# Approximate Representations of Random Intervals for Hybrid Uncertainty Quantification in Engineering Modeling

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**Abstract:**

We review our approach to the representation and propagation of hybrid uncertainties through high-complexity models, based on quantities known as random intervals [15, 20, 21]. These structures have a variety of mathematical descriptions, for example as interval-valued random variables [4], statistical collections of intervals [17], or Dempster-Shafer bodies of evidence on the Borel field [19]. But methods which provide simpler, albeit approximate, representations of random intervals are highly desirable, including p-boxes and traces. Each random interval, through its cumulative belief and plausibility measures functions [36], generates a unique p-box whose constituent CDFs are all of those consistent with the random interval. In turn, each p-box generates an equivalence class of random intervals consistent with it. Then, each p-box necessarily generates a unique trace which stands as the fuzzy set representation of the p-box or random interval. In turn each trace generates an equivalence class of p-boxes. The heart of our approach is to try to understand the tradeoffs between error and simplicity introduced when p-boxes or traces are used to stand in for various random interval operations. For example, Joslyn [18] has argued that for elicitation and representation tasks, traces can be the most appropriate structure, and has proposed a method for the generation of canonical random intervals from elicited traces. But alternatively, models built as algebraic equations of uncertainty-valued variables (in our case, random-interval-valued) propagate uncertainty through convolution operations on basic algebraic expressions, and while convolution operations are defined on all three structures, we have observed that the results of only some of these operations are preserved as one moves through these three levels of specificity. We report on the status and progress of this modeling approach concerning the relations between these mathematical structures within this overall framework.

**Keywords:** Dempster-Shafer theory, random sets, random intervals, p-boxes, probability bounds, fuzzy arithmetic.

## 1. INTRODUCTION

Engineering modeling problems are frequently characterized by a large number of inputs with different forms and levels of uncertainty present on them. For example, it might be desirable in a given context to combine uncertainties characterized by coarse-grained probability distributions, strong or weak statistical data, interval data, possibility distributions, or linguistic information represented as fuzzy sets. Propagating such hybrid

uncertainties through high-complexity models (whether analytical or computational) is thereby especially challenging, as are elicitations and interpretations of both input and output uncertainties by domain experts and customers.

We have been developing an approach to the representation and propagation of hybrid uncertainties in engineering modeling applications based on quantities known as random intervals [15, 20, 21]. These structures have a variety of mathematical descriptions, for example as interval-valued random variables [4], statistical collections of intervals [17], or Dempster-Shafer bodies of evidence on the Borel field [19].

One of the advantages of random interval structures is their ability to generalize more specific kinds of uncertainty quantities with a relative minimum of computational and mathematical complexity. Nonetheless, random intervals are not especially simple structures to represent or manipulate, and therefore methods which provide simpler, albeit approximate, representations of them are highly desirable. In this paper we report on a framework we are developing to accomplish this. In our approach, random interval quantities can be represented in increasingly simplified and approximate forms through first p-box, and then trace, structures.

A p-box [8] is an ordered pair of monotonically increasing functions which together bound a collection of cumulative probability distribution functions. Each random interval, through its cumulative belief and plausibility measures functions [36], generates a unique p-box whose constituent CDFs are all of those consistent with the random interval. In turn, each p-box generates an equivalence class of random intervals consistent with it.

A trace [17] is defined in this context as a fuzzy quantity on the real line. Each p-box necessarily generates a unique trace which stands as the fuzzy set representation of the p-box or random interval. Under different conditions it can take on the properties of a probability distribution, possibility distribution, or so-called "fuzzy interval" quantity (used in fuzzy arithmetic). In turn each trace generates an equivalence class of p-boxes.

The heart of our approach is to try to understand the tradeoffs between error and simplicity introduced when p-boxes or traces are used to stand in for various random interval operations. For example, Joslyn [18] has argued that for elicitation and representation tasks, traces can be the most appropriate structure, and has proposed a method for the generation of canonical random intervals from elicited traces.

But alternatively, models built as algebraic equations of uncertainty-valued variables (in our case, random-interval-valued) propagate uncertainty through convolution operations on basic algebraic expressions. But while convolution operations are defined on all three structures (random intervals, p-boxes, and traces), we have observed that the results of only some of these operations are preserved as one moves through these three levels of specificity.

In this paper, we report on the status and progress of this modeling approach concerning the relations between these mathematical structures within this overall framework.

## 2. GENERALIZED UNCERTAINTY QUANTIFICATION FOR ENGINEERING MODELING

Consider the situation where we have a model, perhaps a large computer code, which acts as a function $f$ mapping inputs $X$ to outputs $Y$. This model $f$ might be quite complex, with high run times, and more significantly multiple input parameters (expressed as the dimensionality of the space $X$), with different kinds of uncertainty represented on them. Given the necessity for many "gaps" between the information present in our simulations from those of reality (model incompleteness and error, and inherent system variability and imprecision), we wish to represent amounts, degrees, and kinds of these uncertainties in formal systems.

But information available on inputs may be rich or sparse, so-called "aleatory" (related to well-known, but chance, outcomes) or "epistemic" (related to a less-than-well-known outcome), and may be made known through objective measurements or through the subjective elicitation of experts. Mathematically, inputs might be represented as probability distributions, paramaterized classes of probability distributions (e.g. $N(\mu, \sigma)$), by a strong statistical collection of data points, by a sparse such collection, by simple intervals, statistical collections of such intervals, or even by non-quantified linguistic expressions.
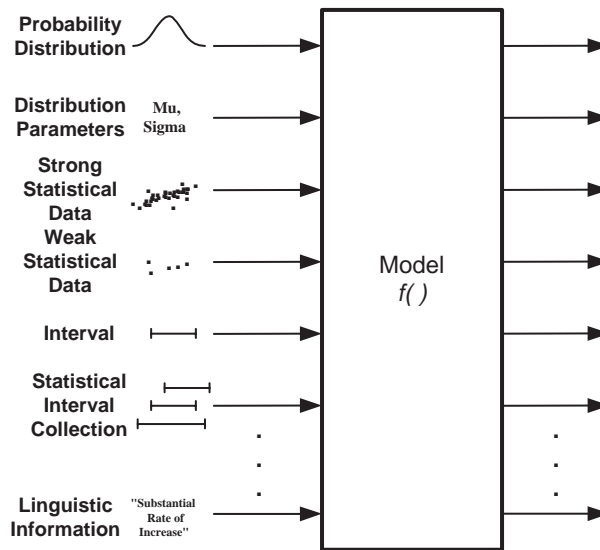


**Figure 1.** Hybrid uncertainty quantification for an engineering modeling problem.

So given a risk or reliability problem related to our model $f$ as charicatured in Fig. 1, how can we quantify this uncertainty on the input space $X$, and furthermore propagate it through $f$ to the output space $Y$? More to the point, how can we do so in a way which respects all the original uncertainty quantifications as provided, making no unnecessary assumptions? Paraphrasing Klir [25], how can we do such in a way which uses no less than, but also no more than, all available information; that is, uses *only*, but *all of*, what we are given?

So wherever possible, we should fit formalism to available information, and not *vice versa*.

Through the 20th century, uncertainty modeling has been dominated by the mathematics of probability, and since Shannon and Weaver [32], information has been defined as a statistical measure of a probability distribution. But also starting in the 1960s, alternative formalisms have arisen. Some of these were intended to stand in contrast to probability theory; others are deeply linked to probability theory, but depart from or elaborate on it in various ways. In the intervening time, there has been a proliferation of methodologies (including interval analysis [28], fuzzy systems [26], fuzzy and monotone measures [24], Dempster-Shafer evidence theory [13], random sets and intervals [22], possibility theory [7], probability bounds [8], rough sets [30], imprecise probabilities [34], and info-gap theory [2]) along with concomitant movements to synthesize and generalize them. Together, following Klir [19, 23], we call these **Generalized Information Theory** (GIT). These methods are increasingly accepted in engineering modeling [15, 33], and our approach is squarely centered here.

As a very simple example motivating our approach, consider that for one of the variables $x$ in our space $X$, we're given only an interval, that $x$ might be between two quantities $a$ and $b$, so that $x \in [a, b]$. How do we represent the uncertainty $U(I)$ in $I$? A standard answer might be to use a uniform probability ditribution $U(I) := p(x)$ with

$$p(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & x \notin [a, b] \end{cases},$$

as shown by the horizontal line in Fig. 2. No doubt this answer is justified (by maximum entropy, insufficient reason, and related principles) *when it is necessary to use a single probability distribution*. But this was not specified in the problem. Indeed, one could argue that *any* probability distribution with support on $[a, b]$ can be justified, perhaps denoted $U(I) \in \mathcal{P}([a, b])$ (perhaps the truncated normal shown in Fig. 2); but better yet, why isn't our uncertainty *all* such distributions: $U(I) = \mathcal{P}([a, b])$ (the box bounded by the dashed lines in Fig. 2).
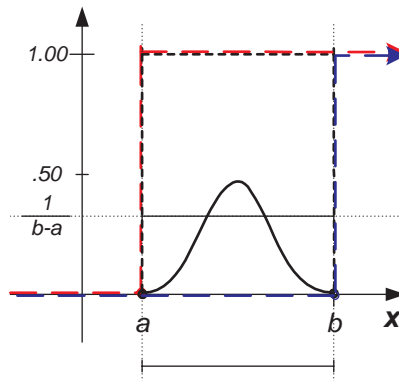


**Figure 2.** Representations of $x \in [a, b]$.

In its purest form, our answer should in fact be *none* of these, but rather that $U(I)$ is best represented by the information as provided us, that is, by the interval itself: $U(I) = [a, b]$.

However, when it is necessary to combine information for some variable $x \in X$ with another $y \in Y$, then these other forms may be vailable. We argue below that all of these answers are approached consistently within the proper GIT context.

## 3. NOTATION

Throughout the paper, assume a universe of discourse $\Omega$, with $\omega \in \Omega$. Denote $A \perp B := A \cap B = \emptyset$. Given a class $\mathcal{C} = \{A\} \subseteq 2^\Omega$, define the core and support respectively as

$$\mathbf{C}(\mathcal{C}) := \bigcap_{A \in \mathcal{C}} A, \qquad \mathbf{U}(\mathcal{C}) := \bigcup_{A \in \mathcal{C}} A.$$

We begin considering $\Omega = \{\omega_i\}, 1 \le i \le n$ to be finite, but move to recognize $\Omega = \mathbb{R}$, and consider Borel sets (half-open interval subsets), elements of a interval Borel field.

DEFINITION 3.1 (INTERVAL BOREL FIELD). *Let*

$$\mathcal{I} := \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a \le b\},$$

*where* $\forall a \le b \in \mathbb{R}, -\infty < a \le b < \infty, [-\infty, b) := \lim_{a \longrightarrow -\infty}[a, b) \in \mathcal{I}, [a, \infty) := \lim_{b \longrightarrow \infty}[a, b) \in \mathcal{I}, [-\infty, \infty) = \mathbb{R} \in \mathcal{I},$ *and* $[-\infty, -\infty) = [\infty, \infty) := \emptyset \in \mathcal{I}$ *by convention.*

In general, let $I := [a, b) \in \mathcal{I}$.

A vector denoted $\vec{a} = \langle a_i \rangle = \langle a_1, a_2, \ldots, a_m \rangle$ is a structure of length $|\vec{a}| := m$ where each element $a_i$ of the vector is an element of some set $a_i \in X$. The $a_i$ are ordered and may include duplicates. Let an element $b \in X$ be said to be included in a vector $b \in \vec{a}$ if $\exists a_i, b = a_i$. Define subtraction of an element $a_i$ from a vector $\vec{a}$ as a new vector

$$\vec{a} - a_i := \langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \rangle$$

so that $|\vec{a} - a_i| = m - 1$.

Since a vector may contain duplicate elements $a_{i_1}, a_{i_2} \in \vec{a}$ with $a_{i_1} = a_{i_2}$, therefore each vector $\vec{a}$ determines a unique non-empty set $A$ constructed by including one instance of each element $a_i \in \vec{a}$, so that $b \in \vec{a} \leftrightarrow b \in A$, $1 \le |A| \le m$, and the quantity $|\vec{a}| - |A|$ is the number of elements of $\vec{a}$ which are duplicates.

Generalized convolution operators will be introduced, and denoted $\oplus \in \{+, -, \times, \div, {}^\wedge\}$ for addition, subtraction, multiplication, division, and exponentiation respectively. Let $\vee$ be the maximum and $\wedge$ the minimum operator.

## 4. RANDOM SETS, RANDOM INTERVALS, AND EVIDENCE THEORY

We now introduce the fundamental ideas of random sets and intervals.

DEFINITION 4.1 (GENERAL RANDOM SET). *Given a probability space $\langle X, \Sigma, \Pr \rangle$, then a function $S \colon X \to 2^\Omega - \{\emptyset\}$, where $-$ is set subtraction, is a random subset of $\Omega$ if $S$ is $\Pr$-measurable, so that $\forall A \subseteq \Omega, A \ne \emptyset, S^{-1}(A) \in \Sigma$.*

Random sets were originally developed as a branch of stochastic geometry, and their mathematics in general can be quite complex [1, 22]. But for our purposes, and especially in the finite case, they can be seen more simply as random variables taking values on subsets of $\Omega$. Further, they are mathematically isomorphic to bodies of evidence in Dempster-Shafer evidence theory [3, 31]. We now reintroduce random sets in this context.

DEFINITION 4.2 (EVIDENCE FUNCTION, BASIC ASSIGNMENT). *A function* $m: 2^\Omega \to [0, 1]$ *is an evidence function (basic assignment) when* $m(\emptyset) = 0$ *and* $\sum_{A \subseteq \Omega} m(A) = 1$.

DEFINITION 4.3 (FINITE RANDOM SET). *Given an evidence function* $m$, *then*

$$\mathcal{S} := \{\langle A_j, m_j \rangle : m_j > 0\}, \tag{1}$$

*is a finite random set where* $A_j \subseteq \Omega, m_j := m(A_j)$, *and* $1 \le j \le N := |\mathcal{S}| \le 2^n - 1$. *Denote the focal set of* $\mathcal{S}$ *as the class* $\mathcal{F}(\mathcal{S}) := \{A_j : m_j > 0\} \subseteq 2^\Omega$.

NOTE 4.4. *Each finite random set* $\mathcal{S}$ *determines a unique general random set* $S: \mathcal{F}(\mathcal{S}) \to 2^\Omega - \emptyset$ *defined on the probability space* $\langle \mathcal{F}(\mathcal{S}), 2^{\mathcal{F}(\mathcal{S})}, \Pr \rangle$, *where* $\Pr$ *is the measure determined by* $m$ *acting as its density function [16]. Moreover,* $\mathcal{S}$ *simply is a Dempster-Shafer body of evidence [13].*

We recognize random sets with the following special structures:

**Consistent:** The global intersection is non-empty: $\forall A_{j_1}, A_{j_2} \in \mathcal{F}(\mathcal{S}), A_{j_1} \not\perp A_{j_2} \leftrightarrow \mathbf{C}(\mathcal{F}(\mathcal{S})) \ne \emptyset$.

**Consonant:** Focal elements are all nested: $\forall A_{j_1}, A_{j_2} \in \mathcal{F}(\mathcal{S}), A_{j_1} \subseteq A_{j_2}$ or $A_{j_1} \subseteq A_{j_2}$.

**Disjoint:** No focal elements intersect: $\forall A_{j_1}, A_{j_2} \in \mathcal{F}(\mathcal{S}), A_{j_1} \perp A_{j_2}$.

**Specific:** All focal elements are singletons: $\forall A_j \in \mathcal{F}(\mathcal{S}), \exists! \omega \in \Omega, A_j = \{\omega\}$.

Note that consonance implies consistency, and specificity implies disjointness. Finally, disjointness implies a lack of consistency, and *vice versa*.

DEFINITION 4.5 (MONOTONE MEASURE, MONOTONE MEASURE TRACE). *[35] Assume a general universe of discourse* $\Omega$, *a class of subsets* $\mathcal{C} \subseteq 2^\Omega$, *and a sequence of such sets* $\{A_1, A_2, \ldots\} \subseteq \mathcal{C}$. *Then* $\nu: \mathcal{C} \to [0, 1]$ *is a monotone measure if*

*1.* $\nu(\emptyset) = 0$

*2.* **Monotonicity:**
$$\forall A, B \subseteq \Omega, \quad A \subseteq B \to \nu(A) \le \nu(B) \tag{2}$$

*3.* **Continuity from Below:**

$$A_1 \subseteq A_2 \subseteq \ldots \text{ and } \mathbf{U}(\mathcal{C}) \in \mathcal{C} \to \lim_{i \longrightarrow \infty} \nu(A_i) = \nu\left(\bigcup_{i=1}^\infty A_i\right).$$

*4.* **Continuity from Above:**

$$A_1 \supseteq A_2 \supseteq \ldots \ \text{ and } \ \mathbf{C}(\mathcal{C}) \in \mathcal{C} \to \lim_{i \longrightarrow \infty} \nu(A_i) = \nu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

$\nu$ *is normal when* $\nu(\Omega) = 1$*. Define the trace of a monotone measure* $\nu$ *as its "one-point covering function"* $q_\nu : \Omega \to [0,1]$*, with* $q_\nu(\omega) := \nu(\{\omega\})$*.*

DEFINITION 4.6 (EVIDENCE MEASURES). *The plausibility and belief measures on* $\forall A \subseteq \Omega$ *are*

$$\mathrm{Pl}(A) := \sum_{A_j \not\perp A} m(A_j), \qquad \mathrm{Bel}(A) := \sum_{A_j \subseteq A} m(A_j),$$

Pl and Bel are generally normal, non-additive monotone measures [35], and are dual, in that $\forall A \subseteq \Omega, \mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A})$.

Random intervals were introduced by Dempster [4].

DEFINITION 4.7 (FINITE RANDOM INTERVAL). *A finite random interval, denoted* $\mathcal{A}$*, is a finite random set on* $\Omega = \mathbb{R}$ *for which* $\mathcal{F}(\mathcal{A}) = \{I_j\} \subseteq \mathcal{I}, 1 \le j \le N$*.*

Thus a finite random interval is a finite random left-closed interval subset of $\mathbb{R}$.

Previously $\Omega$ had been postulated as a finite set, which leads to a great deal of mathematical simplicity. However, even though $\Omega$ is now uncountable, complications can still be avoided as long as $\mathcal{A}$ is finite, that is as long as only finitely many ($N$) focal elements are present. This is because each $I = [l, r) \subseteq \mathbb{R}$ is characterized completely by the two endpoints $l$ and $r$. With each new focal element $A_j$, $N$ grows by 1, and the total number of endpoints present in $\mathcal{F}(\mathcal{A})$ grows by at most 2. Thus the focal set of a finite random interval can be completely represented by the finite collection of these endpoints: $\mathcal{F}(\mathcal{A}) = \{I_j\} = \{[l_j, r_j)\}$. It is only these endpoints that need to be considered, and none of the properties of the continuum of points between them is significant.

On this basis we can describe the various components of a random interval. In general denote $I_j = [l_j, r_j)$. Then, denote the vector of all endpoints $\vec{L} := \langle l_1, r_1, l_2, l_2, \ldots, l_j, r_j, \ldots, l_N, r_N \rangle$, and let $L := \{x_k\}$ be the set derived from eliminating duplicates from $\vec{L}$, with $\forall x_k \in L, \exists x_j \in \vec{L}, x_k = x_j$ and $1 \le k \le Q := |L|, N + 1 \le Q \le 2N = |\vec{L}|$.

The elements of $L$ determine a class $\Gamma = \{G_k\} \subseteq \mathcal{I}$, now with $1 \le k \le Q - 1$, which is the finest partition of the support $\mathbf{U}(\mathcal{A})$ induced by the total intersections of the $I_j$ with each other and with all their intersections recursively. In practice, the $G_k$ are determined simply by ordering the $x_k \in L$ and then traversing them from $\min x_k$ rightward, forming an interval for each point in turn.

An example is shown in Fig. 3, with $N = 4$, $\mathcal{F}(\mathcal{A}) = \{[3.5, 4), [1, 2), [3, 4), [2, 3.5)\}$, and $m$ is as shown. Here $Q = 5$, with $\vec{L} = \langle 3.5, 4, 1, 2, 3, 4, 2, 3.5 \rangle$, $L = \{1, 2, 3, 3.5, 4\}$, and thus $\Gamma = \{[1, 2), [2, 3), [3, 3.5), [3.5, 4)\}$.

Our definition differs somewhat from others in the literature [9] who use fully closed intervals. But not only is the Borel field $\mathcal{I}$ more consistent with that of measure theory [14, 35], it also makes the algebraic manipulations of the $I_j$ much easier, since e.g. for $x \le y \le z, [x, y) \cap [y, z) = \emptyset$.
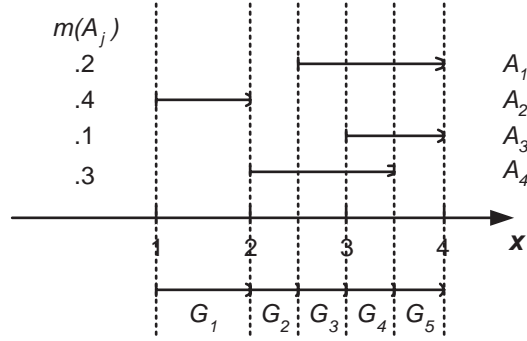
**Figure 3.** Example of a finite random interval.

In real problems, random intervals are derived from collections of observed intervals. In Joslyn's formalism for random interval measurement [17], the values $m(I_j)$ are derived by their relative frequency in this observation record. But depending on the application, it may or may not be likely that two identical intervals $I, I' = [l, r)$ will be observed, as distinct from another interval $I'$ "very close" to $I$, for example $I' = [l + \epsilon, r - \delta)$ for some very small $\epsilon, \delta$. In this case, as $N$ grows each $I_j$ is observed once, but with increasing refinement among the endpoints $\{l_j, r_j\}$. Thus it is common in real applications to deal with random intervals where all the focal elements $I_j \in \mathcal{F}(\mathcal{A})$ are distinct with $Q \sim 2N$, and therefore each with frequency $m(I_j) = 1/N$.

Yager [36] introduced convolution operators on random intervals.

DEFINITION 4.8 (RANDOM INTERVAL CONVOLUTION). *Assume two independent random intervals $\mathcal{A}_1 = \{\langle I_j, m_j \rangle\}, 1 \le j \le N_1, \mathcal{A}_2 = \{\langle I_k, m_k \rangle\}, 1 \le k \le N_2$, and a convolution operator $\oplus$. Then $\mathcal{A}_1 \oplus \mathcal{A}_2 := \{\langle I_l, m_l \rangle\}$ where:*

$$1 \le l \le N_1 N_2, \qquad I_l = \{z = x \oplus y, x \in I_j, y \in I_k\}, \qquad m_l = m_j m_k.$$

## 5. PROBABILITY BOXES

Random intervals can be difficult structures to elicit, represent, and manipulate. The first of the approximations we introduce are so-called **probability boxes**, or just **p-boxes**.

DEFINITION 5.1 (PROBABILITY BOX (P-BOX)). *A p-box [8] is a structure $\mathcal{B} := \langle \underline{B}, \overline{B} \rangle$, where $\underline{B}, \overline{B} : \mathbb{R} \to [0, 1]$ with:*

*1.* $\lim_{x \longrightarrow -\infty} \underline{B}(x) \longrightarrow 0, \quad \lim_{x \longrightarrow \infty} \underline{B}(x) \longrightarrow 1$

*2.* $\lim_{x \longrightarrow -\infty} \overline{B}(x) \longrightarrow 0, \quad \lim_{x \longrightarrow \infty} \overline{B}(x) \longrightarrow 1$

*3.* $\underline{B}(x), \overline{B}(x)$ *are non-decreasing in $x$, and*

*4.* $\underline{B} \le \overline{B}$.

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$\underline{B}$ and $\overline{B}$ are interpreted as bounds on cumulative distribution functions (CDFs). In other words, $\mathcal{B} = \langle \underline{B}, \overline{B} \rangle$ can be identified with the set of all functions $\{F : \underline{B} \leq F \leq \overline{B}\}$ such that $F$ is the CDF of some probability measures Pr on $\mathbb{R}$. For each such $F$, denote $F \in \mathcal{B}$. In this way, each p-box defines such a class of probability measures.

DEFINITION 5.2 (P-BOX CONVOLUTION). *Assume two p-boxes $\mathcal{B}_1, \mathcal{B}_2$, and a convolution operator $\oplus$. When $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent, then define*

$$(\mathcal{B}_1 \oplus \mathcal{B}_2)(z) := \left\{ \int_{x \oplus y \leq z} d\Pi(F(x), G(y)) : F \in \mathcal{B}_1, G \in \mathcal{B}_2 \right\},$$

*where $\Pi(u, v) = uv$ is the product copula [29].*

Each random interval naturally generates a p-box.

THEOREM 5.3. *Given a random interval $\mathcal{A}$, then $\mathcal{B}(\mathcal{A}) := \langle \mathrm{BEL}, \mathrm{PL} \rangle$ is a P-Box, where BEL and PL are the "cumulative belief and plausibility distributions" $\mathrm{PL}, \mathrm{BEL} \colon \mathbb{R} \to [0, 1]$ originally defined by Yager [36]*

$$\mathrm{BEL}(x) := \mathrm{Bel}([-\infty, x)), \quad \mathrm{PL}(x) := \mathrm{Pl}([-\infty, x)).$$

*Proof.* Assume a random interval $\mathcal{A}$. We need to show:

1. First,

$$\lim_{x \to -\infty} \mathrm{BEL}(x) = \lim_{x \to -\infty} \mathrm{Bel}([-\infty, x)) = \mathrm{Bel}\left( \lim_{x \to -\infty} [-\infty, x) \right) = \mathrm{Bel}([-\infty, -\infty)) = \mathrm{Bel}(\emptyset) = 0.$$

Similarly,

$$\lim_{x \to \infty} \mathrm{BEL}(x) = \lim_{x \to \infty} \mathrm{Bel}([-\infty, x)) = \mathrm{Bel}\left( \lim_{x \to \infty} [-\infty, x) \right) = \mathrm{Bel}([-\infty, \infty)) = \mathrm{Bel}(\mathbb{R}) = 1.$$

The results $\lim_{x \to -\infty} \mathrm{PL}(x) = 0, \lim_{x \to \infty} \mathrm{PL}(x) = 1$ follow identically.

2. Since $x \leq y \to [-\infty, x) \subseteq [-\infty, y)$, and since Bel and Pl are monotone measures, therefore from monotone measure monotonicity $x \leq y \to \mathrm{BEL}(x) \leq \mathrm{BEL}(y)$, therefore BEL(x) is monotone non-decreasing in $x$. And similarly for PL.

3. $\forall I \in \mathcal{I}, \mathrm{Bel}(I) \leq \mathrm{Pl}(I)$, and thus in particular $\forall x \in \mathbb{R}, \mathrm{Bel}([-\infty, x)) \leq \mathrm{Pl}([-\infty, x))$, and so $\forall x \in \mathbb{R}, \mathrm{BEL}(x) \leq \mathrm{PL}(x)$.

Therefore $\langle \mathrm{BEL}, \mathrm{PL} \rangle$ is a p-box.

The p-box generated from the example random interval is shown in Fig. 4. Since $\overline{B}$ and $\underline{B}$ partially overlap, the diagram is somewhat ambiguous on its far left and right portions, but note that

$$\overline{B}([-\infty, 1)) = 0, \quad \underline{B}([-\infty, 2, )) = 0, \quad \overline{B}([3, \infty)) = 1, \quad \underline{B}([3.5, \infty)) = 1.$$
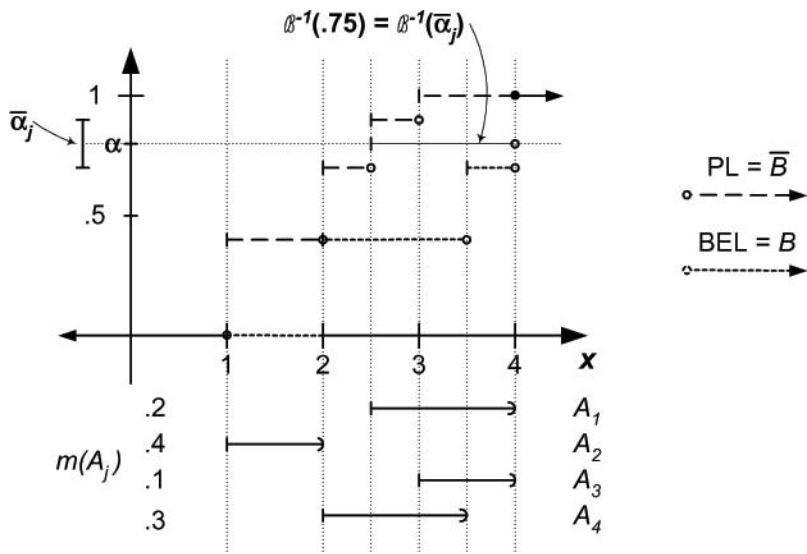
**Figure 4.** A finite random interval and its piecewise-constant p-box $\mathcal{B}(\mathcal{A})$.
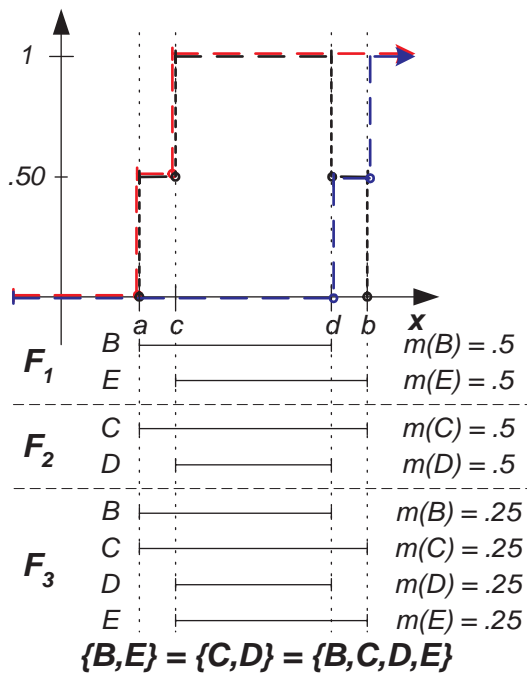


**Figure 5.** Three different random intervals and their common p-box and trace.

But for the converse, each p-box determines only an equivalence class of random intervals. Consider the example shown in Fig. 5 for $a < c < d < b \in \mathbb{R}, \mathcal{C} = \{B = [a,d), C = [a,b), D = [c,d), E = [c,b)\}$, and three different focal classes $\mathcal{F}_1 = \{B, E\}, \mathcal{F}_2 = \{C, D\}$, and $\mathcal{F}_3 = \{B, C, D, E\}$ with their respective $m_1, m_2$, and $m_3$ are shown. We have $\mathcal{B}(\mathcal{A}_1) = \mathcal{B}(\mathcal{A}_2) = \mathcal{B}(\mathcal{A}_3)$.

Thus for a given p-box $\mathcal{B}$, we can denote $\mathcal{A}(\mathcal{B})$ as the equivalence class of random intervals consistent with it: $\mathcal{A}(\mathcal{B}) := \{\mathcal{A} : \mathcal{B}(\mathcal{A}) = \mathcal{B}\}$.

$\underline{B}$ and $\overline{B}$ have inverses under reasonable conditions. Assume that $\underline{B}$ and $\overline{B}$ are piecewise continuous from the left. Then define the quasi-inverses

$$\underline{B}^{-1}(\alpha) := \underset{x \in \mathbb{R}}{\operatorname{argmin}} |\alpha - \underline{B}(x)|, \qquad \overline{B}^{-1}(\alpha) := \underset{x \in \mathbb{R}}{\operatorname{argmin}} |\alpha - \overline{B}(x)|,$$

for $\alpha \in [0,1]$, and

DEFINITION 5.4 (P-BOX INVERSE). *Given a p-box $\mathcal{B}$, let $\mathcal{B}^{-1} \colon [0,1] \to \mathcal{I}$ where $\forall \alpha \in [0,1]$*

$$\mathcal{B}^{-1}(\alpha) := \left\{ \left[ \overline{B}^{-1}(\alpha), \underline{B}^{-1}(\alpha) \right) \right\}.$$

Condition 4 of (5.1) guarantees that for each $\alpha = [0,1], \mathcal{B}^{-1}(\alpha)$ exists and is a member of $\mathcal{I}$. When $\underline{B}$ and $\overline{B}$ are piecewise-constant, $\mathcal{B}^{-1}$ naturally partitions $[0,1]$ into disjoint intervals denoted $\bar{\alpha}_j$ over which $\forall \alpha, \alpha' \in \bar{\alpha}_j, \mathcal{B}^{-1}(\alpha) = \mathcal{B}^{-1}(\alpha')$. In practice, denote $\bar{\alpha}_j := \left[ \alpha_j^l, \alpha_j^r \right]$, where

$$\alpha_j^l = \underset{x \in \mathbb{R}}{\operatorname{argmin}} \overline{B}(x) \geq \alpha, \qquad \alpha_j^u = \underset{x \in \mathbb{R}}{\operatorname{argmax}} \underline{B}(x) \leq \alpha.$$

This is shown in Fig. 4.

Given a piecewise-constant p-box, there is a canonical way to construct a random interval consistent with it.

DEFINITION 5.5 (CANONICAL RANDOM INTERVAL FROM P-BOX). *Assume a p-box $\mathcal{B}$. Then construct $\mathcal{A}^*(\mathcal{B}) := \{\langle \mathcal{B}^{-1}(\bar{\alpha}_j), m_j \rangle\}$, where $\mathcal{B}^{-1}(\bar{\alpha}_j) := \mathcal{B}^{-1}(\alpha_j^l) = \mathcal{B}^{-1}(\alpha_j^r)$ and $m_j = \alpha_j^l - \alpha_j^r$.*

THEOREM 5.6. *$\mathcal{A}^*(\mathcal{B})$ is a random interval, and $\mathcal{A}^*(\mathcal{B}) \in \mathcal{A}(\mathcal{B})$.*

*Proof.* It is evident from the definitions (5.4) and (5.5) that each $\bar{\alpha}_j \in \mathcal{I}$. Also, since the $\hat{\alpha}_j$ partition $[0,1]$, therefore

$$\sum_j m_j = \sum_j \left( \alpha_j^l - \alpha_j^r \right) = 1.$$

It is relatively easy to see in Fig. 4 that $\mathcal{A}^*(\mathcal{B}(\mathcal{A})) = \mathcal{A}$, although we know that this is not always so.

## 6. RANDOM INTERVAL TRACES

A fuzzy (sub)set of $\Omega$, denoted $\widetilde{A} \widetilde{\subseteq} \Omega$, is determined by its membership function, which is any function of the form $\mu_{\widetilde{A}} \colon \Omega \to [0,1]$. Denote the core of a fuzzy set as $\mathbf{C}(\mu) := \{\omega \in \Omega : \mu(\omega) = 1\}$.

The value of $\mu_{\widetilde{F}}(\omega_i)$ indicates the degree or extent to which $\omega_i \in \Omega$. Fuzzy sets generalize classical (crisp) sets in that a subset $A \subseteq \Omega$ has a memberhsip function defined as the characteristic function $\mu_A := \chi_A$. In the sequel, let each fuzzy set be considered to be a fuzzy subset of the reals $\widetilde{A} \widetilde{\subseteq} \mathbb{R}$.

The trace of any monotone measure defined on $\mathbb{R}$ is a fuzzy set.

COROLLARY 6.1. *Given a monotone measure $\nu$, then $q_\nu$ is a membership function.*

*Proof.* Follows trivially from the definition of trace (4.5).

Fuzzy sets also have convolutions.

DEFINITION 6.2 (FUZZY SET CONVOLUTION). *Assume two fuzzy intervals $\widetilde{A}_1, \widetilde{A}_2$, a convolution operator $\oplus$, and a T-norm $\sqcap$. Let $\widetilde{A}_3 = \widetilde{A}_1 \oplus \widetilde{A}_2$. Then*

$$\mu_{\widetilde{A}3}(z) := \bigvee_{x \oplus y = z} \mu_{\widetilde{A}1}(x) \sqcap \mu_{\widetilde{A}2}(y).$$

There are two special kinds of fuzzy subsets which are of particular interest to us.

DEFINITION 6.3 (FUZZY INTERVAL). *[5, 6] A fuzzy subset of the real line $\widetilde{F} \widetilde{\subseteq} \mathbb{R}$ is a* **fuzzy interval** *if $\widetilde{F}$ is maximally normalized and convex, so that*

$$\forall x, y \in \mathbb{R}, \quad \forall z \in [x, y], \quad \mu_{\widetilde{F}}(z) \geq \mu_{\widetilde{F}}(x) \wedge \mu_{\widetilde{F}}(y).$$

Note that convexity here implies unimodality in the weak sense that $\mathbf{C}(\widetilde{F})$ is a closed interval. This goes to a limit for fuzzy numbers.

DEFINITION 6.4 (FUZZY NUMBER). *A* **fuzzy number** *is a fuzzy interval $\widetilde{F}$ where $\exists x \in \mathbb{R}, \mathbf{C}(\widetilde{F}) = \{x\}$.*

So each random interval naturally generates a trace.

DEFINITION 6.5 (RANDOM INTERVAL TRACE). *Given a random interval $\mathcal{A}$, define the function $\rho_{\mathcal{A}} \colon \mathbb{R} \to [0,1]$ as the plausibilistic trace, or just trace, of $\mathcal{A}$, where $\rho_{\mathcal{A}} = q_{\mathrm{Pl}}$. Therefore*

$$\forall x \in \mathbb{R}, \quad \rho_{\mathcal{A}}(x) := \mathrm{Pl}(\{x\}) = \sum_{A_j \ni x} m_j. \tag{3}$$

An example is shown in Fig. 6, with $\mathcal{A}$ as before, and $\rho$ shown in the top of the figure.

But for the converse, each fuzzy subset of $\mathbb{R}$ determines only an equivalence class of random intervals. Consider again the example shown in Fig. 5. Each of the three random intervals $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ generates exactly the same trace, here shown in the bold, dashed, "step-pyramid" shaped curve.

So for a given fuzzy set $\widetilde{F}$, denote $\mathcal{A}\left(\widetilde{F}\right)$ as the equivalence class of random intervals consistent with it: $\mathcal{A}(\widetilde{F}) := \left\{\mathcal{A} : \rho(\mathcal{A}) = \widetilde{F}\right\}$. The structure of this equivalence class
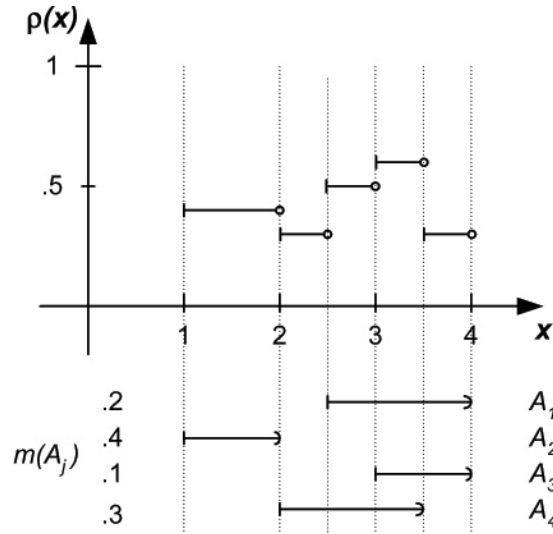
**Figure 6.** Example random interval with its trace and its components.

is not simple, and has been dealt with in depth by Goodman and his colleagues [10–12]. Furthermore, they have shown that operations on fuzzy sets are preserved when projected through the random set space.

Joslyn has shown the basis to derive fuzzy mathematics from (empirically derived) random intervals [17]. First, $\rho$ is constant over each $G_k \subseteq \mathbb{R}$. But moreoever:

THEOREM 6.6. *[17] The trace $\rho_{\mathcal{A}}$ of a random interval $\mathcal{A}$ is a fuzzy interval iff $\mathcal{A}$ is consistent.*

This is important because fuzzy intervals generalize crisp intervals as fuzzy sets generalize crisp sets. They are also the basis for "fuzzy arithmetic", since the set of fuzzy intervals is closed under convolution. In addition:

PROPOSITION 6.7. *Given two fuzzy intervals $\widetilde{F}_1, \widetilde{F}_2$, a convolution operator $\oplus$, and a T-norm $\sqcap$, then $\widetilde{F}_1 \oplus \widetilde{F}_2$ is not necessarily a fuzzy interval. However,*

$$\mathbf{C}\left(\widetilde{F}_1 \oplus \widetilde{F}_2\right) = \mathbf{C}\left(\widetilde{F}_1\right) \oplus \mathbf{C}\left(\widetilde{F}_2\right), \quad \mathbf{U}\left(\widetilde{F}_1 \oplus \widetilde{F}_2\right) = \mathbf{U}\left(\widetilde{F}_1\right) \oplus \mathbf{U}\left(\widetilde{F}_2\right).$$

## 7. P-BOXES AND TRACES

We now begin to explore the relations among the categories of random intervals and their trace and p-box representations. These are diagrammed in Fig. 7.

First, a given p-box determine a trace uniquely.

DEFINITION 7.1 (TRACE OF A P-BOX). *Assume a p-box $\mathcal{B}$. Then its trace, denoted $\rho(\mathcal{B})$, is determined by $\rho(\mathcal{B}) := \overline{B} - \underline{B}$.*

The trace determined in this way from the p-box of a random interval is the same as the trace of the random interval itself, as we will now show.
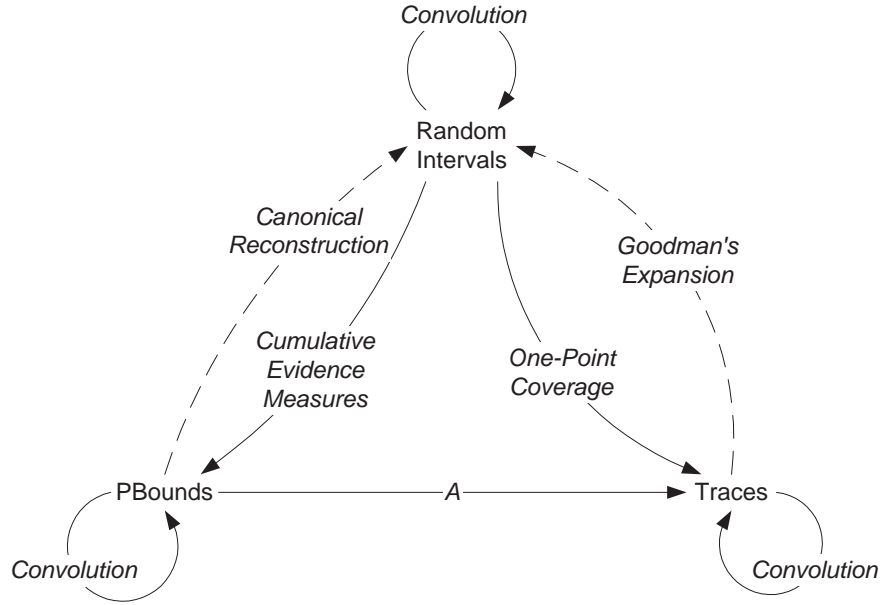
**Figure 7.** Relations among random intervals, p-boxes, and traces.

THEOREM 7.2. *For all random intervals $\mathcal{A}$, $\rho(\mathcal{B}(\mathcal{A})) = \rho(\mathcal{A})$.*

*Proof.* Assume a random interval $\mathcal{A}$. Fix a point $x \in \mathbb{R}$. Then

$$\mathrm{PL}(x) = \mathrm{Pl}((\infty, x)) = \sum_{I_j \not\subset (\infty, x]} m(I_j) = \sum_{l_j \leq x} m(I_j)$$

$$\mathrm{BEL}(x) = \mathrm{Bel}((\infty, x]) = \sum_{I_j \subseteq (\infty, x]} m(I_j) = \sum_{x \geq r_j} m(I_j). \tag{4}$$

Then from (5.5) and (7.1),

$$\rho(\mathcal{B}(\mathcal{A}))(x) = \mathrm{PL}(x) - \mathrm{BEL}(x) = \sum_{l_j \leq x < r_j} m(I_j) = \sum_{x \in I_j} m(I_j) = \rho(\mathcal{A})(x). \tag{5}$$

Note how crucial the use of half-open intervals is. The weak inequality in (4) results through subtraction in the appropriate half-open interval in (5), and this would have been the case whether the $I_j$ were closed or not. These results can be checked with some simple diagrammatic reasoning between Fig. 4 and Fig. 6.

But conversely, it might be that the trace of a random interval has multiple p-boxes which could generate it.

## 8. FUTURE WORK

Future development requires the following considerations:

- Given that $\mathcal{B} \to \rho$, then it should be that $\mathcal{A}(\mathcal{B}) \subseteq \mathcal{A}(\mathcal{B}(\rho))$. What about the converse?

- For a given $\mathcal{A}$, compare $\mathcal{A}(\mathcal{B}(\mathcal{A}))$ and $\mathcal{A}(\rho(\mathcal{A}))$.

Comparison of canonical reconstructions:

- For a given $\mathcal{A}$, compare $\mathcal{A}^*(\mathcal{B}(\mathcal{A}))$ and $\mathcal{A}^*(\rho(\mathcal{A}))$.

- Keep going: compare $\rho(\mathcal{A}^*(\mathcal{B}))$ and $\mathcal{B}(\mathcal{A}^*(\rho))$.

Convolutions. Similar questions for convolutions all around. In particular:

- Compare $\mathcal{B}(\mathcal{A}_1 \oplus \mathcal{A}_2)$ with $\mathcal{B}(\mathcal{A}_1) \oplus \mathcal{B}(\mathcal{A}_2)$.

- Compare $\mathcal{A}^*(\mathcal{B}(\mathcal{A}_1 \oplus \mathcal{A}_2))$ with $\mathcal{A}^*(\mathcal{B}(\mathcal{A}_1) \oplus \mathcal{B}(\mathcal{A}_2))$.

- Compare $\rho(\mathcal{A}_1 \oplus \mathcal{A}_2)$ with $\rho(\mathcal{A}_1) \oplus \rho(\mathcal{A}_2)$.

- Compare $\mathcal{A}^*(\rho(\mathcal{A}_1 \oplus \mathcal{A}_2))$ with $\mathcal{A}^*(\rho(\mathcal{A}_1) \oplus \rho(\mathcal{A}_2))$.

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