Characteristic Impedance, Causality, and Microwave Circuit Theory

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INTRODUCTION

A new waveguide equivalent-circuit theory guarantees causality of the network parameters of passive devices. It fixes both the magnitude and phase of the characteristic impedance of a waveguide by marrying a power normalization with constraints that enforce simultaneity of the theory's voltages and currents and the actual fields in the circuit.

THE CAUSAL CIRCUIT THEORY

The power-normalized waveguide equivalent-circuit theory of [1] begins with a waveguide that is uniform in the axial direction and supports only a single mode of propagation at the reference plane where v and i are defined. The voltage v is defined by

$$\boldsymbol{E}_{\mathrm{t}}(\boldsymbol{r},z) = [c_+(z)e^{-\gamma z} + c_-(z)e^{\gamma z}]\,\boldsymbol{e}_{\mathrm{t}}(\boldsymbol{r}) \equiv \frac{v(z)}{v_0}\boldsymbol{e}_{\mathrm{t}}(\boldsymbol{r}) \quad \textbf{(1)}$$

and the current i by

$$\boldsymbol{H}_{\mathrm{t}}(\boldsymbol{r},z) = \left[c_+(z)e^{-\gamma z} - c_-(z)e^{\gamma z}\right]\boldsymbol{h}_{\mathrm{t}}(\boldsymbol{r}) \equiv \frac{i(z)}{i_0}\boldsymbol{h}_{\mathrm{t}}(\boldsymbol{r}), \quad \textbf{(2)}$$

where $\boldsymbol{r} = (x,y)$ is the transverse coordinate, $\boldsymbol{E}_{\mathrm{t}}$ and $\boldsymbol{H}_{\mathrm{t}}$ are the total electric and magnetic fields in the guide, $\boldsymbol{e}_{\mathrm{t}}$ and $\boldsymbol{h}_{\mathrm{t}}$ are the modal electric and magnetic fields of the single propagating mode, γ is the modal propagation constant, and c_+ and c_- are the forward and reverse amplitudes of the mode. The time dependence $e^{j\omega t}$ in **(1)** and **(2)** has been suppressed, and all of the parameters are functions of ω. The two factors v_0 and i_0 define v and i in terms of the fields, and can be thought of as voltage and current normalization factors.

The total time-averaged power p in the waveguide is found by integrating the Poynting vector over the guide's cross section S:

$$p \equiv \frac{1}{2}\int_S \boldsymbol{E}_{\mathrm{t}} \times \boldsymbol{H}_{\mathrm{t}}^* \cdot \boldsymbol{z}\,\mathrm{d}S = \frac{1}{2}\frac{v\,i^*}{v_0 i_0^*}\int_S \boldsymbol{e}_{\mathrm{t}} \times \boldsymbol{h}_{\mathrm{t}}^* \cdot \boldsymbol{z}\,\mathrm{d}S. \quad \textbf{(3)}$$

The power normalization in [1] is achieved by imposing the constraint

$$v_0\ i_0^* \ = \ p_0 \ \equiv \int_S \boldsymbol{e}_{\mathrm{t}} \times \boldsymbol{h}_{\mathrm{t}}^* \cdot \boldsymbol{z}\,\mathrm{d}S, \quad \textbf{(4)}$$

which ensures that the time-averaged power is $p = ½vi^*$.

The characteristic impedance Z_0 of a waveguide is defined by

$$Z_0 \ \equiv \ \left.\frac{v}{i}\right|_{c_-=0}. \quad \textbf{(5)}$$

Equations **(1)**, **(2)**, and **(4)** give

$$Z_0 \ = \ \frac{v_0}{i_0} \ = \ \frac{|v_0|^2}{p_0^*} \ = \ \frac{p_0}{|i_0|^2}. \quad \textbf{(6)}$$

Equation **(6)** shows that the phase of Z_0 in the power-normalized circuit theory is equal to the phase of p_0, which is a fixed property of the guide uniquely determined by the modal field solutions $\boldsymbol{e}_{\mathrm{t}}$ and $\boldsymbol{h}_{\mathrm{t}}$.

The causal circuit theory of [2] requires that $Z_0(\omega)$ be causal. That is, the theory requires that $\hat{Z}_0(t) = 0$ for $t < 0$, where $\hat{Z}_0(t)$ is the inverse Fourier transform of $Z_0(\omega)$. This condition ensures that the voltage in the waveguide responds to input currents after, not before, the onset of the current.

The theory also requires that $Y_0 \equiv 1/Z_0(\omega)$ be causal so that the current in the waveguide responds to input voltages after, not before, the onset of the voltage. These two constraints imply that $Z_0(\omega)$ is minimum phase [3], [4], [5].

The minimum phase constraint is a strong one. The real and imaginary parts of the complex logarithm of a minimum phase function are a Hilbert transform pair: that is, $\ln|Z_0|$ and $\arg(Z_0)$ are a Hilbert transform pair. The result is that we can determine $\ln|\lambda Z_0|$, where λ is a constant, from the inverse Hilbert transform of $\arg(Z_0)$.

The scalar multiplier λ sets the overall impedance of the system, is linked to the units of voltage and current chosen in the theory, and is the only free parameter not determined by the causal theory of [2].

Lossless Coaxial Transmission Line

The power flow p_0 is real in a lossless coaxial transmission line, so the phase of Z_0 is 0. The set of constant functions form the null space of the Hilbert transform, so in the causal circuit theory $|Z_0|$ must be constant.

Dominant TE_{10} Mode of Lossless Rectangular Waveguide

The power flow p_0 and therefore Z_0 are real in a lossless rectangular waveguide above cutoff and imaginary below cutoff. So $\arg(Z_0)$ is equal to $\pm\pi/2$ below cutoff, and 0 above. The inverse Hilbert transform of a function that is equal to $-\pi/2$ for $-\omega_c < \omega < 0$, $\pi/2$ for $0 < \omega < \omega_c$, and 0 elsewhere is [6]

$$\frac{1}{2} \ln\left|\frac{\omega^2}{\omega^2 - \omega_c^2}\right|. \qquad \textbf{(7)}$$

The causality constraint therefore requires that

$$|Z_0| \propto \sqrt{\left|\frac{\omega^2}{\omega^2 - \omega_c^2}\right|}, \qquad \textbf{(8)}$$

where $\propto$ indicates proportionality. That is, Z_0 must be proportional to the wave impedance of the guide: the choice $|Z_0| = 1$ is not admissible in the causal theory.

References

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