizations for which explicit expressions for the modal fields and wave impedance are available. In treating these modes we have placed no restriction on the form of the wave impedance Z_w : it need not have a rational approximation.

We used the construction algorithm of Appendix 1 to overcome the TEM, TE, and TM restriction of the classic circuit theory. That algorithm requires that the c_m in (11) be rational functions of ω . Nevertheless the form of (11) is general enough to represent any piecewise continuous modal field up to any finite frequency to any desired accuracy. This is because rational functions are sufficient to approximate to arbitrary precision any function analytic in a half-plane and either regular at infinity or possessing an isolated pole there [7]. Thus we can approximate the wave solutions in guides constructed entirely of materials with finite loss as accurately as we wish with this expansion, and we see that it is not overly restrictive in practice. However we may not be able to treat some lossless idealizations that are neither TEM, TM, nor TE with the two approaches suggested here.

CHARACTERISTIC IMPEDANCE

Figure 1 shows a source connected to an infinite waveguide with the reference plane chosen far enough away from the source to satisfy the single-mode restrictions of this theory. Since only the forward mode is present $c_- = 0$,

$$\begin{aligned} v_+(\omega, z) &= v_0(\omega) c_+(\omega) e^{-\gamma z}, \\ i_+(\omega, z) &= i_0(\omega) c_+(\omega) e^{-\gamma z}, \end{aligned} \quad (13)$$

and so

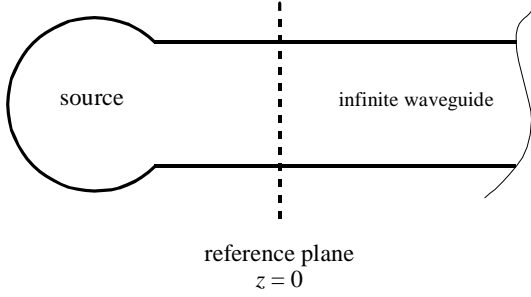


Fig. 1. A source connected to an infinite waveguide.

$$Z_0(\omega) \equiv \left. \frac{v(\omega, z)}{i(\omega, z)} \right|_{z=0} = \frac{v_0(\omega)}{i_0(\omega)}, \quad (14)$$

which is indeed independent of z . Thus

$$\arg(Z_0(\omega)) = \arg\left(\frac{v_0(\omega)}{i_0(\omega)}\right) = \arg(v_0(\omega)i_0^*(\omega)) \quad (15)$$

is fixed by the power normalization (6) [2].

Maxwell's equations imply that, when only the forward mode is present, \hat{E}_t and \hat{H}_t arrive simultaneously (see Appendix 3), so \hat{v} and \hat{i} must as well. Thus in our causal theory $Z_0 = v/i$ must be a minimum phase function, and

$$\mathcal{H}[\ln(|Z_0(\omega)|)] = \arg[Z_0(\omega)]. \quad (16)$$

This fixes Z_0 within a positive scalar multiplier, which we determine when we choose λ . As a corollary, if the guide has a unique wave impedance, that wave impedance will be minimum phase.

UNIQUENESS

The causal power-normalization is unique. Imagine that there are two possible voltage normalizations v_{01} and v_{02} in the theory. We require that for any excitation

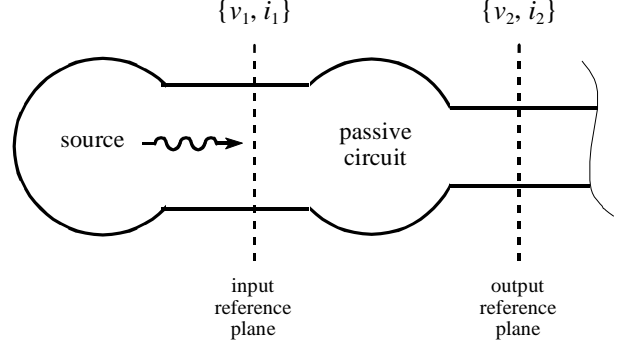


Fig. 2. A source exciting a passive circuit connected to another waveguide.

in the guide the temporal voltage \hat{v} must start simultaneously with the electric field, so the two temporal voltages \hat{v}_1 and \hat{v}_2 associated with v_{01} and v_{02} will always start at the same time as well.

Simultaneous starting times of \hat{v}_1 and \hat{v}_2 for any excitation implies that $v_1/v_2 = v_{01}/v_{02}$ is minimum phase, so $\arg(v_{01}/v_{02}) = \mathcal{H}[\ln(v_{01}/v_{02})]$. However, once we have chosen the constant multiplier λ , and thus fixed Z_0 , we must also have $|v_0|^2 = |v_0(p_0/i_0^*)| = |Z_0 p_0|$, which implies that $|v_{01}/v_{02}| = 1$, so $\arg(v_{01}/v_{02}) = 0$, and $v_{01} = v_{02}$. A similar argument shows that i_0 is unique.

CAUSALITY CONDITION

Consider the passive circuit of Fig. 2. It connects an input waveguide with voltages and currents v_1 and i_1 at the reference plane on the left far enough from the source and circuit to satisfy the single-mode assumption of this theory to an output waveguide with voltages and currents v_2 and i_2 at the reference plane on the right, again far enough from the circuit to satisfy our single-mode assumption.

If the voltage or current at the output were to start before the voltage and current at the input, the fields at the output would have to have started before the fields at the input. This is clearly not possible, so we conclude that the voltage at the output always starts after the

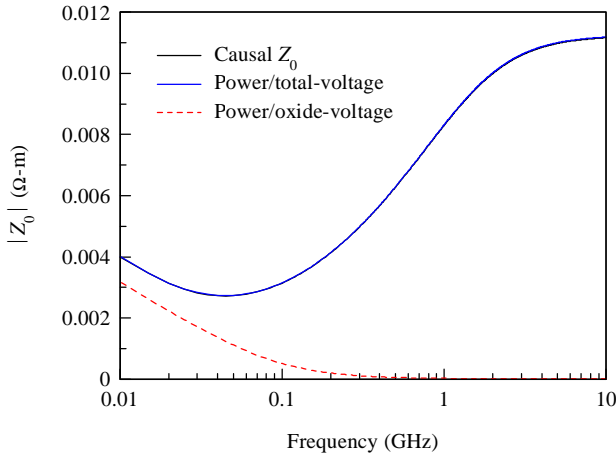


Fig. 3. $|Z_0|$ for the metal-insulator-semiconductor transmission line of [9]. The two solid curves are so close as to be indistinguishable.

voltage at the input. This shows that transfer functions such as Z_{21} that determine voltage or current at the output from voltage or current at the input are causal. A similar argument shows that the “driving-point” impedances [8] of the system are minimum phase, and that our voltages and currents cannot propagate faster than the speed of light.

In essence, by enforcing simultaneity in our causal circuit theory, the causal properties of the actual circuits are preserved as well. This is significant because causal network parameters are a basic requirement for stable time-domain circuit simulation.

MIS TRANSMISSION LINE

Metal-insulator-semiconductor (MIS) transmission lines are neither TEM, TE, nor TM. The theories of [2] and [3] suggest combining either a voltage normalization

$$v_0 = - \int_{\text{path}} \mathbf{e}_t \cdot d\mathbf{l} \quad (17)$$

or a current normalization

$$i_0 = \oint_{\text{closed path}} \mathbf{h}_t \cdot d\mathbf{l}. \quad (18)$$

with the power constraint of (6) to construct v and i . However, different choices of voltage and current paths in (17) and (18) result in different characteristic impedances. Now we will show that not all of these choices are consistent with our causal circuit theory.

Figure 3 compares three characteristic impedances for the TM_{01} mode of the infinitely wide MIS line investigated in [9]. This MIS line consists of a $1.0 \mu\text{m}$ thick metal signal plane with a conductivity of $3 \times 10^7 \text{ S/m}$ separated from the $100 \mu\text{m}$ thick $100 \Omega\text{-cm}$ silicon supporting substrate by a $1.0 \mu\text{m}$ thick oxide with conductivity of 10^{-3} S/m . The ground conductor on the back of the silicon substrate is infinitely thin and perfectly conducting.

The two solid curves in Fig. 3, which are labeled “Causal Z_0 ” and “Power/total-voltage,” agree so closely as to be indistinguishable on the graph. The curve “Causal Z_0 ” is the magnitude of the characteristic impedance determined from the phase of p_0 and the minimum phase properties of Z_0 . The curve “Power/total-voltage” is the magnitude of the characteristic impedance defined with a power-voltage definition. Here the power normalization is based on (6) (the integral of the Poynting vector over the guide cross section) and the voltage normalization of (17), where the path in (17) begins at the ground on the back of the silicon substrate and terminates on the conductor metal on top of the oxide.

The conventional theories of [2] and [3] do not specify the voltage path uniquely, and the choice is not obvious. For example, devices embedded in MIS lines are fabricated on the silicon surface; they are connected to the signal line with vias through the oxide and to the ground with ohmic contacts at the silicon surface. This suggests that a voltage path in the MIS line from the silicon surface through the oxide to the signal line, which is equally consistent with the conventional theories, might correspond more closely to the actual voltage seen by the device than the total voltage across the MIS line.

However, Fig. 3 shows that the characteristic impedance defined from the power constraint of (6) and the voltage across the oxide, which is labeled “Power/oxide-voltage,” differs significantly from the characteristic impedance required by the causal theory presented here.

Figure 4 shows the Fourier transform of the characteristic impedance defined with the voltage path through the oxide and illustrates the difficulty with this definition: the guide will respond to input signals before the excitation reaches it.

This example illustrates an important contribution of the causal theory presented here: it replaces the subjective and sometimes misleading “common-sense” criteria for defining Z_0 in guides that are neither TEM, TE, or TM with a clear and unambiguous procedure that guarantees causal responses. This new approach should be especially useful in complex transmission structures where the choice of voltage and current paths is not intuitively obvious.

ERROR IN $|Z_0|$

This causal circuit theory determines $|Z_0|$ from the phase of p_0 through a Hilbert transform relationship. Evaluating the Hilbert transform requires integrating over all frequencies. Ignorance of the phase of p_0 at frequencies above those at which the theory is to be applied will result in errors in the $|Z_0|$ at the frequencies where the theory is applied.

Appendix 4 develops a bound for the error in $|Z_0|$ at a given frequency ω when the $\arg(p_0)$ is known exactly up to some greater frequency ω_0 . The result is

$$\frac{\omega_0^2 - \omega^2}{\omega_0^2} \leq \left| \frac{Z_0'}{Z_0} \right| \leq \frac{\omega_0^2}{\omega_0^2 - \omega^2}, \quad (19)$$

where Z_0 is the actual characteristic impedance and Z_0' is the value of characteristic impedance we determine from incorrect assumptions about the high frequency behavior of $\arg(p_0)$.

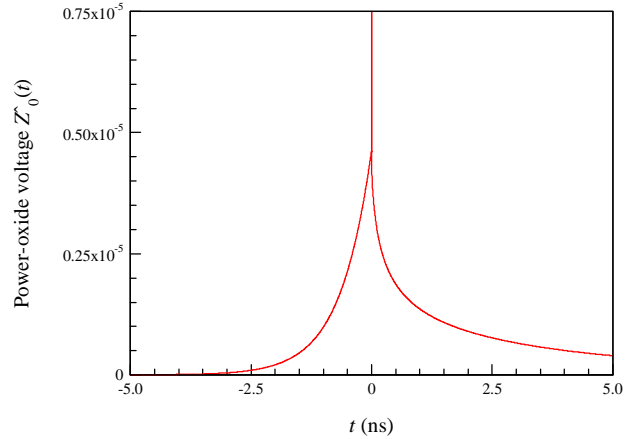


Fig. 4. The Fourier transform of the characteristic impedance labeled “Power/oxide-voltage” of Fig. 3.

The expression in (19) shows that the error in $|Z_0|$ can be made as small as required if we are willing to restrict the frequencies ω to which we apply the theory to frequencies much smaller than ω_0 , the frequency to which we evaluate the phase of p_0 . Although the convergence indicated by (19) is slow, it corresponds to a worst case scenario: convergence for more forgiving phase errors will be better.

It should perhaps be emphasized that, while small errors in $|Z_0|$ will sometimes be unavoidable, the resulting model will nevertheless be consistent both with the actual values of $\arg(p_0)$ and the actual fields for $|\omega| < |\omega_0|$. Therefore this circuit theory should lead to useful time-domain simulations when higher frequencies, where the waveguide behavior is poorly characterized, contain minimal energy.

CONCLUSION

We have presented a causal power-normalized waveguide circuit theory that overcomes the TEM, TE, and TM restrictions of classic waveguide circuit theories. The network parameters of the causal circuit theory presented here preserve the causal properties of the actual circuit and the power in the network. This is significant because these properties are required for stable time-domain circuit simulation. Since classical waveguide circuit theories also enforce these properties

in TEM, TE, and TM guides, we can say that this theory conserves the essential attributes of the classical waveguide circuit theory in a more general setting.

In the causal circuit theory the magnitude of the characteristic impedance is related to its temporal properties, not to its properties in the frequency domain. This adds a new perspective to the debate over the relative merits of the various impedance normalizations possible in waveguide equivalent-circuit theories.

We could have applied causality constraints to an analogous reciprocity-normalized circuit theory [10]. However, the new reciprocity-normalized theory would fail to enforce the passivity condition that ensures that the real part of the impedance of passive circuits is always positive, which will make stable circuit simulation impossible in certain circumstances. Our causal power-normalized theory, on the other hand, explicitly enforces the passivity and causality conditions, both of which are requirements for stable time-domain simulation.

APPENDIX 1: CONSTRUCTION OF v_0

Referring to Fig. 1, we seek a normalizing voltage $v_0'(\omega)$ such that the temporal voltage $\hat{v}'(t,0)$ will start exactly when the electric field arrives at $z = 0$ and \mathbf{e}_t can be written in the form of (11). That is, if $\hat{v}'(t,0) = 0$ for $t < 0$, then the electric field at $z = 0$ vanishes for times $t < 0$, and vice versa.

Consider the normalizing voltage

$$v_0'(\omega) \equiv \sum_{m=1}^n a_m(\omega) \int \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{e}_t(\omega, \mathbf{r}) d\mathbf{r}, \quad (20)$$

where the $a_m(\omega)$ are polynomials in ω . This normalizing voltage is defined so that

$$v_0'(\omega) \equiv \sum_{m=1}^n a_m(\omega) c_m(\omega). \quad (21)$$

Referring again to Fig. 1, only the single forward mode is present, so the voltage v' associated with the normalizing voltage v_0' at $z = 0$ is

$$\begin{aligned} v'(\omega, 0) &\equiv c_+(\omega) v_0'(\omega) \\ &= \sum_{m=1}^n a_m(\omega) \int \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{E}_t(\omega, \mathbf{r}, 0) d\mathbf{r}. \end{aligned} \quad (22)$$

In the time domain (22) is

$$\hat{v}'(t, 0) = \sum_{m=1}^n \hat{a}_m(t) \otimes \int \mathbf{f}_m(\mathbf{r}) \cdot \hat{\mathbf{E}}_t(t, \mathbf{r}, 0) d\mathbf{r}. \quad (23)$$

Since the a_m are polynomials, they have no poles at all and are analytic everywhere. As a result, $\hat{a}_m(t) = 0$ for $t < 0$ (see Appendix 2). So, if the electric field vanishes for $t < 0$, then so do its moments with respect to the \mathbf{f}_m , and we see that, by construction, a vanishing electric field for $t < 0$ implies that $\hat{v}'(t, 0) = 0$ for $t < 0$.

We will now show that it is possible to construct the polynomials a_m so that the inverse is true as well. That is, so that $\hat{v}'(t, 0) = 0$ for $t < 0$ implies that the moments of the electric field with respect to the \mathbf{f}_m , and hence the electric field itself, vanish for $t < 0$. In essence, we will show that there are enough degrees of freedom available in the choice of the polynomials a_m that we can eliminate all of the poles in the lower half of the ω plane from an expression that determines the moments of the electric field from v' . This will ensure that the expression is analytic in the lower half plane, and so that their Fourier transforms are 0 for $t < 0$.

The m th moment of the total electric field with respect to \mathbf{f}_m is

$$\begin{aligned} \int \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{E}_t(\omega, \mathbf{r}, 0) d\mathbf{r} &= \frac{v'(\omega, 0)}{v_0'(\omega)} \int \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{e}_t(\omega, \mathbf{r}) d\mathbf{r} \\ &= \frac{v'(\omega, 0)}{v_0'(\omega)} c_m(\omega) = I_m^{-1}(\omega) v'(\omega). \end{aligned} \quad (24)$$

If, for some m , I_m^{-1} has no poles in the lower half of the ω plane, then $\hat{v}'(t,0) = 0$ for $t < 0$ implies that the m th moment of the total electric field vanishes for $t < 0$. Our aim is to show that we can pick the a_m so that none of the I_m^{-1} have any poles at all. We will do this by showing that we can construct the a_m so that none of the I_m have zeroes.

We can write the c_m as $c_m(\omega) \equiv P_m(\omega)/Q_m(\omega)$, where the P_m and Q_m are polynomials in ω , and expand the I_m as

$$\begin{aligned} I_m(\omega) &\equiv \frac{v_0'(\omega)}{c_m(\omega)} = \frac{\sum_{j=1}^n a_j(\omega) c_j(\omega)}{c_m(\omega)} \\ &= \frac{Q_m(\omega)}{P_m(\omega)} \sum_{j=1}^n a_j(\omega) \frac{P_j(\omega)}{Q_j(\omega)}. \end{aligned} \quad (25)$$

We can rearrange (25) to obtain a single common denominator:

$$\begin{aligned} I_m(\omega) &= \frac{\sum_j \left(a_j(\omega) P_j(\omega) \prod_{k \neq j} Q_k(\omega) \right)}{P_m(\omega) \prod_{l \neq m} Q_l(\omega)} \\ &= \frac{\sum_j a_j(\omega) I_j'(\omega)}{I_m'(\omega)}, \end{aligned} \quad (26)$$

where

$$I_j'(\omega) \equiv P_j(\omega) \prod_{k \neq j} Q_k(\omega). \quad (27)$$

The numerator of (26) is independent of the index m .

Define $G(\omega)$ to be a greatest common divisor of the I_j' . That is, G is a polynomial of largest possible order such that $I_j' = I_j'' G$, where $I_j''(\omega)$ is a polynomial of order less than or equal to the order of I_j' . The Euclidian algorithm provides a procedure for finding a set of a_j so that $\sum a_j(\omega) I_j'(\omega) = G(\omega)$ [11]. So we can write (26) as

$$\begin{aligned} I_m(\omega) &= \frac{\sum_j a_j(\omega) I_j'(\omega)}{I_m'(\omega)} \\ &= \frac{G(\omega)}{I_m'(\omega)} = \frac{1}{I_m''(\omega)}. \end{aligned} \quad (28)$$

We have just shown that it is possible to construct a_m so that the I_m in (24) have no zeroes. This guarantees that we can construct a normalizing voltage v_0' from the modal fields such that the voltage v' associated with it is 0 for times $t < 0$ whenever the electric field is 0 for $t < 0$, and vice versa. That is, we have constructed a voltage $\hat{v}'(t,0)$ that starts simultaneously with the electric field.

APPENDIX 2:

MINIMUM PHASE FUNCTIONS

Throughout this work we denote the frequency-domain representation of a function as $F(\omega)$, and its time-domain representation as $\hat{F}(t)$, where ω is the angular frequency and t is the time. Here $\hat{F}(t)$ is the inverse Fourier transform of $F(\omega)$:

$$\hat{F}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega, \quad (29)$$

where t is real, and the integration in (29) is performed over real values of ω . $F(\omega)$ is the Fourier transform of $\hat{F}(t)$:

$$F(\omega) \equiv \int_{-\infty}^{\infty} \hat{F}(t) e^{-j\omega t} dt, \quad (30)$$

where ω may be complex. If either $F(\omega)$ or $\hat{F}(t)$ in (29) or (30) has poles for real ω or t , we take the principal value of the integrals.

Causal function: A causal function $\hat{F}(t)$ equals 0 for $t < 0$. This implies that $F(\omega)$ is analytic for

$\text{Im}(\omega) \leq 0$ and that $\text{Im}(F(\omega)) = \mathcal{H}[\text{Re}(F(\omega))]$, where \mathcal{H} is the Hilbert transform [12], [13].

Minimum phase function: We call a function $F(\omega)$ minimum phase if both $F(\omega)$ and its reciprocal $1/F(\omega)$ correspond to causal functions in the time domain [13]. Since neither the dependent nor independent variables in the time domain related by a minimum phase function in the frequency domain can occur before the other, *two nonzero signals related by a minimum phase function start simultaneously*.

A minimum phase function is causal, so has the property that its real and imaginary parts are a Hilbert transform pair. In addition, the real and imaginary parts of the complex logarithm of a minimum phase function are a Hilbert transform pair [13]. That is, $\arg(F(\omega)) = \mathcal{H}[\ln|F(\omega)|]$. The minimum phase constraint is much stronger than the causality constraint: it allows the phase of the function to be determined from the Hilbert transform of the logarithm of its magnitude and the magnitude of the function to be determined within a constant multiplier from its phase.

Rational function: A rational function $F(\omega)$ can be written as

$$F(\omega) = \frac{P(\omega)}{Q(\omega)} = \lambda \frac{\prod (\omega - \alpha_i)}{\prod (\omega - \beta_i)}, \quad (31)$$

where ω may be complex, λ is a scalar, and $P(\omega)$ and $Q(\omega)$ are polynomials in ω with complex roots α_i and β_i . Except for the multiplier λ , any rational function $F(\omega)$ is entirely described by its zeroes α_i and poles β_i .

Pole and zero positions: Since causal rational functions are analytic in the lower half of the ω plane defined by $\text{Im}(\omega) < 0$, all of the poles of a causal function $F(\omega)$ must lie in the upper half of the ω plane [12]. That is, $\text{Im}(\beta_i) > 0$ for all the β_i in (31).

If $F(\omega)$ is minimum phase, then its reciprocal $1/F(\omega)$ is also causal, and its zeroes must also lie in the upper half of the ω plane. That is, both $\text{Im}(\beta_i) > 0$ and $\text{Im}(\alpha_i) > 0$ for all of the α_i and β_i in (31) [13].

APPENDIX 3:

SIMULTANEITY OF \hat{E}_t AND \hat{H}_t

We will now show that $\hat{E}_t(t, \mathbf{r}, 0)$ and $\hat{H}_t(t, \mathbf{r}, 0)$ due to the source in Fig. 1 start simultaneously. Assume that the transverse electric field due to the source has not yet arrived at some transverse coordinate \mathbf{r} at the reference plane of Fig. 1 for $t < 0$. That is, we will assume that $\hat{E}_t(t, \mathbf{r}, z) = 0$ for $t < 0$ and $z > 0$. The fields in the region $z > 0$ must satisfy $\nabla \times \hat{\mathbf{E}} = -\partial \hat{\mathbf{B}} / \partial t$ for $t < 0$, which implies

$$\frac{\partial \hat{E}_z}{\partial y} \mathbf{x} - \frac{\partial \hat{E}_z}{\partial x} \mathbf{y} = -\frac{\partial \hat{\mathbf{B}}}{\partial t}. \quad (32)$$

As a result, $\hat{\mathbf{B}}_z(t, \mathbf{r}, z) = 0$ for $t < 0$ and $z > 0$.

The fields must also satisfy $\nabla \times \hat{\mathbf{H}} = \epsilon \partial \hat{\mathbf{E}} / \partial t$ for $t < 0$ and $z > 0$, where ϵ is the position-dependent permittivity. Since $\hat{E}_t(t, \mathbf{r}, z) = \hat{B}_z(t, \mathbf{r}, z) = 0$ for $t < 0$ and $z > 0$,

$$-\frac{\partial \hat{H}_y}{\partial z} \mathbf{x} + \frac{\partial \hat{H}_x}{\partial z} \mathbf{y} + \left(\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} \right) \mathbf{z} = \epsilon \frac{\partial \hat{E}_z}{\partial t}. \quad (33)$$

This in turn implies that

$$\frac{\partial \hat{H}_y}{\partial z} = \frac{\partial \hat{H}_x}{\partial z} = 0 \quad (34)$$

for $t < 0$ and $z > 0$. This shows that, except for a dc component, $\hat{H}_t(t, \mathbf{r}, z) = 0$ for $t < 0$ and $z > 0$. So we see that $\hat{E}_t(t, \mathbf{r}, z) = 0$ for $t < 0$ and $z > 0$ implies $\hat{H}_t(t, \mathbf{r}, z) = 0$ there as well, and the transverse magnetic field starts at the reference plane no earlier than the transverse electric field.

A similar argument shows that the transverse electric field starts no earlier than the transverse magnetic field. This completes the argument, showing that neither $\hat{E}_t(t, \mathbf{r}, 0)$ nor $\hat{H}_t(t, \mathbf{r}, 0)$ precedes the other, and thus that they start simultaneously.

APPENDIX 4: ERROR BOUND FOR $|Z_0|$

Assume that we have determined exactly the phase of p_0 up to some frequency ω_0 and that we wish to determine $|Z_0(\omega)|$ at frequencies $\omega < \omega_0$. We will develop an expression bounding the error with which we calculate $|Z_0|$.

The logarithm of $|Z_0|$ is the inverse Hilbert transform of $\arg(p_0)$:

$$\ln|Z_0(\omega)| \equiv -\pi^{-1} \int_{-\infty}^{\infty} \frac{\arg(p_0(\sigma))}{\sigma - \omega} d\sigma. \quad (35)$$

If $b(\omega)$ is the error we make in determining the phase of p_0 , we calculate the characteristic impedance Z_0'' from

$$\ln|Z_0''(\omega)| \equiv -\pi^{-1} \int_{-\infty}^{\infty} \frac{\arg(p_0(\sigma)) + b(\sigma)}{\sigma - \omega} d\sigma. \quad (36)$$

We will always use a condition such as (17) to match the low frequency limits of $|Z_0|$ and $|Z_0''|$, so we can write the magnitude of the characteristic impedance Z_0' we will use in the theory as

$$|Z_0'(\omega)| = \left| \frac{Z_0(0)}{Z_0''(0)} \right| |Z_0''(\omega)|. \quad (37)$$

Expanding $\eta(\omega) \equiv \ln|Z_0'| - \ln|Z_0|$ using (35) and (36), we obtain

$$\eta(\omega) = -\omega\pi^{-1} \int_{-\infty}^{\infty} \frac{b(\sigma)}{\sigma(\sigma - \omega)} d\sigma. \quad (38)$$

Since $b(\omega)$ is odd and equal to 0 for $|\omega| < \omega_0$, we can rewrite (38) as

$$\eta(\omega) = -2\omega^2\pi^{-1} \int_{\omega_0}^{\infty} \frac{b(\sigma)}{\sigma(\sigma^2 - \omega^2)} d\sigma. \quad (39)$$

The sign of the real part of p_0 indicates the direction of the real time-averaged power carried by the mode down the guide. If the real part of p_0 for the forward (decaying) mode were negative, the mode would no longer dissipate energy as it propagated down the guide, and violate conservation of energy. So the phase of p_0 can only vary between $\pm\pi/2$, and the error $b(\omega)$ we make in evaluating the phase of p_0 cannot be greater than $\pm\pi$. Since the denominator of (39) is odd, the worst case error is made when $b(\omega) = \pm\pi$. So we can bound η with

$$|\eta(\omega)| \leq 2\omega^2 \int_{\omega_0}^{\infty} \frac{1}{\sigma(\sigma^2 - \omega^2)} d\sigma = \ln \frac{\omega_0^2}{|\omega_0^2 - \omega^2|}. \quad (40)$$

Straightforward manipulation gives (19).

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