Searching for low weight pseudo-codewords

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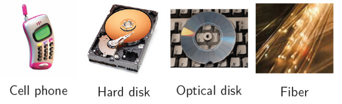
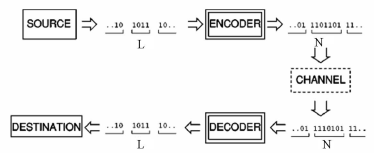
Outline

Error Correction



Scheme:



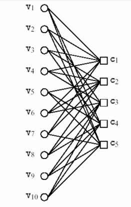
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding:
  reconstruct most probable codeword by noisy (polluted) channel

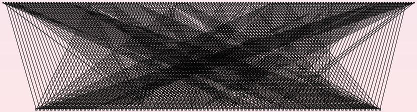
Low Density Parity Check Codes

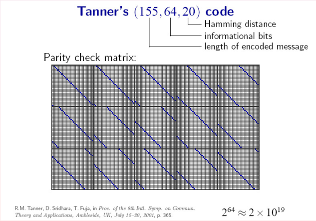


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse





Decoding=Statistical Inference

Maximum Likelihood — Maximum-a-Posteriori

$$\text{ML} = \arg\max_{\boldsymbol{\sigma}=\text{codeword}} P(\mathbf{x}_{\text{out}}|\boldsymbol{\sigma}) \qquad \text{MAP}_i = \text{sign}\left(\frac{\sum_{\boldsymbol{\sigma}} \sigma_i P(\mathbf{x}_{\text{out}}|\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} P(\mathbf{x}_{\text{out}}|\boldsymbol{\sigma})}\right)$$

MAP≈BP=Belief-Propagation (Bethe-Pieirls) — Gallager '61

- Exact on a tree
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

- Message Passing = iterative BP

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Bethe free energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = -\sum_i h_i \sum_{\sigma_i} \sigma_i b_i(\sigma_i) + \sum_\alpha \sum_{\boldsymbol{\sigma}_\alpha} b_\alpha(\boldsymbol{\sigma}_\alpha) \ln b_\alpha(\boldsymbol{\sigma}_\alpha) - \sum_i (q_i - 1) \sum_{\sigma_i} b_i(\sigma_i) \ln b_i(\sigma_i)$$

constraints:

$$\forall\ i,\alpha:\quad 0 \le b_i(\sigma_i), b_\alpha(\boldsymbol{\sigma}_\alpha) \le 1$$

$$\forall\ \alpha:\quad \sum_{\boldsymbol{\sigma}_\alpha} b_\alpha(\boldsymbol{\sigma}_\alpha) = 1$$

$$\forall\ i;\ \alpha \in i:\quad b_i(\sigma_i) = \sum_{\boldsymbol{\sigma}_\alpha \setminus \sigma_i} b_\alpha(\boldsymbol{\sigma}_\alpha)$$

Belief-Propagation Equations:

$$\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$$

- Relaxation to minimum of the Bethe Free energy enforces convergence of iterative BP (Stepanov, Chertkov '06)

LP decoding
Feldman, Wainwright, Karger '03

- LP decoding = minimization of a linear function over a bounded domain described by linear constraints
- "Large SNR" limit of BP: $F \approx E = -\sum_i h_i \sum_{\sigma_i} \sigma_i b_i(\sigma_i)$
- "small" polytope = get rid of b_α

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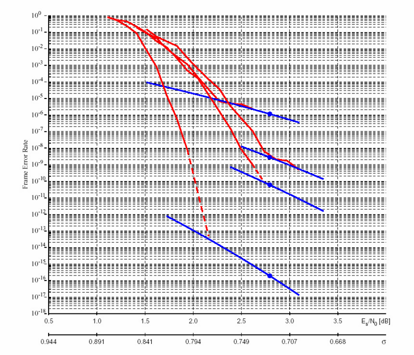
Error-Floor



T. Richardson, Allerton '03

- BER vs SNR = measure of performance
- Waterfall $\leftrightarrow$ Error-floor
- Suboptimal decoding causes error-floor: at $s^2 = E_s/N_0 \to \infty$, $FER_{\mathrm{ML}} \sim \exp(-d_{\mathrm{ML}} s^2/2)$ vs $FER_{\mathrm{sub}} \sim \exp(-d_{\mathrm{sub}} s^2/2)$ where $d_{\mathrm{ML}} \geq d_{\mathrm{sub}}$
- Monte-Carlo is useless at FER $\lesssim 10^{-8}$
- Need an efficient method to analyze error-floor

Pseudo-codewords and Instantons

Error-floor is caused by Pseudo-codewords:

Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '03; Vontobel, Koetter '04-'06

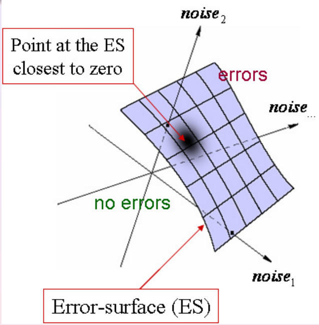
Instanton = optimal conf of the noise

$$BER = \int d(noise)\ WEIGHT(noise)$$

$$BER \sim WEIGHT \left(\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array} \right)$$

$$\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array} = \begin{array}{c} Point\ \ at\ \ the\ \ ES \\ closest\ \ to\ \ "0" \end{array}$$

Instantons are decoded to Pseudo-Codewords

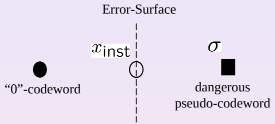


Instanton-amoeba

= optimization algorithm

Stepanov, et.al '04,'05

Stepanov, Chertkov '06



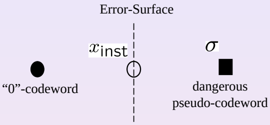
Weighted Median: $x_{\text{inst}} = \arg\min_{\mathbf{x}} \mathcal{P}(\mathbf{0}|\mathbf{x})\big|_{E(\mathbf{x};\boldsymbol{\sigma})=0}$

$$\mathbf{x}_{\text{inst}} = \frac{\boldsymbol{\sigma}}{2}\frac{\sum_i \sigma_i}{\sum_i \sigma_i^2}, \quad d = \frac{(\sum_i \sigma_i)^2}{\sum_i \sigma_i^2}, \quad \text{FER} \sim \exp(-d\cdot s^2/2)$$

Wiberg '96; Forney et.al '99; Vontobel, Koetter '03,'05

Pseudo-Codeword-Search Algorithm — Chertkov, Stepanov '06

- **Start:** Initiate $\mathbf{x}^{(0)}$.
- **Step 1:** $\mathbf{x}^{(k)}$ is decoded to $\boldsymbol{\sigma}^{(k)}$.
- **Step 2:** Find $\mathbf{y}^{(k)}$ - weighted median between $\boldsymbol{\sigma}^{(k)}$, and "0"
- **Step 3:** If $\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)}$, $k_* = k$ End. Otherwise go to **Step 2** with $\mathbf{x}^{(k+1)} = \mathbf{y}^{(k)} + 0$.



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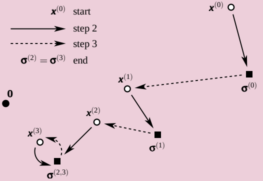

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Multiple repetitions ⇒ instanton frequency spectra

LP complexity grows exponentially with check degree

Current solutions:

- Adaptive LP (Taghavi, Siegel '06)
- BP-style relaxation of LP (Vontobel, Koetter '06)

Dendro-trick = Graph Modification (our solution)

Chertkov,Stepanov'07

- MAP solutions are identical
- Set of Pseudo-codewords are identical
- Instanton spectra are very alike, ≈

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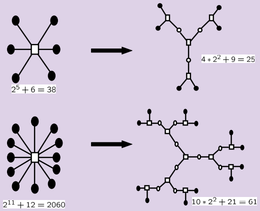

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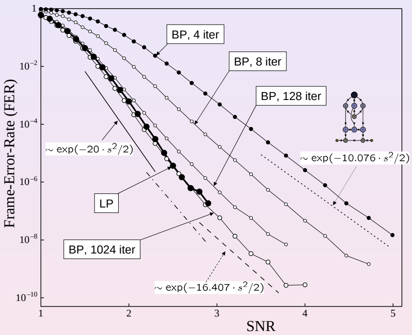
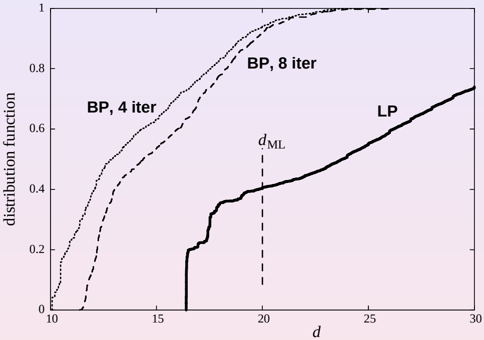
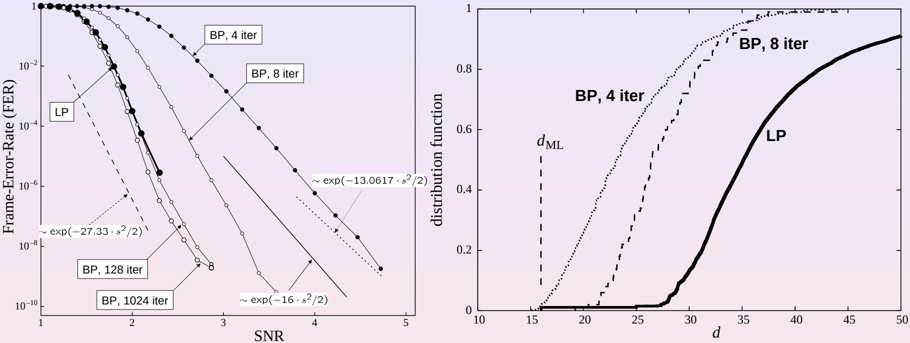




- $d_{\text{min;inst}} < d_{\text{ML}} = 20$
- Dangerous instantons are frequent



- $d_{\text{min;inst;LP}} > d_{\text{ML}} = 16$
- Dangerous codewords are rare $\Rightarrow$ emergence of a steep transient asymptotic of FER vs SNR

Conclusions:

- BP is faster but LP is easier to analyze
- Instanton-amoeba and especially Pseudo-codeword-search are of a practical value for the error-floor domain exploration
- Dendro-LDPC is a convenient trick reducing complexity of LP

Future Challenges:

- Improving LP/BP (Facet Guessing of Dimakis,Wainwright '06 and LP-erasure Chertkov,Chernyak '06 are good candidates)
- Analyzing really long codes
- Analyzing error-floor in correlated channels (e.g. 1d and 2d ISI with and without coding)
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All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm