Analyzing and **Correcting** Effects of Noise and Disorder in Optics Communications
Analyzing and Decoding LDPC codes

Michael Chertkov[1]
Vladimir Chernyak [2,1], Mikhail Stepanov [3,1], Bane Vasic [3]

[1]CNLS & T-13, LANL

[2]Wayne State, Detroit & [3]U of Arizona, Tucson

Seattle, 09/11/06

Outline

1. Error Correction in communications and other technologies
   - Intro
   - General scheme
   - LDPC codes
2. Understanding Error-Floor
   - Emergence of the Error-Floor is a caveat of BP
   - Instanton (optimal fluctuation) method
   - Tanner code. Gaussian channel.
   - Tanner code. Laplacian channel.
   - Linear Programming Decoding
3. Improving Belief Propagation/Bethe-Peierls
   - Factor Graph Model
   - Bethe Free Energy and BP
   - Loop Calculus
   - Loop Corrected Belief Propagation
   - BP for Channels with Correlations
4. Path forward

Noise and disorder in communication lines

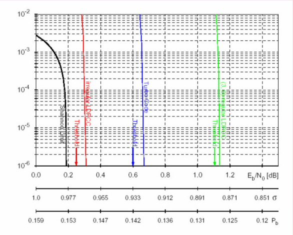
- Amplifier Noise
- Cross-channel (and inter-channel) Interference
- Variation in dispersion
- Variation in birefringence
- ...

Why error-correction?

- "Cheap" alternative to fiber line improvements
- Error-free transmission is achievable – in theory

Challenges:

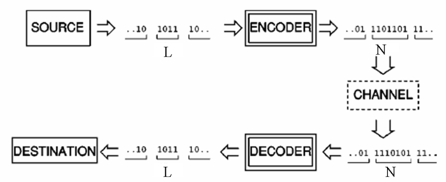
- Extremely low BER (error-floor)
- Case specific correlations
- Optical implementation (switch)



Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



Examples of Gaussian and Laplacian White Channels:

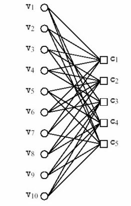
$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p_{Gauss}(x|y) \sim \exp(-s^2(x-y)^2/2)$$

$$p_{Lap}(x|y) \sim \exp(-s|x-y|)$$

- Channel (fiber)
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding:
  reconstruct most probable codeword by noisy (polluted) channel

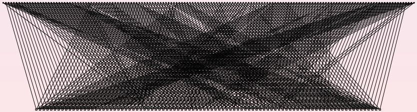
Low Density Parity Check Codes

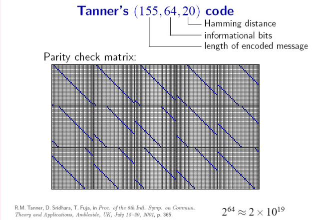


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LPDC = graph (parity check matrix) is sparse





Decoding Low Density Parity Check Codes

Maximum Likelihood/Maximum-a-Posteriori

Exhaustive search for pre-image = the best one can possibly do

BP=Belief-Propagation (Bethe-Pieirls)

- Exact on a tree
- Applies to a general inference problem on a (sparse) graph
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

- Message Passing = iterative BP

Decoding Low Density Parity Check Codes

Maximum Likelihood/Maximum-a-Posteriori

Exhaustive search for pre-image = the best one can possibly do

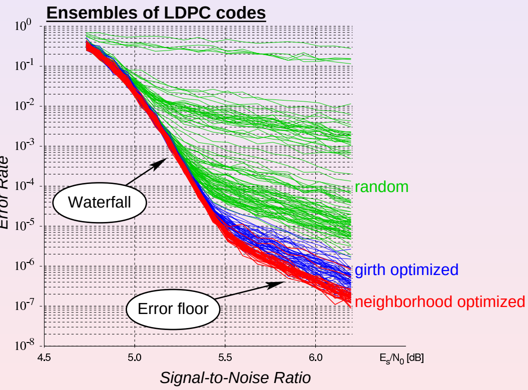
BP=Belief-Propagation (Bethe-Pieirls)

- Exact on a tree
- Applies to a general inference problem on a (sparse) graph
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{j\alpha} = h_j + \sum_{\beta \neq \alpha}^{j \in \beta} \tanh^{-1} \left(\prod_{i \neq j}^{i \in \beta} \tanh \eta_{i\beta} \right)$$

- Message Passing = iterative BP

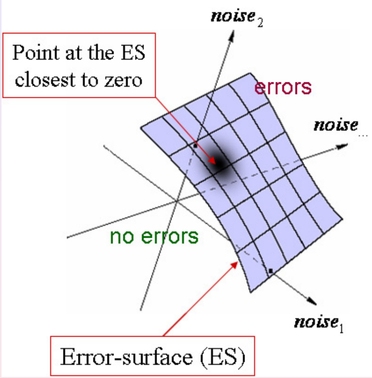
Error-Floor



- BER vs SNR = measure of performance
- Waterfall $\leftrightarrow$ Error-floor
- Suboptimal decoding causes error-floor
- Fluctuations within an expurgated ensemble of codes are stronger in the error-floor domain
- Monte-Carlo is useless at BER $\lesssim 10^{-8}$
- Need an efficient method to analyze error-floor

Optimal Fluctuation

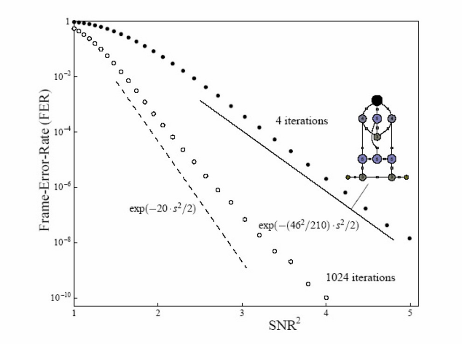
- Laplace method = Large deviation = Steepest descent = instanton = optimal fluctuation method
- BER $= \int d\text{noise Weight(noise)}$
- BER $\sim$ Weight $\begin{pmatrix} \text{optimal configuration} \\ \text{of the noise} \end{pmatrix}$
- $\begin{pmatrix} \text{optimal configuration} \\ \text{of the noise} \end{pmatrix} = \begin{pmatrix} \text{point at the ES} \\ \text{closest to zero} \end{pmatrix}$

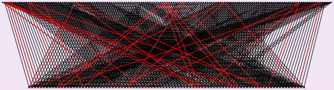


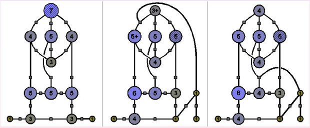
Chernyak, Chertkov, Stepanov, Vasic Phys.Rev.Lett **93**, 198702 (2004)

(155,64,20) Tanner code. Gaussian Channel.

- Instanton-amoeba (numerical minimization) implemented for 4-iterations of iterative BP



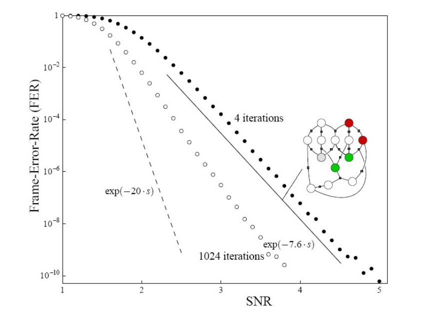


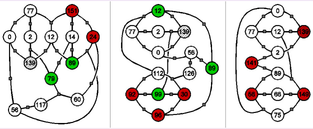


$l_{eff}^2 = \frac{46^2}{210} \approx 10.076$ $l_{eff}^2 = \frac{806}{79} \approx 10.203$ $l_{eff}^2 = \frac{44^2}{188} \approx 10.298$

Stepanov, Chertkov, Chernyak, Vasic, Phys Rev Lett **95**, *228701 (2005) + cond-mat/0506037 + ISIT 2006*

(155,64,20) Tanner code. Laplacian Channel.





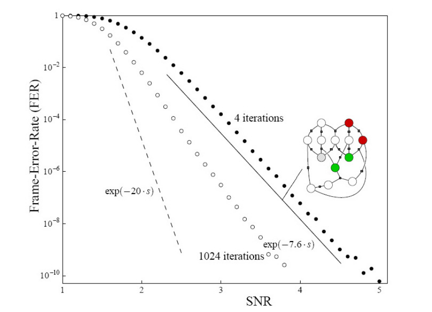
$l_{eff}^2 = 7.6$ $l_{eff}^2 = 8$ $l_{eff}^2 = 8$

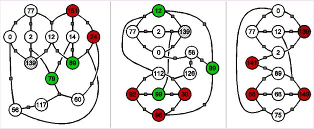
Lesson:

Strong dependence of the error-floor performance on the channel

Stepanov, Chertkov, 43rd Allerton conference (2005), arXiv:cs.IT/0507031

(155,64,20) Tanner code. Laplacian Channel.





$l^2_{eff} = 7.6$ $l^2_{eff} = 8$ $l^2_{eff} = 8$

Lesson:

Strong dependence of the error-floor performance on the channel

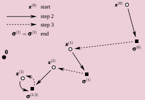
Stepanov, Chertkov, 43rd Allerton conference (2005), arXiv:cs.IT/0507031

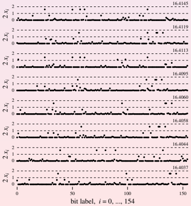
LP decoding — Feldman, Wainwright, Karger '03

- LP decoding = minimization of a linear function over a bounded domain described by linear conditions
- Fast and Discrete
- "Large SNR" limit of BP

Instanton for LP-decoding — Chertkov, Stepanov '06

- Find instanton in small number of discrete steps





$(155, 64, 20)$ test:

~ 200 instantons were found within the gap between error-floor (BP) and Hamming distance (MAP) asymptotics, $16.4037 < E < 20$

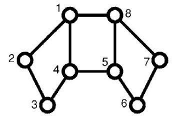
Chertkov, Stepanov – submitted to IEEE IT, arXiv:cs.IT/0601113

Factorization (Forney '01, Loeliger '01)

$$P\{\boldsymbol{\sigma}\} = \prod_{a\in X} f_a(\boldsymbol{\sigma}_a),$$

$$Z = \sum_{\{\boldsymbol{\sigma}\}} P\{\boldsymbol{\sigma}\},$$

$$X = \text{edges}$$



$$\sigma_{ab} = \sigma_{ba} = \pm 1$$

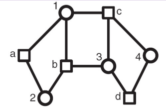
$$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$$

$$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$$

Example: Error-Correction (bipartite)

$$f_i(\boldsymbol{\sigma}_i) = \begin{cases} 1, & \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise} \end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha} \sigma_i, +1\right) \exp\left(\sum_{i\in\alpha} \sigma_i h_i / q_i\right)$$



h_i - log-likelihoods
q_i-connectivity degrees

Bethe free energy: variational approach (Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

- $F = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) + \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})$
- constraints:

$$\forall\ a, c;\ c \in a: \quad 0 \leq b_a(\boldsymbol{\sigma}_a), b_{ac}(\sigma_{a,c} \leq 1$$

$$\forall\ a, c;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = \sum_{\sigma_{a,c}} b_{ac}(\sigma_{a,c}) = 1$$

$$\forall\ a, c;\ c \in a: \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a) = \sum_{\sigma_c \setminus \sigma_{ac}} b_c(\boldsymbol{\sigma}_c)$$

- Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

- LP=BP at SNR $\rightarrow 0$
- Convergence of iterative BP is not guaranteed
- Relaxation to minimum of the Bethe Free energy enforces convergence of iterative BP

(Stepanov, Chertkov – 44th Allerton 09/2006)

Bethe free energy: variational approach (Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

- $F = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) + \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})$
- constraints:

$$\forall\ a, c;\ c \in a:\quad 0 \le b_a(\boldsymbol{\sigma}_a), b_{ac}(\sigma_{a,c} \le 1$$

$$\forall\ a, c;\ c \in a:\quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = \sum_{\sigma_{a,c}} b_{ac}(\sigma_{a,c}) = 1$$

$$\forall\ a, c;\ c \in a:\quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a) = \sum_{\sigma_c \setminus \sigma_{ac}} b_c(\boldsymbol{\sigma}_c)$$

- Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

- LP=BP at SNR $\to 0$
- Convergence of iterative BP is not guaranteed

- Relaxation to minimum of the Bethe Free energy enforces convergence of iterative BP

(Stepanov, Chertkov – 44th Allerton 09/2006)

Loop Calculus: (Chertkov,Chernyak '06)

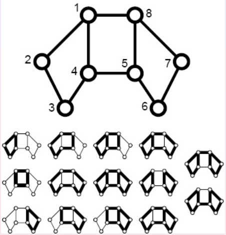
- Exact expression (for partition function, etc) in terms of BP

$$Z = Z_0\left(1+\sum_C r(C)\right), \quad r(C) = \frac{\prod_{a\in C}\mu_a}{\prod_{(ab)\in C}(1-m_{ab}^2)}$$

$$m_{ab} = \int d\boldsymbol{\sigma}_a b_a(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a = \int d\boldsymbol{\sigma}_a b_a(\boldsymbol{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$

$b_{ab}, b_a, Z_0 \equiv -\ln F$ – all calculated within BP



- BP is exact on a tree
- BP is a Gauge fixing condition

Utility of the loop calculus for improving BP

LP-version of the loop-improved algorithm

- **1.** Run LP algorithm. Terminate if LP succeeds.
- **2.** If LP fails, find the most relevant loop C correspondent to the maximum in amplitude $r(C)$.
- **3.** Modify the log-likelihoods along the loop C introducing "erasures" along the loop.
- **4.** Run LP with modified log-likelihoods.

$(155, 64, 20)$ test

All "dangerous" pseudo-codewords (~ 200 of them) previously found by LP-instanton method were successfully corrected by the loop-improved LP algorithm.

- The loop-imroved decoding shows NO error-floor.

Chertkov, Chernyak – invited talk at 44th Allerton conference, 09/2006

Utility of the loop calculus for improving BP

LP-version of the loop-improved algorithm

- **1.** Run LP algorithm. Terminate if LP succeeds.
- **2.** If LP fails, find the most relevant loop C correspondent to the maximum in amplitude $r(C)$.
- **3.** Modify the log-likelihoods along the loop C introducing "erasures" along the loop.
- **4.** Run LP with modified log-likelihoods.

$(155, 64, 20)$ test

All "dangerous" pseudo-codewords (~ 200 of them) previously found by LP-instanton method were successfully corrected by the loop-improved LP algorithm.

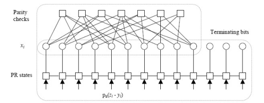
- The loop-imroved decoding shows NO error-floor.

Chertkov, Chernyak – invited talk at 44th Allerton conference, 09/2006

Example: Inter-Symbol interference (Partial Response) channel

Di-code formulation

- output $= y_i = \sum_j J_{ij}\sigma_j + \xi_i$
  $\langle \xi_i \rangle = 0, \quad \langle \xi_i \xi_j \rangle = \delta_{ij}/s^2$
- $\boldsymbol{\sigma}$ is encoded by an LDPC code
- di-code=Inter-Symbol Interference + LDPC code
- Decoding of the di-code = solving inference problem on an extended Tanner graph



Our approach

- Formulate di-BP = minimum of the joint (di-) Bethe free energy
- Test iterative version of the di-BP against Monte-Carlo simulations
- Apply instanton (and instanton-LP) approach to analysis of di-BP (di-code) error-floor
- Develop loop-improved di-BP/LP

Anguita, Chertkov – in progress

Analyzing, Developing and Improving Inference Algorithms

Methods

- Instanton toolbox for Coding
- Approximate algorithms improving BP
  - (a) Enforcing convergence to BP
  - (b) Loop series truncation
  - (c) Correcting BP gauges based on bare BP
- Other gauge fixing ideas

Applications

- Error-Correction over "interesting" channels (e.g. in fiber optics)
- Inter-symbol interference on 2d, 3d and networks (wireless, magnetic/holographic memory)
- Other problems in information and computer sciences (community detection, coding and routing on networks, combinatorial optimization, cryptography, etc)

Towards Designing Better Error-Correction Codes