Physics of Algorithms
Loop Calculus in Information Theory and Statistical Physics

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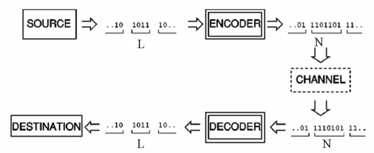
Thanks to M. Stepanov (UofA, Tucson)

Outline

Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



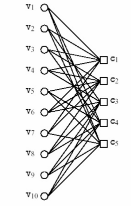
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding [Algorithm]:
  reconstruct most probable codeword by noisy (polluted) channel

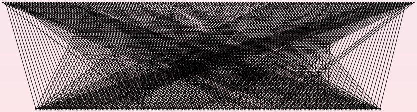
Low Density Parity Check Codes

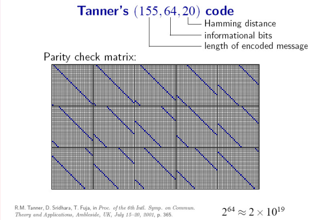


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

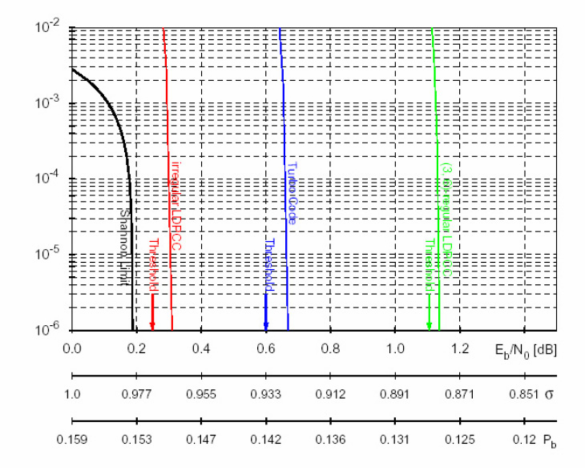
$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse





Shannon Transition



- Phase Transition
- Ensemble of Codes [analysis & design]
- Thermodynamic limit but ...

Statistical Models

Ising model $\sigma_i = \pm 1$

$$\mathcal{P}(\boldsymbol{\sigma}) = Z^{-1} \exp\left(\sum_{i,j} J_{ij}\sigma_i\sigma_j\right)$$

J_{ij} define the graph (lattice)

Decoding $\sigma_i = \pm 1$

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}(\mathbf{x}) \prod_\alpha \delta\left(\prod_{i\in\alpha}\sigma_i, +1\right) \prod_i p(\sigma_i|x_i)$$

Hard (check) constraints define the graph/code

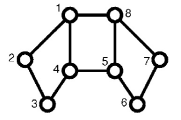
N.Sourlas '89; A.Montanari '00: Error-correction as a Statistical Mechanics

Graphical models

Factorization (Forney '01, Loeliger '01)

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a)$$

$$\underbrace{Z(\mathbf{x}) = \sum_{\boldsymbol{\sigma}}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a))}_{\text{partition function}}$$



$f_a \geq 0$

$\sigma_{ab} = \sigma_{ba} = \pm 1$

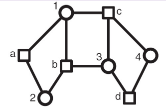
$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$

$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$

Example: Error-Correction (linear code, bipartite Tanner graph)

$$f_i(h_i|\boldsymbol{\sigma}_i) = \exp(\sigma_i h_i)\cdot\begin{cases} 1, & \forall\alpha,\beta \ni i, \quad \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise} \end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha}\sigma_i, +1\right)$$



h_i - log-likelihoods

Statistical Inference

$\boldsymbol{\sigma}_{\text{orig}}$	$\Rightarrow$	$\mathbf{x}$	$\Rightarrow$	$\boldsymbol{\sigma}$
original *data* $\boldsymbol{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\mathbf{x}\|\boldsymbol{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\boldsymbol{\sigma} \in \mathcal{C}$

Maximum Likelihood [ground state]

Maximum-a-Posteriori [magnetization]

$$\text{ML} = \arg\max_{\boldsymbol{\sigma}} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma}) \qquad \text{MAP}_i = \arg\max_{\sigma_i} \sum_{\boldsymbol{\sigma}\setminus\sigma_i} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma})$$

Exhaustive search is generally expensive:

Statistical Inference

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$$\boldsymbol{\sigma}=(\sigma_1,\cdots,\sigma_N),\quad N\ \textit{finite},\quad \sigma_i=\pm 1\ (\textit{example})$$

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Exhaustive search is generally expensive:
complexity of the algorithm $\sim 2^N$

Variational Method in Statistical Mechanics

$$P(\boldsymbol{\sigma}) = \frac{\prod_a f_a(\boldsymbol{\sigma}_a)}{Z}, \quad Z \equiv \sum_{\boldsymbol{\sigma}} \prod_a f_a(\boldsymbol{\sigma}_a)$$

Exact Variational Principle — Kullback-Leibler '51

$$F\{b(\boldsymbol{\sigma})\} = -\sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \sum_a f_a(\boldsymbol{\sigma}_a) - \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \ln b(\boldsymbol{\sigma})$$

$$\left.\frac{\delta F}{\delta b(\boldsymbol{\sigma})}\right|_{b(\boldsymbol{\sigma})=p(\boldsymbol{\sigma})} = 0 \quad \text{under} \quad \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) = 1$$

Variational Ansatz

- Mean-Field: $p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \prod_i b_i(\sigma_i)$
- Belief Propagation:

$$p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \frac{\prod_a b_a(\boldsymbol{\sigma}_a)}{\prod_{(a,b)} b_{ab}(\sigma_{ab})} \quad \text{(exact on a tree)}$$

$$b_a(\boldsymbol{\sigma}_a) = \sum_{\boldsymbol{\sigma} \setminus \boldsymbol{\sigma}_a} b(\boldsymbol{\sigma}), \quad b_{ab}(\sigma_{ab}) = \sum_{\boldsymbol{\sigma} \setminus \sigma_{ab}} b(\boldsymbol{\sigma})$$

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Bethe free energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = \underbrace{-\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)}_{\text{self-energy}} + \underbrace{\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})}_{\text{configurational entropy}}$$

$\forall\ a;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1, \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$

$\Rightarrow$ Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

MAP≈BP=Belief-Propagation (Bethe-Pieirls): iterative ⇒ Gallager '61; MacKay '98

- Exact on a tree ▸ Derivation Sketch
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{\alpha j} = h_j + \sum_{\beta \neq \alpha}^{j \in \beta} \tanh^{-1}\left(\prod_{i \neq j}^{i \in \beta} \tanh \eta_{\beta i} \right) \quad \Leftarrow \text{LDPC representation}$$

- Message Passing = iterative BP
- Convergence of MP to minimum of Bethe Free energy can be enforced

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Linear Programming version of Belief Propagation

In the limit of large SNR, $\ln f_a \to \pm\infty$: BP→LP

$$\text{Minimize } F \approx E = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) = \text{self energy}$$

under set of linear constraints

LP decoding of LDPC codes Feldman, Wainwright, Karger '03

- ML can be restated as an LP over a codeword polytope
- LP decoding is a "local codewords" relaxation of LP-ML
- Codeword convergence certificate
- Discrete and Nice for Analysis
- Large polytope $\{b_\alpha, b_i\} \Rightarrow$ Small polytope $\{b_i\}$

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BP does not account for Loops

Questions:

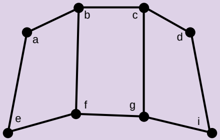
- Is BP just a heuristic in a loopy case?
- Why does it (often) work so well?
- Does exact inference allow an expression in terms of BP?
- Can one correct BP systematically?

Previous Considerations:

- Rizzo, Montanari '05 - Corrections to BP approximation
- Parisi, Slanina '05 - BP as a saddle-point + corrections

Chertkov,Chernyak '06

Local Gauge, G, Transformations



$$Z=\sum_{\boldsymbol{\sigma}}\prod_a f_a(\boldsymbol{\sigma}_a),\;\; \boldsymbol{\sigma}_a=(\sigma_{ab},\sigma_{ac},\cdots),\;\; \sigma_{ab}=\sigma_{ba}=\pm 1$$

$$f_a(\boldsymbol{\sigma}_a=(\sigma_{ab},\cdots))\to\sum_{\sigma'_{ab}} G_{ab}\left(\sigma_{ab},\sigma'_{ab}\right) f_a(\sigma'_{ab},\cdots)$$

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The partition function is invariant under any G-gauge!

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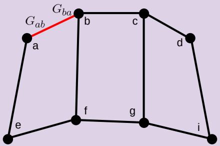

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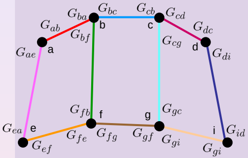

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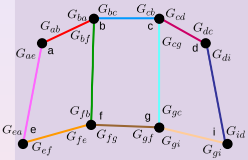

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Gauge Transformation: Binary Representation

$$Z=\sum_{\boldsymbol\sigma}\prod_a f_a(\boldsymbol\sigma_a)=\sum_{\boldsymbol\sigma'}\prod_a f_a(\boldsymbol\sigma_a)\prod_{bc}\frac{1+\sigma_{bc}\sigma_{cb}}{2},\quad \sigma_{bc}\neq\sigma_{cb}$$

The binary trick: $1+\sigma_{bc}\sigma_{cb}=$

$$\frac{\exp(\sigma_{bc}\eta_{cb}+\sigma_{cb}\eta_{cb})}{\cosh(\eta_{bc}+\eta_{cb})}\left(1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})\right)$$

$$\tilde f_a(\boldsymbol\sigma_a)=f_a(\boldsymbol\sigma_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}(\sigma_{bc},\sigma_{cb})=1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

Graph Coloring

$$Z=\left(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\right)^{-1}\sum_{\boldsymbol\sigma'}\prod_a\tilde f_a(\boldsymbol\sigma_a)\prod_{bc}V_{bc}$$

$$Z=\underbrace{Z_0(\eta)}_{\substack{\text{ground state}\\ \boldsymbol\sigma=+1}}+\underbrace{\sum_{\text{all possible colorings of the graph}}Z_c(\eta)}_{\text{excited states}}$$

Gauges and BP

Fixing the gauges $\Rightarrow$ BP equations!!

Two alternative ways to understand BP-gauges:

BP equations

Color Principe:

no loose ends

$$Z = Z_0(\eta) + \sum_{c=colorings} Z_c(\eta)$$

$$Z_c(\eta) = \prod_{a\in C} \Psi_{a;C}(\eta)$$

Variational Principe:

ground state is η-independent

$$Z \to Z_0(\eta)$$

$$\left.\frac{\delta Z_0}{\delta \eta_{ab}}\right|_{\eta^{(bp)}} = 0$$

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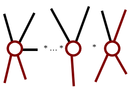

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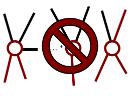

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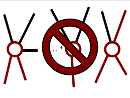

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Loop Series: Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

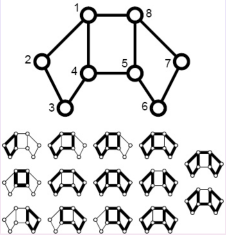
$$Z=\sum_{\boldsymbol{\sigma}_\sigma}\prod_a f_a(\boldsymbol{\sigma}_a)=Z_0\left(1+\sum_C r(C)\right)$$

$$r(C)=\frac{\prod_{a\in C}\mu_a}{\prod_{(ab)\in C}(1-m_{ab}^2)}=\prod_{a\in C}\tilde{\mu}_a$$

$C\in$ Generalized Loops = Loops without loose ends

$$m_{ab}=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$



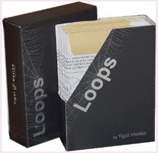
- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.
- ▸ Features of the Loop Calculus/Series

Loops are important ...

CHICAGO LOOP ALLIANCE

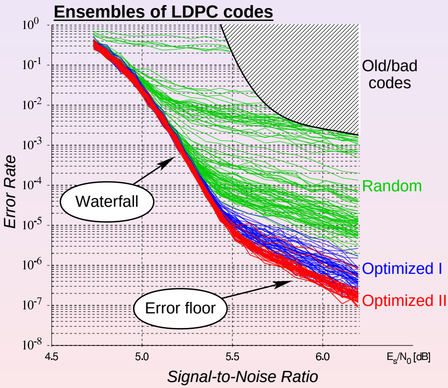
1. Introduction
   - Enabling Example: Error Correction
   - Statistical Inference
   - Bethe Free Energy and Belief Propagation (BP)

2. Loop Calculus
   - Gauge Transformations and BP
   - Loop Series in terms of BP



3. Applications
   - Analysis and Improvement of LDPC-BP/LP Decoding
   - Long Correlations and Loops in Statistical Mechanics

Error-Floor



- BER vs SNR = measure of performance
- Finite size effects
- Waterfall ↔ Error-floor
- Error-floor typically emerges due to sub-optimality of decoding
- Monte-Carlo is useless at FER $\lesssim 10^{-8}$
- Need an efficient method to analyze error-floor

Pseudo-codewords and Instantons

Error-floor is caused by Pseudo-codewords:

Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '03; Vontobel, Koetter '04-'06

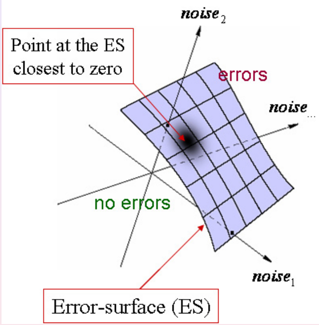
Instanton = optimal conf of the noise

$$BER = \int d(noise)\ WEIGHT(noise)$$

$$BER \sim WEIGHT\begin{pmatrix} optimal\ \ conf \\ of\ \ the\ \ noise \end{pmatrix}$$

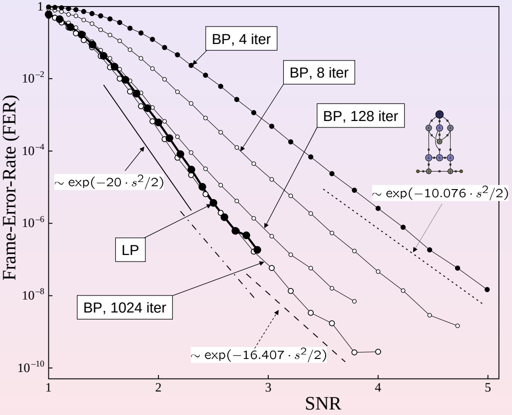
$$\begin{matrix} optimal\ \ conf \\ of\ \ the\ \ noise \end{matrix} = \begin{matrix} Point\ \ at\ \ the\ \ ES \\ closest\ \ to\ \ "0" \end{matrix}$$

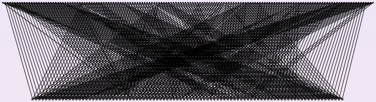
Chernyak, Chertkov, Stepanov, Vasic '04;'05



Instantons are decoded to Pseudo-Codewords

Pseudo-codeword & instanton search





Pseudo-codeword search

- Instanton-amoeba

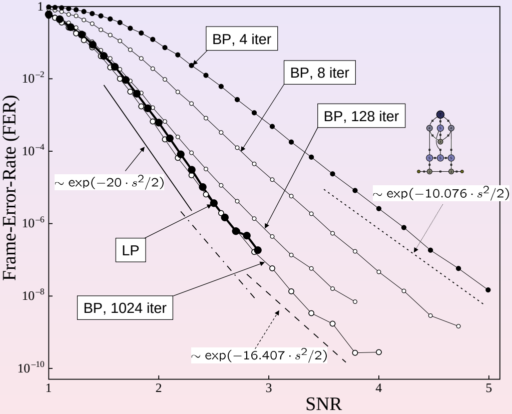
Stepanov, et.al '04,'05,'06

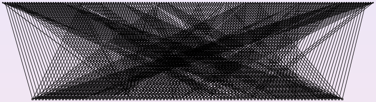
- LP-search

Chertkov, Stepanov '06,'07

What does Loop Calculus show for dangerous Pseudo-codewords?

Pseudo-codeword & instanton search





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What does Loop Calculus show for dangerous Pseudo-codewords?

Why loops?

If BP/LP fails while ML/MAP would not [pseudo-codewords]
... one needs to account for Loops

- How many loops are critical to recover from the failure?
- Will accounting for a single most important loop be sufficient?
- How long is the critical loop?
- Will it be difficult to find the critical loop?
- If there are many ...
  how are the critical loops distributed over scales?

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Loop Calculus & Pseudo-Codeword Analysis

Single loop truncation

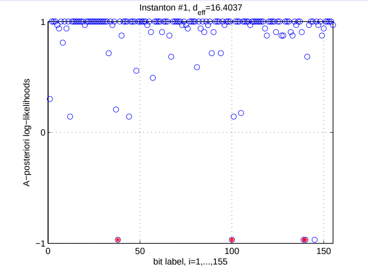
$$Z = Z_0(1+\sum_C r_C) \approx Z_0(1+r(\Gamma))$$

Synthesis of Pseudo-Codeword Search Algorithm (Chertkov, Stepanov '06) & Loop Calculus

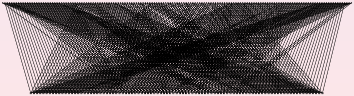
- Consider pseudo-codewords one after other
- For an individual pseudo-codeword/instanton identify a critical loop, Γ, giving major contribution to the loop series.
- Hint: look for single connected loops and use local "triad" contributions as a tester: $r(\Gamma)=\prod_{\alpha\in\Gamma}\tilde{\mu}_\alpha^{(bp)}$

Proof-of-Concept test [(155, 64, 20) code over AWGN]

- $\forall$ pseudo-codewords with $16.4037 < d < 20$ (~ 200 found) there always exists a simple single-connected critical loop(s) with $r(\Gamma)\sim 1$.
- Pseudo-codewords with the lowest d show $r(\Gamma)=1$
- Invariant with respect to other choices of the original codeword



Bigger Set



Extended Variational Principe & Loop-Corrected BP

Bare BP Variational Principe:
$$\left.\frac{\delta Z_0}{\delta \eta_{ab}}\right|_{\eta^{(bp)}} = 0$$

New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\delta \exp(-\mathcal{F})}{\delta \eta_{ab}}\right|_{\eta_{\mathrm{eff}}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

$$\left.\frac{\sum_{\sigma_a}(\tanh(\eta_{ab}+\eta_{ba})-\sigma_{ab})P_a(\sigma_a)}{\sum_{\sigma_a} P_a(\sigma_a)}\right|_{\eta_{\mathrm{eff}}} = \text{explicitly known contribution}\Big|_{\eta_{\mathrm{eff}}} \neq 0 \quad [\text{along } \Gamma]$$

Loop-Corrected BP Algorithm

- 1. Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found).
- 2. If BP fails find the most relevant loop Γ that corresponds to the maximal $|r_\Gamma|$. Triad search is helping.
- 3. Solve the modified-BP equations for the given Γ. Terminate if the improved-BP succeeds.
- 4. Return to **Step 2** with an improved Γ-loop selection.

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- **1.** Run LP algorithm. Terminate if LP succeeds (i.e. a valid code word is found).
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$(155, 64, 20)$ Test

- **IT WORKS!**
  All troublemakers ($\sim$ 200 of them) previously found by LP-based Pseudo-Codeword-Search Algorithm method were successfully corrected by the LP-erasure algorithm.
- Method is invariant with respect the choice of the codeword (used to generate pseudo-codewords).

General Conjecture:

- Loop-erasure algorithm is capable of reducing the error-floor
- Bottleneck is in finding the critical loop
- Local adjustment of the algorithm, anywhere along the critical loop, in the spirit of the Facet Guessing (Dimakis, Wainwright '06), may be sufficient

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Error-correction is probably (?) easy

- BP is improvable with few loops
- Pseudo-codewords are correctable

How about difficult applications?

- SAT, spin-glasses, ...
- E.g. ... If there are many critical loops,
  how are the critical loops distributed over scales?

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Dilute Gas of Loops: $Z = Z_0(1+\sum_C r_C) \approx Z_0 \cdot \prod_{C_{sc}=\text{single connected}} (1+r_{sc})$

Applies to

- Lattice problems in high spatial dimensions
- Large Erdös-Renyi problems (random graphs with controlled connectivity degree)

- The approximation allows an easy multi-scale re-summation
- In the para-magnetic phase and $\mathbf{h}=0$: the only solution of BP is a trivial one $\boldsymbol{\eta}=0$, $Z_0 \to 1$, and the Loop Series is reduced to the high-temperature expansion [Domb, Fisher, et al '58-'90]

Ising model in the factor graph terms

$$Z = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=(i,j)\in X} \exp\left(J_{ij}\sigma_i\sigma_j\right) = \sum_{\boldsymbol{\sigma}} \prod_{a\in\{i\}\cup\{\alpha\}} f_a(\boldsymbol{\sigma}_a)$$

$$f_i(\boldsymbol{\sigma}_i) = \begin{cases} \exp(h_i\sigma_i), & \sigma_{i\alpha}=\sigma_{i\beta}=\sigma_i \quad \forall \alpha,\beta \ni i \\ 0, & \text{otherwise;} \end{cases}$$

$$f_\alpha\left(\boldsymbol{\sigma}_\alpha = (\sigma_{\alpha i},\sigma_{\alpha j})\right) = \exp\left(J_{ij}\sigma_{\alpha i}\sigma_{\alpha j}\right)$$

Loop Series trivially pass the common "loop" tests (from Rizzo, Montanari '05)

- Evaluation of the critical temperature in the constant exchange, zero field Ising model
- Leading $1/N$ corrections to the Free Energy of the Viana-Bray model in the vicinity of the critical point (glass transition)

Results

- BP is better then just a heuristic in the loopy case ... BP is the special Gauge condition eliminating all contributions but loops.
- Exact Marginal probability allows explicit Loop Series expression in terms of a solution of the Belief Propagation equations.
- Truncation and/or Re-summation of the Loop Series provide hierarchy of systematically improvable approximations/algorithms. Standard BP/LP is a first member in the hierarchy.
- Local example (truncation). Finding a critical loop, or a small number of critical loops, can be algorithmically sufficient for drastic improvement of BP decoding in the error-floor domain.
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Future Challenges

- Better Algorithms: Loop Series Truncation/Resummation
- Generalizations. q-ary and continuous alphabets. Quantum spins, Quantum error-correction.
- Loop calculus based analysis of graph ensembles, e.g. understanding and improving the cavity method [Mézard, Parisi '85-'03]
- Extending the list of Loop Calculus Applications, e.g. SAT and cryptography
- Non-BP gauges, e.g. for stat problems on regular and irregular lattices
- Relation to graph ζ-functions [Koetter, Li, Vontobel, Walker '05]

Other complementary developments, e.g. wrt Algorithms:

- Improving BP [Survey Propagation = Mézard et.al '02; Generalized BP = Yedidia et.al '01]
- Correcting for Loops in BP [Montanarri, Rizzo '05; Parisi, Slanina '05]
- Accelerating convergence of bare BP-LDPC [Stepanov, Chertkov '06]
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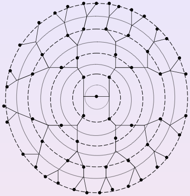
All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm



$$Z(\mathbf{h}) = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^{M} \delta\left(\prod_{i\in\alpha}\sigma_i, 1\right) \exp\left(\sum_{i=1}^{N} h_i\sigma_i\right)$$

h_i is a log-likelihood at a bit (outcome of the channel)

$$Z^{\pm}_{j\alpha}(\mathbf{h}^{>}) \equiv \sum_{\boldsymbol{\sigma}^{>}}^{\sigma_j=\pm 1} \prod_{\beta^{>}} \delta\left(\prod_{i\in\beta}\sigma_i, 1\right) \exp\left(\sum_{i^{>}} h_i\sigma_i\right)$$

$$Z^{\pm}_{j\alpha} = \exp(\pm h_j) \prod_{\beta\neq\alpha}^{j\in\beta} \frac{1}{2}\left(\prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} + Z^{-}_{i\beta}) \pm \prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} - Z^{-}_{i\beta})\right)$$

$$\eta_{j\alpha} \equiv \frac{1}{2}\ln\left(\frac{Z^{+}_{j\alpha}}{Z^{-}_{j\alpha}}\right), \quad \eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh\eta_{i\beta}\right)$$

◂ Bethe Free Energy

Gauges and BP equations

Partition function in the colored representation

$$Z=\Big(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\Big)^{-1}\sum_{\boldsymbol{\sigma}'}\prod_a\tilde f_a\prod_{bc}V_{bc},\quad \tilde f_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a)=f_a(\boldsymbol{\sigma}_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}(\sigma_{bc},\sigma_{cb})=1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

Fixing the gauges ⇒ BP equations!!

$$\sum_{\boldsymbol{\sigma}_a}\left(\tanh(\eta_{ab}^{(bp)}+\eta_{ba}^{(bp)})-\sigma_{ab}\right)\tilde f_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a)=0\quad\Rightarrow\quad \underbrace{\eta_{\alpha j}^{bp}=h_j+\sum_{\beta\neq\alpha}^{j\in\beta}\tanh^{-1}\Big(\prod_{i\neq j}^{i\in\beta}\tanh\eta_{\beta i}^{bp}\Big)}_{\text{LDPC case}}$$

◂ Gauges and BP

$Z = Z_0(1 + \sum_C r_C),\ r_C = \prod_{a\in C} \tilde{\mu}_a$

- Bethe Free Energy is related to the "ground state" term in the partition function: $F(b^*(\eta)) = -\ln Z_0(\eta)$, where
  $b_a^*(\boldsymbol{\sigma}_a) = \frac{f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b\in a}\eta_{ab}\sigma_{ab})}{\sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b\in a}\eta_{ab}\sigma_{ab})}, \quad b_{ab}^*(\sigma_{ab}) = \frac{\exp((\eta_{ab}+\eta_{ba})\sigma_{ab})}{2\cosh(\eta_{ab}+\eta_{ba})}$
- Extrema of $F(b)$ are in one-to-one correspondence with extrema of $Z_0(\eta)$.
- Loop series can be built around any extremum (minimum, maximum or saddle-point) of the Bethe Free energy.
- $-1 \le r_C, \tilde{\mu}_a \le 1$. The tasks of finding all $\tilde{\mu}_a$ (over the graph) and r_C for a given loop are (computationally) not difficult. All that suggests simple heuristic for finding loops with large r_C.
- Linear Programming limit of the Loop Calculus is well defined.
- Any marginal probability, e.g. magnetization (a-posteriori log-likelihood) at an edge, is expressed as modified Loop Series.

Loop Series

$Z = Z_0(1 + \sum_C r_C),\ r_C = \prod_{a\in C} \tilde{\mu}_a$

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LP decoding ($\sigma_i = 0, 1$ AWGN channel)

Minimize, $E = \sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) \sum_{i\in\alpha} \sigma_i(1-2x_i)/q_i$, under $0 \le b_i(\sigma_i), b_\alpha(\sigma_\alpha) \le 1$

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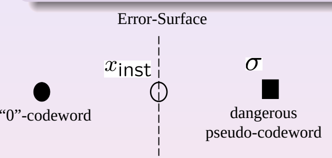

Weighted Median:

$$\mathbf{x}_{\text{inst}} = \frac{\boldsymbol{\sigma}}{2}\frac{\sum_i \sigma_i}{\sum_i \sigma_i^2}, \quad d = \frac{(\sum_i \sigma_i)^2}{\sum_i \sigma_i^2}$$

$$\text{FER} \sim \exp(-d \cdot s^2/2)$$

Wiberg '96; Forney et.al '01
Vontobel, Koetter '03,'05

(155, 64, 20), AWGN test:

- Fast Convergence

$\sim$ 200 pseudo-codewords within $16.4037 < d < 20$

Pseudo-Codeword-Search Algorithm

Chertkov, Stepanov '06

- **Start:** Initiate $\mathbf{x}^{(0)}$.
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Pseudo-codewords & Instantons

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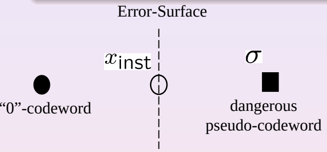

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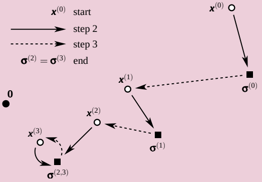

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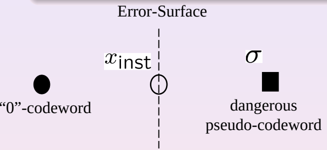

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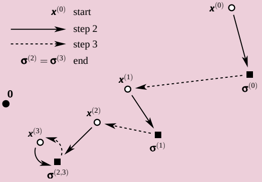

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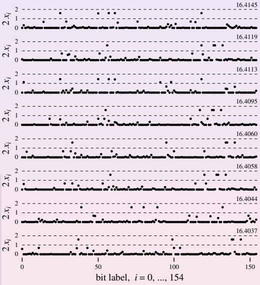

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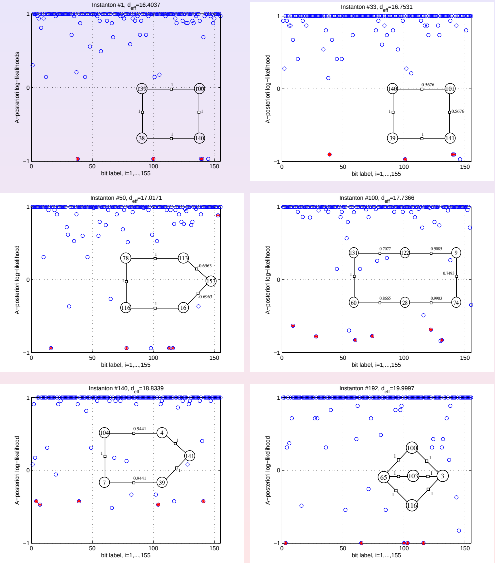
Instanton #1, d_{eff}=16.4037
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Instanton #50, d_{eff}=17.0171
Instanton #100, d_{eff}=17.7366
Instanton #140, d_{eff}=18.8339
Instanton #192, d_{eff}=19.9997
A−posteriori log−likelihoods
A−posteriori log−likelihood
bit label, i=1,...,155

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