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in intense Charged Particle Beams
with Large Energy Anisotropy**

by

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Electromagnetic Weibel instability in intense charged particle beams with large energy anisotropy

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Abstract

In plasmas with strongly anisotropic distribution functions, collective instabilities may develop if there is sufficient coupling between the transverse and longitudinal degrees of freedom. Our previous numerical and theoretical studies of intense charged particle beams with large temperature anisotropy [E. A. Startsev, R. C. Davidson and H. Qin, *PRSTAB*, **6**, 084401 (2003); *Phys. Plasmas* **9**, 3138 (2002)] demonstrated that a fast, electrostatic, Harris-like instability develops, and saturates nonlinearly, for sufficiently large temperature anisotropy ($T_{\perp b}/T_{\parallel b} \gg 1$). The total distribution function after saturation, however, is still far from equipartitioned. In this paper the linearized Vlasov-Maxwell equations are used to investigate detailed properties of the transverse electromagnetic Weibel-type instability for a long charge bunch propagating through a cylindrical pipe of radius r_w . The kinetic stability analysis is carried out for azimuthally symmetric perturbations about a two-temperature thermal equilibrium distribution in the smooth-focusing approximation. The most unstable modes are identified, and their eigenfrequencies, radial mode structure and instability thresholds are determined. The stability analysis shows that, although there is free energy available to drive the electromagnetic Weibel instability, the finite transverse geometry of the charged particle beam introduces a large threshold value for the temperature anisotropy ($(T_{\perp b}/T_{\parallel b})^{Weibel} \gg (T_{\perp b}/T_{\parallel b})^{Harris}$) below which the instability is absent. Hence, unlike the case of an electrically neutral plasma, the Weibel instability is not expected to play as significant a role in the process of energy isotropization of intense unneutralized charged particle beams as the electrostatic Harris-type instability.

I. INTRODUCTION

Periodic focusing accelerators, transport systems and storage rings [1–6] have a wide range of applications ranging from basic scientific research in high energy and nuclear physics, to applications such as heavy ion fusion, spallation neutron sources, tritium production and nuclear waste transmutation, to mention a few examples. Of particular importance at the high beam currents and charge densities of practical interest, are the effects of the intense self fields produced by the beam space charge and current on determining the detailed equilibrium, stability and transport properties. While considerable progress has been made in understanding the self-consistent evolution of the beam distribution function, $f_b(\mathbf{x}, \mathbf{p}, t)$, and self-generated electric and magnetic fields, $\mathbf{E}^s(\mathbf{x}, t)$ and $\mathbf{B}^s(\mathbf{x}, t)$, in kinetic analysis based on the nonlinear Vlasov-Maxwell equations [1, 7–12], in numerical simulation studies of intense beam propagation [13–23], and in macroscopic warm-fluid models [24–27], the effects of finite geometry and space-charge effects often make predictions of detailed stability behavior difficult. It is therefore important to develop an improved understanding of fundamental collective stability properties, including the case where a large temperature anisotropy ($T_{\perp b} \gg T_{\parallel b}$) can drive electrostatic Harris-like [28] and/or electromagnetic Weibel-like [29] instabilities, familiar in the study of electrically neutral plasmas.

It is well known that in neutral plasmas with strongly anisotropic distributions ($T_{\parallel b}/T_{\perp b} \ll 1$) collective instabilities may develop if there is sufficient coupling between the transverse and longitudinal degrees of freedom [28, 29]. Such anisotropies develop naturally in accelerators. Indeed, due to conservation of energy for particles with charge e_b and mass m_b accelerated by a voltage V , the energy spread of particles in the beam does not change, and (nonrelativistically) $\Delta E_{bi} = m_b \Delta v_{bi}^2 / 2 = \Delta E_{bf} \simeq m_b V_b \Delta v_{bf}$, where $V_b = (e_b V / m_b)^{1/2}$ is the average beam velocity after acceleration. Therefore, the longitudinal velocity spread-squared, or equivalently the temperature, changes according to $T_{\parallel bf} \simeq T_{\parallel bi}^2 / 2e_b V$ (for a nonrelativistic beam). At the same time, the transverse temperature may increase due to nonlinearities in the applied and self-field forces, nonstationary beam profiles, and beam mis-

match. These processes provide the free energy to drive collective instabilities and may lead to a deterioration of beam quality [22, 30, 31]. Such instabilities may also lead to an increase of longitudinal velocity spread, which will make the focusing of the beam difficult and may impose a limit on the minimum spot size achievable in heavy ion fusion experiments.

Previous studies have mostly focused on electrostatic Harris-like anisotropy-driven instability for long, coasting beams [9, 12–14, 25, 32–36]. It has been shown that moderately intense beams with normalized beam intensity $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_f^2 \gtrsim 0.5$ are linearly unstable to short-wavelength, perturbations with $k_z^2 r_b^2 \gtrsim 1$, provided the ratio of longitudinal to transverse temperatures ($T_{\parallel b}/T_{\perp b}$) is smaller than some small threshold value. Here, $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b e_b^2/\gamma_b m_b$ is the relativistic plasma frequency-squared, \hat{n}_b is the on-axis number density of beam particles, $\gamma_b = (1 - V_b^2/c^2)^{-1/2}$ is the relativistic mass factor, $V_b = \beta_b c$ is the average beam velocity in the axial direction and $\omega_f = \text{const.}$ is the smooth-focusing frequency associated with the applied focusing field. Our recent studies [12–14] have demonstrated that, although the electrostatic, Harris-like instability develops in intense charged particle beams with $T_{\perp b}/T_{\parallel b} \gg 1$, it saturates for relatively large values of temperature anisotropy, and the total distribution function after saturation is still far from equipartitioned. It is well known in large-size electrically neutral plasmas, that the transverse electromagnetic Weibel instability can occur even for small temperature anisotropy. The purpose of the present paper is to study the Weibel instability mechanism for intense noneutral charged particle beams. While the Weibel instability [29] has been extensively investigated for counterstreaming, anisotropic, electrically neutral plasmas [37], intense electron beam propagation through a large-volume neutralizing plasma background [38–40], for fast ignition schemes where an intense charged particle beam interacts with a dense target plasma [41–45], and for interpenetrating electron-positron plasmas [46], to the best of our knowledge the present analysis represents the first theoretical investigation of the Weibel instability for a one-component, anisotropic ($T_{\perp b} \gg T_{\parallel b}$) charged particle beam, including the important effects of intense space-charge fields.

In Sec. II, a simplified kinetic theory of the Weibel instability is presented for axially

symmetric ($\partial/\partial\theta = 0$) transverse electromagnetic perturbations about a bi-Maxwellian distribution of beam particles. The analysis leads to the matrix dispersion equation (20) derived from the linearized Vlasov-Maxwell equations, which constitutes a relatively straightforward generalization of the analysis of the Weibel instability for an electrically neutral plasma to the case of a charged particle beam with intense self fields. The most unstable modes are identified, and their eigenfrequencies, and radial mode structure are determined. The instability threshold is obtained, and the relative importance of the Weibel instability and the Harris instability for anisotropic, intense charged particle beams is discussed.

II. LINEAR STABILITY THEORY

We briefly outline here a simple derivation of the dispersion relation for the Weibel instability in an intense charged particle beam. The analysis assumes transverse electromagnetic perturbations about a thermal equilibrium distribution with temperature anisotropy ($T_{\perp b} > T_{\parallel b}$) described in the beam frame ($V_b = 0$ and $\gamma_b = 1$) by the self-consistent axisymmetric Vlasov equilibrium [1, 11]

$$f_b^0(r, \mathbf{p}) = \frac{\hat{n}_b}{(2\pi m_b T_{\perp b})} \exp\left(-\frac{H_{\perp}}{T_{\perp b}}\right) \frac{1}{(2\pi m_b T_{\parallel b})^{1/2}} \exp\left(-\frac{p_z^2}{2m_b T_{\parallel b}}\right). \quad (1)$$

Here, $H_{\perp} = p_{\perp}^2/2m_b + (1/2)m_b\omega_f^2(x^2 + y^2) + e_b\phi^0(r)$ is the single-particle Hamiltonian for the transverse particle motion, $p_{\perp} = (p_x^2 + p_y^2)^{1/2}$ is the transverse particle momentum, $r = (x^2 + y^2)^{1/2}$ is the radial distance from the beam axis, $\omega_f = \text{const.}$ is the transverse frequency associated with the applied focusing field in the smooth-focusing approximation, and $\phi^0(r)$ is the equilibrium space-charge potential determined self-consistently from Poisson's equation,

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \phi^0}{\partial r} = -4\pi e_b n_b^0, \quad (2)$$

where $n_b^0(r) = \int d^3p f_b^0(r, \mathbf{p})$ is the equilibrium number density of beam particles. For simplicity, the analysis is carried out in the beam frame ($V_b = 0$ and $\gamma_b = 1$). Furthermore, setting $\phi^0(r=0) = 0$, the constant \hat{n}_b occurring in Eq. (1) can be identified with the on-axis density $n_b^0(r=0)$, and the constants $T_{\perp b}$ and $T_{\parallel b}$ can be identified with the transverse and

longitudinal temperatures (energy units), respectively. Finally, e_b and m_b are the charge and mass, respectively, of a beam particle.

For present purposes, we consider small-amplitude, axisymmetric ($\partial/\partial\theta = 0$) electromagnetic perturbations of the form

$$\delta\mathbf{A}(\mathbf{x}, t) = \delta A_\theta(\mathbf{x}, t)\mathbf{e}_\theta = \mathbf{e}_\theta\delta\hat{A}_\theta(r)\exp(ik_z z - i\omega t), \quad (3)$$

where $\delta\mathbf{A}(\mathbf{x}, t)$ is the perturbed vector potential, k_z is the axial wavenumber, and ω is the complex oscillation frequency, with $\text{Im}\omega > 0$ corresponding to instability (temporal growth). Note that we have chosen the Coulomb gauge with $\nabla \cdot \delta\mathbf{A} = 0$. The nonzero components of the electromagnetic field perturbations can be expressed in terms of the vector potential as

$$\begin{aligned} \delta E_\theta &= \frac{i\omega}{c}\delta A_\theta, \\ \delta B_r &= -ik_z\delta A_\theta, \\ \delta B_z &= \frac{1}{r}\frac{\partial}{\partial r}(r\delta A_\theta). \end{aligned} \quad (4)$$

Note that the fields in Eq. (4) are purely electromagnetic with transverse polarization. Indeed, $\nabla \cdot \delta\mathbf{E} = 4\pi e_b\delta n_b = 0$, and there is no charge perturbation, although the current perturbation, $\delta\mathbf{J}_b = e_b \int d^3p \mathbf{v} \delta f_b$ is generally non-zero. The linearized Maxwell's equation for δA_θ can be expressed as

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r \delta\hat{A}_\theta(r) - k_z^2 \delta\hat{A}_\theta(r) + \frac{\omega^2}{c^2} \delta\hat{A}_\theta(r) = -\frac{4\pi e_b}{c} \int d^3p v_\theta \delta\widehat{f}_b(r, \mathbf{p}), \quad (5)$$

where $\delta\widehat{f}_b(r, \mathbf{p})$ is the Fourier amplitude of the perturbed distribution function $\delta f_b(r, z, \mathbf{p}, t)$, i.e.,

$$\delta f_b(r, z, \mathbf{p}, t) = \delta\widehat{f}_b(r, \mathbf{p}) \exp(ik_z z - i\omega t). \quad (6)$$

The perturbed distribution function $\delta f_b(r, z, \mathbf{p}, t)$ satisfies the linearized Vlasov equation [1]

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - (e_b \nabla_\perp \phi^0 + m_b \omega_f^2 \mathbf{x}_\perp) \cdot \frac{\partial}{\partial \mathbf{p}_\perp} \right\} \delta f_b = -e_b (\delta\mathbf{E} + \frac{1}{c} \mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{p}} f_b^0. \quad (7)$$

Using Eqs. (4), we can rewrite the right-hand side of Eq. (7) as

$$\begin{aligned} e_b(\delta\mathbf{E} + \frac{1}{c}\mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial}{\partial\mathbf{p}}f_b^0 &= \frac{e_b(xp_y - yp_x)}{m_b r} \left\{ \delta E_\theta \frac{\partial f_b^0}{\partial H_\perp} + \frac{\delta B_r}{c} \left(v_z \frac{\partial}{\partial H_\perp} - \frac{\partial}{\partial p_z} \right) f_b^0 \right\} \\ &= \frac{ie_b}{m_b c} \frac{\delta A_\theta}{r} P_\theta \left\{ (\omega - k_z v_z) \frac{\partial f_b^0}{\partial H_\perp} + k_z v_z \frac{\partial f_b^0}{\partial H_\parallel} \right\}. \end{aligned} \quad (8)$$

Here, $H_\parallel = p_z^2/2m_b$, $v_z = p_z/m_b$, and $P_\theta = rp_\theta = xp_y - yp_x = \text{const.}$ is the canonical angular momentum of a beam particle moving in the axisymmetric equilibrium configuration. Finally, using the method of characteristics [1], the solution to the linearized Vlasov equation (7) can be expressed as

$$\begin{aligned} \widehat{\delta f_b}(r, \mathbf{p}) &= -\frac{e_b}{cm_b} P_\theta \left[(\omega - k_z v_z) \frac{\partial f_b^0}{\partial H_\perp} + k_z v_z \frac{\partial f_b^0}{\partial H_\parallel} \right] \\ &\quad \times i \int_{-\infty}^t dt' \frac{\delta \widehat{A}_\theta[r'(t')]}{r'(t')} \exp\{i(k_z v_z - \omega)(t' - t)\} \end{aligned} \quad (9)$$

for perturbations about the choice of the anisotropic thermal equilibrium distribution function in Eq. (1). In the orbit integral in Eq. (9), $Im\omega > 0$ is assumed, and $r'(t') = [x'^2(t') + y'^2(t')]^{1/2}$ is the transverse orbit in the equilibrium field configuration such that $[\mathbf{x}'_\perp(t'), \mathbf{p}'_\perp(t')]$ passes through the phase-space point $(\mathbf{x}_\perp, \mathbf{p}_\perp)$ at time $t' = t$ [1]. The function $r'(t')$ is a periodic function of t' with period $T_r = 2\pi/\omega_r$, where $\omega_r(H_\perp, P_\theta)$ is the frequency of radial oscillations. We expand in the Fourier series representation

$$\frac{\delta \widehat{A}_\theta[r'(t')]}{r'(t')} = \sum_n \exp[in\omega_r(t' - t'_{min})] \int_0^{T_r} \frac{d\tau}{T_r} \frac{\delta \widehat{A}_\theta[\widehat{r}(\tau)]}{\widehat{r}(\tau)} \exp\{-in\omega_r\tau\}, \quad (10)$$

where $\widehat{r}(\tau)$ is defined by the equation

$$\tau = \int_{r'_{min}}^{\widehat{r}(\tau)} \frac{d\bar{r}}{\sqrt{2m_b[H_\perp - P_\theta^2/2m_b\bar{r}^2 - \psi_0(\bar{r})]}}, \quad (11)$$

and $\psi_0(\bar{r}) = m_b\omega_j^2\bar{r}^2/2 + e_b\phi_0(\bar{r})$. Here, t'_{min} is defined by $r'(t'_{min}) = r'_{min}(H_\perp, P_\theta)$, where $r'_{min}(H_\perp, P_\theta)$ is the minimum radial excursion of the particle trajectory undergoing periodic motion. Substituting Eq. (10) into Eq. (9) and integrating over t' , we obtain

$$\widehat{\delta f_b}(r, \mathbf{p}) = -\frac{e_b}{m_b c} P_\theta \left[(\omega - k_z v_z) \frac{\partial f_b^0}{\partial H_\perp} + k_z v_z \frac{\partial f_b^0}{\partial H_\parallel} \right] \sum_n \frac{\exp(in\omega_r t)}{n\omega_r + k_z v_z - \omega} I^n(H_\perp, P_\theta). \quad (12)$$

In Eq. (12), the quantity \hat{t} can be expressed as

$$\hat{t}(r, H_{\perp}, P_{\theta}) = \int_{r'_{min}}^r \frac{d\bar{r}}{\sqrt{2m_b[H_{\perp} - P_{\theta}^2/2m_b\bar{r}^2 - \psi_0(\bar{r})]}}. \quad (13)$$

Furthermore, $I^n(H_{\perp}, P_{\theta})$ is defined by

$$I^n(H_{\perp}, P_{\theta}) = \int_0^{T_r} \frac{d\tau}{T_r} \frac{\hat{A}_{\theta}[\hat{r}(\tau)]}{\hat{r}(\tau)} \exp\{-in\omega_r\tau\}. \quad (14)$$

In Eq. (5), we express the perturbation amplitude as $\delta\hat{A}_{\theta}(r) = \sum_n \alpha_n A_n(r)$, where $\{\alpha_n\}$ are constants, and the complete set of vacuum eigenfunctions $\{A_n(r)\}$ is defined by $A_n(r) = C_n J_1(\lambda_n r/r_w)$. Here, λ_n is the n 'th zero of $J_1(\lambda_n) = 0$, and $C_n = 1/r_w$ is a normalization constant such that $\int_0^{r_w} dr r A_n(r) A_{n'}(r) = [J_2^2(\lambda_n)/2]\delta_{n,n'}$. We substitute $\delta\hat{A}_{\theta}(r) = \sum_n \alpha_n A_n(r)$ into Maxwell's equation (5) and operate with $\int_0^{r_w} dr r A_{n'}(r) \dots$. This gives the matrix dispersion equation

$$\sum_n \alpha_n D_{n,n'}(\omega) = 0, \quad (15)$$

where $D_{n,n'}(\omega)$ is defined by

$$D_{n,n'}(\omega) = \frac{J_2^2(\lambda_n)}{2} \left(\lambda_n^2 + k_z^2 r_w^2 - r_w^2 \frac{\omega^2}{c^2} \right) \delta_{n,n'} + \chi_{n,n'}(\omega), \quad (16)$$

and the beam-induced susceptibility $\chi_{n,n'}(\omega)$ is defined by

$$\chi_{n,n'}(\omega) = -\frac{4\pi e_b}{c} r_w^2 \int_0^{r_w} dr r A_{n'}(r) \int d^3 p v_{\theta} \delta f_b^n(r, \mathbf{p}). \quad (17)$$

Here, $\delta f_b^n(r, \mathbf{p})$ is defined by Eq. (12) with $\delta\hat{A}_{\theta} \rightarrow A_n$.

By changing the integration variables in Eq. (17) from $\{r, p_r, p_{\theta}\}$ to $\{\hat{t}, H_{\perp}, P_{\theta}\}$, where $\hat{t}(r, H_{\perp}, P_{\theta})$ is the time measured along the particle trajectory from the point where the radial distance is equal to r'_{min} defined in Eq. (13), the integration volume transforms according to $r dr dp_r dp_{\theta} = dP_{\theta} dH_{\perp} d\hat{t}$. Using Eq. (12), it follows that Eq. (17) can be rewritten as

$$\begin{aligned} \chi_{n,n'}(\omega) &= \frac{\hat{\omega}_{pb}^2}{c^2} \sum_p \int \frac{dP_{\theta} P_{\theta}^2}{m_b^2 \omega_r} \frac{dH_{\perp}}{T_{\perp b}^2} dp_z \exp\left[-\frac{H_{\perp}}{T_{\perp b}}\right] f_M(p_z) \\ &\times \left[\frac{\omega - k_z v_z}{\omega - p\omega_r - k_z v_z} + \frac{k_z^2 T_{\perp b}/m_b}{(\omega - p\omega_r - k_z v_z)^2} \right] (I_n^p)^* I_{n'}^p, \end{aligned} \quad (18)$$

where $\hat{\omega}_{pb}^2 = 4\pi e_b^2 \hat{n}_b / m_b$ is the plasma frequency-squared, and $f_M(p_z) = (2\pi m_b T_{\parallel b})^{-1/2} \exp(-p_z^2 / 2m_b T_{\parallel b})$. In Eq. (18), I_n^p is defined by

$$I_n^p(H_{\perp}, P_{\theta}) = \int_0^{T_r} \frac{d\tau}{T_r} \frac{J_1 \left[\frac{\lambda_n \hat{r}(\tau)}{r_w} \right]}{\hat{r}(\tau)} \exp\{-ip\omega_r \tau\}. \quad (19)$$

In Eq. (18), $(\)^*$ denotes complex conjugate. The condition for a nontrivial solution to Eq. (15) is

$$\det\{D_{n,n'}(\omega)\} = 0, \quad (20)$$

which plays the role of a matrix dispersion relation that determines the complex oscillation frequency ω .

In the following analysis, it is convenient to introduce the effective *depressed* betatron frequency $\omega_{\beta\perp}$. It can be shown [1] that for the equilibrium distribution in Eq. (1), the mean-square beam radius r_b^2 defined by

$$r_b^2 = \langle r^2 \rangle = \frac{\int dr r^3 n_b^0(r)}{\int dr r n_b^0(r)}, \quad (21)$$

is related exactly to the line density $N_b = 2\pi \int dr r n_b^0(r)$, where $n_b^0(r) = \int d^3p f_b^0(r, \mathbf{p})$, and the transverse beam temperature $T_{\perp b}$ by the equilibrium radial force balance equation [1]

$$\omega_f^2 r_b^2 = \frac{N_b e_b^2}{m_b} + \frac{2T_{\perp b}}{m_b}. \quad (22)$$

Equation (22) can be rewritten as

$$\left(\omega_f^2 - \frac{1}{2} \bar{\omega}_{pb}^2 \right) r_b^2 = \frac{2T_{\perp b}}{m_b}, \quad (23)$$

where we have introduced the effective *average* beam plasma frequency $\bar{\omega}_{pb}$ defined by

$$r_b^2 \bar{\omega}_{pb}^2 \equiv \int_0^{r_w} dr r \omega_{pb}^2(r) = \frac{2e_b^2 N_b}{m_b}. \quad (24)$$

Then, Eq. (23) can be used to introduce the effective *depressed* betatron frequency $\omega_{\beta\perp}$ defined by

$$\omega_{\beta\perp}^2 \equiv \left(\omega_f^2 - \frac{1}{2} \bar{\omega}_{pb}^2 \right) = \frac{2T_{\perp b}}{m_b r_b^2}. \quad (25)$$

If, for example, the beam density were uniform over the beam cross-section, then Eq. (25) corresponds to the usual definition of the depressed betatron frequency for a Kapchinskij-Vladimirskij (KV) beam equilibrium, and it is readily shown that the radial orbit $\hat{r}(\tau)$ occurring in Eqs. (11), (14) and (19) can be expressed as

$$\hat{r}^2(\tau) = \frac{H_{\perp}}{m_b \omega_{\beta\perp}^2} \left[1 - \sqrt{1 - \left(\frac{\omega_{\beta\perp} P_{\theta}}{H_{\perp}} \right)^2 \cos(2\omega_{\beta\perp} \tau)} \right]. \quad (26)$$

In general, for the choice of equilibrium distribution function in Eq. (1), there will be a spread in transverse depressed betatron frequencies $\omega_{\beta\perp}(H_{\perp}, P_{\theta})$, and the particle trajectories will not be described by the simple trigonometric function in Eq. (26). For present purposes, however, we consider a simple *model* in which the radial orbit $\hat{r}(\tau)$ occurring in Eqs. (11), (14) and (19) is approximated by Eq. (26) with the constant frequency $\omega_{\beta\perp}$ defined in Eq. (25), and the *approximate* equilibrium density profile is defined by $n_b^0(r) = \hat{n}_b \exp(-m_b \omega_{\beta\perp}^2 r^2 / 2T_{\perp b})$. For a nonuniform beam, $\omega_{\beta\perp}^{-1}$ is the characteristic time for a particle with thermal speed $v_{th\perp} = (2T_{\perp b}/m_b)^{1/2}$ to cross the rms radius r_b of the beam. In this case, it is shown in Appendices A and B that $D_{n,n'}(\omega)$ can be evaluated in closed analytical form provided the conducting wall is sufficiently far removed from the beam ($r_w/r_b \gtrsim 3$, say). In the limit of an anisotropic beam distribution that is *cold* in the longitudinal direction, i.e.,

$$\frac{T_{\parallel b}}{T_{\perp b}} \rightarrow 0, \quad (27)$$

the result is

$$\begin{aligned} D_{n,n'}(\omega) &= \frac{J_2^2(\lambda_n)}{2} \left(\lambda_n^2 + k_z^2 r_w^2 - r_w^2 \frac{\omega^2}{c^2} \right) \delta_{n,n'} + r_b^2 \frac{\hat{\omega}_{pb}^2}{c^2} z_{n,n'} \exp(-z_{n,n} - z_{n',n'}) \\ &\times \left\{ [I_0^2(z_{n,n'}) - I_1^2(z_{n,n'})] \left(1 + \frac{k_z^2 v_{th\perp}^2}{2\omega^2} \right) + \sum_{q=1}^{\infty} [I_q^2(z_{n,n'}) - I_{q-1}(z_{n,n'}) I_{q+1}(z_{n,n'})] \right. \\ &\quad \left. \times \left(\frac{2\omega^2}{\omega^2 - 4q^2 \omega_{\beta\perp}^2} + k_z^2 v_{th\perp}^2 \frac{\omega^2 + 4q^2 \omega_{\beta\perp}^2}{(\omega^2 - 4q^2 \omega_{\beta\perp}^2)^2} \right) \right\}. \quad (28) \end{aligned}$$

Here, $z_{n,n'} = (r_b/r_w)^2 \lambda_n \lambda_{n'}/4$. In this case, the matrix elements decrease exponentially away from the diagonal, with

$$\left| \frac{D_{n,n+k}}{D_{n,n}} \right| \sim \exp \left(-\frac{\pi^2 k^2}{4} \frac{r_b^2}{r_w^2} \right), \quad (29)$$

where k is an integer, and we have used the approximate relation $\lambda_n \approx \pi(4n + 1)/4$. Also, because $\hat{\omega}_{pb}^2 r_b^2 / c^2 \ll 1$ in the parameter regime of practical interest for intense beam applications, the dispersion matrix in Eq. (28) describes modes with frequencies at even multiples of the depressed betatron frequency, i.e., $\omega \approx 2q\omega_{\beta\perp}$, where $q = 0, \pm 1, \pm 2 \dots$. In the subsequent analysis, we study only the lowest frequency modes with $\omega \approx 0$. Therefore, for $r_w/r_b \gtrsim 3$, and for the lowest frequency modes, we approximate $\{D_{n,n'}(\omega)\}$ by the finite size square matrix of rank N defined by

$$D_{n,n'}^N(\omega) = \frac{J_2^2(\lambda_n)}{2} \left(\lambda_n^2 + k_z^2 r_w^2 \right) \delta_{n,n'} + r_b^2 \frac{\hat{\omega}_{pb}^2}{c^2} z_{n,n'} \exp(-z_{n,n} - z_{n',n'}) \times \left\{ [I_0^2(z_{n,n'}) - I_1^2(z_{n,n'})] \left(1 + \frac{k_z^2 v_{th\perp}^2}{2\omega^2} \right) \right\}, \quad (30)$$

where $n = 1, 2, \dots, N$.

The dispersion matrix (30) can be used to investigate detailed electromagnetic stability properties for strong anisotropy ($T_{\parallel b}/T_{\perp b} \rightarrow 0$) for a wide range of normalized axial wavenumbers ($k_z r_b$) and normalized beam skin depth $\delta_b = c/r_b \hat{\omega}_{pb}$. For sufficiently large values of $k_z r_b$, the large temperature anisotropy ($T_{\parallel b}/T_{\perp b} \rightarrow 0$) in Eq. (30) provides the free energy to drive the classical Weibel-type instability [29], generalized here to include finite transverse geometry and beam space-charge effects. The influence of the finite longitudinal temperature can be taken into account if one assumes $T_{\parallel b} \neq 0$ in Eq. (1). This results in the (collisionless) growth rate reduction of the unstable mode due to resonant wave-particle interactions [1] associated with the axial momentum spread of the beam particles. The dispersion relation for the case of nonzero longitudinal temperature $T_{\parallel b} \neq 0$ is derived in Appendix B.

Typical numerical results obtained from the approximate dispersion relation utilizing Eqs. (30) and (B13) are presented in Figs. 1 – 3 for the case where $r_w = 3r_b$. Only the leading-order terms at frequency $\omega \approx 0$ in Eq. (B13) have been retained in the analysis, and only the results for the most unstable mode are plotted in Figs. 1 – 3. Also, for $T_{\parallel b}/T_{\perp b} = 0$, it is found that the growth rate for the instability is nearly independent of the normalized skin depth for $c/r_b \hat{\omega}_{pb}$ in the range $c/r_b \hat{\omega}_{pb} \geq 1$. Therefore, the results in Figs. 1 and 2 are shown for a characteristic value of the normalized skin depth taken to be $c/r_b \hat{\omega}_{pb} = 100$. The four curves

in Figs. 1 and 2 represent the solution obtained using four different values of the dispersion matrix rank $N = 2, 3, 4, 5$. It is evident that the curves with $N = 4$ and $N = 5$ are almost indistinguishable. Figure 1 shows the normalized growth rate $(Im\omega)/(\hat{\omega}_{pb}v_{\perp b}^{th}/c)$ plotted versus normalized wavenumber $k_z r_b$ for normalized on-axis beam skin depth $c/r_b \hat{\omega}_{pb} = 100$ and temperature ratio $T_{\parallel b}/T_{\perp b} = 0$. Here, $v_{\perp b}^{th} = (2T_{\perp b}/m_b)^{1/2} \ll c$ is the transverse thermal speed of a beam particle. Because, $|\omega| \sim (\hat{\omega}_{pb}v_{\perp b}^{th}/c) \ll \omega_{\beta\perp}$, the results obtained in Fig. 1 justify the approximation that has been made in the derivation of the dispersion matrix in Eq. (30). Note that the maximum growth rate in Fig. 1 is reduced from the growth rate obtained for an infinite plasma with one active component [29].

The normalized eigenfunction $\delta\hat{A}_\theta(r)$ corresponding to $c/r_b \hat{\omega}_{pb} = 100$, $k_z r_b = 3$ and $T_{\parallel b}/T_{\perp b} = 0$ is plotted versus r/r_w in Fig. 2. Note, that the eigenfunction has no zeros (except at $r = 0$ and $r = r_w$) for dispersion matrix ranks $N \geq 3$.

Finally, using Eq. (B13), an important characteristic of the instability, the longitudinal threshold temperature $T_{\parallel b}^{th}$ for the onset of instability normalized to the transverse temperature $T_{\perp b}$, is plotted in Fig. 3 versus the normalized skin depth $c/r_b \hat{\omega}_{pb}$. The threshold temperature dependence on the normalized skin depth is very well approximated by the formula $T_{\parallel b}^{th}/T_{\perp b} \approx 10^{-0.7} r_b^2 \hat{\omega}_{pb}^2 / c^2$ [Fig. 3]. For charged particle beams of practical interest $r_b^2 \hat{\omega}_{pb}^2 / c^2 \sim (v_{\perp b}^{th}/c)^2 \ll 1$, and therefore $T_{\parallel b}^{th}/T_{\perp b} \ll 1$.

Finally, as noted earlier, for $T_{\parallel b}/T_{\perp b} \rightarrow 0$ and $r_b/r_w = 1/3$, the maximum growth rate in Fig. 1 asymptotes at

$$(Im\omega)_{max} = 0.85 \hat{\omega}_{pb} \frac{v_{\perp b}^{th}}{c} \quad (31)$$

for $k_z r_b \gg 1$, where $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2 / m_b)^{1/2}$ is the on-axis plasma frequency and $v_{\perp b}^{th} = (2T_{\perp b}/m_b)^{1/2}$ is the transverse thermal speed. Indeed, a numerical analysis of the dispersion relation shows that the maximum growth estimate in Eq. (31) is valid over the entire range of $\hat{\omega}_{pb}$ and $v_{\perp b}^{th}$ consistent with equilibrium force balance in Eq. (23) provided $c/\hat{\omega}_{pb} > r_b$. It is instructive to rewrite Eq. (31) in terms of the average depressed tune $\bar{\nu}/\nu_0$ defined by

$$\frac{\bar{\nu}}{\nu_0} = \frac{\omega_{\beta\perp}}{\omega_f} = \left(1 - \frac{\bar{\omega}_{pb}^2}{2\omega_f^2}\right)^{1/2}, \quad (32)$$

where ω_f is the applied focusing frequency, and $\bar{\omega}_{pb}$ is defined in Eq. (24). For the model density profile $n_b^0(r) = \hat{n}_b \exp(-m_b \omega_{\beta\perp}^2 r^2 / 2T_{\perp b})$ assumed in the present analysis, it follows that $\bar{\omega}_{pb}^2 = \hat{\omega}_{pb}^2 / 2$. Making use of Eqs. (25) and (32), and $\hat{\omega}_{pb} = \sqrt{2}\bar{\omega}_{pb}$ and $r_b/r_w = 1/3$, it is straightforward to show that Eq. (31) can be expressed in the equivalent form

$$\frac{(Im\omega)_{max}}{\omega_f} = 0.57 \frac{\omega_f r_w}{c} \frac{\bar{\nu}}{\nu_0} \left(1 - \frac{\bar{\nu}^2}{\nu_0^2}\right)^{1/2}, \quad (33)$$

where $\omega_f r_w / c \ll 1$ in the regime of practical interest. Equation (33) displays clearly the dependence of $(Im\omega)_{max}$ on the depressed tune, showing that $(Im\omega)_{max}$ is a maximum for $\bar{\nu}/\nu_0 = 1/\sqrt{2} = 0.707$ (Fig. 4).

III. CONCLUSIONS

To summarize, we have generalized the classical Weibel-type instability to the case of an intense charged particle beam with anisotropic temperature ($T_{\parallel b}/T_{\perp b} < 1$) including the important effects of finite transverse geometry and beam space-charge. Using the simplified assumption of negligible spread in depressed betatron frequency $\omega_{\beta\perp}$, we derived a simple dispersion equation for the lowest-order azimuthally-symmetrical ($\partial/\partial\theta = 0$) eigenmode. By numerically solving the matrix dispersion relation, the mode structure, growth rate and condition for the onset of the instability was obtained. It is found that even a small longitudinal temperature, $T_{\parallel b}^{th}/T_{\perp b} \approx 10^{-0.7} r_b^2 \hat{\omega}_{pb}^2 / c^2 \sim (v_{\perp b}^{th}/c)^2 \ll 1$, is large enough to stabilize the Weibel instability. The presence of this threshold is due to the finite transverse geometry of the charged particle beam. Indeed, the stabilization criteria for an infinite-size one-component plasma is determined from $T_{\parallel}/T_{\perp} < 1/(1 + k^2 c^2 / \hat{\omega}_{pb}^2)$ [2]. In the case of finite transverse geometry, we estimate $k_{min}^2 \sim 1/r_b^2$ and recover the threshold condition in Fig. 3 in the limit $c^2 / \hat{\omega}_{pb}^2 r_b^2 \gg 1$ up to a numerical factor of order unity specific to the cylindrical geometry. In previous work [12–14] we have shown that the fast, electrostatic, Harris-like instability develops and saturates at a much larger longitudinal temperature $T_{\parallel b}^{th}/T_{\perp b} \approx 0.1$. Hence, we conclude that the Weibel instability is not likely to play a significant role in the process of energy isotropization in intense unneutralized charged particle beams.

With regard to future investigations, we plan to carry out detailed kinetic simulations of the Weibel instability employing a version of the BEST code [1, 13, 17] modified to include transverse electromagnetic effects by incorporating a Darwin model [see, for example, Ref. 37] that neglects the displacement current. This will permit detailed investigations of the both the linear and nonlinear phases of the instability; the effects of closer proximity of the conducting wall, where the dispersion matrix defined in Eq. (28) becomes even denser in terms of the importance of the off-diagonal elements; the (stabilizing) influence of a longitudinal momentum spread; and the effects of other choices of distribution function, e.g., a waterbag equilibrium [1].

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APPENDIX A: EVALUATION OF THE ORBIT INTEGRAL

In this Appendix, we evaluate the orbit integral [see Eq. (19)]

$$I_n^p = \int_0^{T_r} \frac{d\tau}{T_r} \frac{J_1[\lambda_n \hat{r}(\tau)/r_w]}{\hat{r}(\tau)} \exp[-ip\omega_r\tau], \quad (\text{A1})$$

where $\hat{r}(\tau)$ is the trajectory of a particle with transverse energy H_\perp and canonical angular momentum P_θ moving in the quadratic potential $\psi(r) = m_b\omega_{\beta\perp}^2 r^2/2$ with initial condition $\hat{r}(\tau = 0) = r_{\min}(H_\perp, P_\theta)$ [see Eq. (11)]. Here, $\omega_r(H_\perp, P_\theta)$ is the frequency of radial motion.

In Cartesian coordinates, the particle trajectory $\{\hat{x}(\tau), \hat{y}(\tau)\}$ can be expressed as

$$\begin{aligned} \hat{x}(\tau) &= \frac{\sqrt{H_\perp/m_b}}{\omega_{\beta\perp}} \cos(\omega_{\beta\perp}\tau) [\cos(\beta) - \sin(\beta)], \\ \hat{y}(\tau) &= \frac{\sqrt{H_\perp/m_b}}{\omega_{\beta\perp}} \sin(\omega_{\beta\perp}\tau) [\sin(\beta) + \cos(\beta)], \end{aligned} \quad (\text{A2})$$

where $\cos(2\beta) = P_\theta\omega_{\beta\perp}/H_\perp$. Therefore, we can express

$$\hat{r}^2(\tau) = \hat{x}^2(\tau) + \hat{y}^2(\tau) = \frac{H_\perp}{m_b\omega_{\beta\perp}^2} [1 - \sin(2\beta) \cos(2\omega_{\beta\perp}\tau)], \quad (\text{A3})$$

From Eq. (A3) it follows that $\omega_r(H_\perp, P_\theta) = 2\omega_{\beta\perp}$. Substituting Eqs. (A3) into Eq. (A1) and changing the integration variable to $\alpha = 2\omega_{\beta\perp}\tau$, we obtain

$$F(a, b) = \frac{r_w}{\lambda_n} I_n^p = \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{J_1(\sqrt{a^2 + b^2 - 2ab \cos(\alpha)})}{\sqrt{a^2 + b^2 - 2ab \cos(\alpha)}} \exp[-ip\alpha]. \quad (\text{A4})$$

Here, $a = (\lambda_n/r_w\omega_{\beta\perp})\sqrt{(H_\perp - \omega_{\beta\perp}P_\theta)/2m_b}$ and $b = (\lambda_n/r_w\omega_{\beta\perp})\sqrt{(H_\perp + \omega_{\beta\perp}P_\theta)/2m_b}$. To evaluate this integral we first evaluate

$$G(a, b) = \int_0^{2\pi} \frac{d\alpha}{2\pi} J_0(\sqrt{a^2 + b^2 - 2ab \cos(\alpha)}) \exp[-ip\alpha]. \quad (\text{A5})$$

To evaluate the integral in Eq. (A5), we use the summation theorem for Bessel functions [47], which gives

$$J_0(w\sqrt{r^2 + \rho^2 - 2r\rho \cos \alpha}) = \sum_{k=-\infty}^{+\infty} J_k(w\rho)J_k(wr) \exp(ik\alpha). \quad (\text{A6})$$

Substituting Eq. (A6) into Eq. (A5), and integrating, we obtain

$$G(a, b) = J_p(a)J_p(b). \quad (\text{A7})$$

Next, we use Eq. (A5) to construct the combination

$$a \frac{\partial G(a, b)}{\partial a} - b \frac{\partial G(a, b)}{\partial b} = (b^2 - a^2)F(a, b). \quad (\text{A8})$$

Where we have used the fact that $J'_0(x) = -J_1(x)$. Finally, using Eqs. (A7) and (A8), and the recursion relation for Bessel functions, $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}$, we obtain

$$I_n^p = \frac{\lambda_n}{r_w} \frac{aJ_{n-1}(a)J_n(b) - bJ_{n-1}(b)J_n(a)}{b^2 - a^2}. \quad (\text{A9})$$

APPENDIX B: EVALUATION OF THE SUSCEPTIBILITY INTEGRAL

To complete the evaluation of the susceptibility defined in Eq. (18), we determine the following integral

$$I = \frac{\hat{\omega}_{pb}^2}{c^2} \int_0^\infty \frac{dH_\perp}{m_b^2 T_{\perp b}^2} \int_{-H_\perp/\omega_{\beta\perp}}^{H_\perp/\omega_{\beta\perp}} \frac{P_\theta^2 dP_\theta}{2\omega_{\beta\perp}} \exp\left[-\frac{H_\perp}{T_{\perp b}}\right] (I_n^p)^* I_{n'}^p, \quad (\text{B1})$$

where the orbit integral I_n^p is defined in Eq. (19) and was evaluated in Appendix A. Using Eqs. (19) and (A9), and introducing the change of integration variables,

$$\begin{aligned}x &= (H_\perp + P_\theta \omega_{\beta\perp})/2T_{\perp b}, \\y &= (H_\perp - P_\theta \omega_{\beta\perp})/2T_{\perp b},\end{aligned}\tag{B2}$$

Eq. (B1) can be rewritten as

$$\begin{aligned}I &= \frac{T_{\perp b}}{m_b c^2} \left(\frac{\hat{\omega}_{pb}}{\omega_{\beta\perp}} \right)^2 \int_0^\infty dx \int_0^\infty dy \exp(-x-y) \\&\times [\sqrt{x} J_{p-1}(k_n \sqrt{x}) J_p(k_n \sqrt{y}) - \sqrt{y} J_{p-1}(k_n \sqrt{y}) J_p(k_n \sqrt{x})] \\&\times [\sqrt{x} J_{p-1}(k_m \sqrt{x}) J_p(k_m \sqrt{y}) - \sqrt{y} J_{p-1}(k_m \sqrt{y}) J_p(k_m \sqrt{x})],\end{aligned}\tag{B3}$$

where $k_n = \lambda_n \sqrt{T_{\perp b}/m_b}/r_w \omega_{\beta\perp}$. The change of variables in Eq. (B2) reduces the two-dimensional integral in Eq. (B1) to the products of two one-dimensional integrals. The integral in Eq. (B3) can be rewritten as

$$I = 2 \frac{T_{\perp b}}{m_b c^2} \left(\frac{\hat{\omega}_{pb}}{\omega_{\beta\perp}} \right)^2 (I_1^{n,m} I_2^{n,m} - I_3^{n,m} I_3^{m,n}),\tag{B4}$$

where

$$I_1^{n,m} = \int_0^\infty dx x J_{p-1}(k_n \sqrt{x}) J_{p-1}(k_m \sqrt{x}) \exp(-x),\tag{B5}$$

$$I_2^{n,m} = \int_0^\infty dx J_p(k_n \sqrt{x}) J_p(k_m \sqrt{x}) \exp(-x) = \exp\left(-\frac{k_n^2 + k_m^2}{4}\right) I_p\left(\frac{k_n k_m}{2}\right),\tag{B6}$$

$$I_3^{n,m} = \int_0^\infty dx \sqrt{x} J_{p-1}(k_n \sqrt{x}) J_p(k_m \sqrt{x}) \exp(-x),\tag{B7}$$

We now use the Bessel function identity, $x J_{p-1}(x) = p J_p(x) + x J_p'(x)$, to express the integrals $I_1^{n,m}$ and $I_3^{n,m}$ in terms of $I_2^{n,m}$. This gives

$$I_3^{n,m} = \left(\frac{d}{dk_n} + \frac{p}{k_n} \right) I_2^{n,m},\tag{B8}$$

and

$$I_1^{n,m} = \left(\frac{d}{dk_n} + \frac{p}{k_n} \right) \left(\frac{d}{dk_m} + \frac{p}{k_m} \right) I_2^{n,m}.\tag{B9}$$

Substituting Eqs. (B6), (B8) and (B9) into Eq. (B4), we obtain after some algebraical manipulation

$$\begin{aligned}
I &= 2 \frac{T_{\perp b}}{m_b c^2} \left(\frac{\hat{\omega}_{pb}}{\omega_{\beta\perp}} \right)^2 \left(I_2^{n,m} \frac{\partial^2}{\partial k_n \partial k_m} I_2^{n,m} - \frac{\partial}{\partial k_n} I_2^{n,m} \frac{\partial}{\partial k_n} I_2^{m,n} \right) \\
&= 2 \frac{T_{\perp b}}{m_b c^2} \left(\frac{\hat{\omega}_{pb}}{\omega_{\beta\perp}} \right)^2 \left(\frac{k_n k_m}{4} \right) \exp \left(-\frac{k_n^2 + k_m^2}{2} \right) \\
&\times \left[I_p^2 \left(\frac{k_n k_m}{2} \right) - I_{p-1} \left(\frac{k_n k_m}{2} \right) I_{p+1} \left(\frac{k_n k_m}{2} \right) \right], \tag{B10}
\end{aligned}$$

where use has been made of the following identities to eliminate the derivatives of the Bessel functions

$$\begin{aligned}
I_p''(x) + \frac{I_p'(x)}{x} - \left(1 + \frac{p^2}{x^2} I_p(x) \right) &= 0, \\
x I_p'(x) &= x I_{p-1}(x) - p I_p(x), \\
x I_p'(x) &= x I_{p+1}(x) + p I_p(x). \tag{B11}
\end{aligned}$$

In Eq. (B3), we have extended the integration limits to infinity. The error in this approximation is proportional to $\exp[-(r_w/r_b)^2]$, which is negligibly small for $r_w/r_b \gtrsim 3$. Collecting the result from Eq. (B10), the expression for the matrix elements becomes

$$\begin{aligned}
D_{n,n'}(\omega) &= \frac{J_2^2(\lambda_n)}{2} (\lambda_n^2 + k_z^2 r_w^2 - r_w^2 \frac{\omega^2}{c^2}) \delta_{n,n'} + r_b^2 \frac{\hat{\omega}_{pb}^2}{c^2} z_{n,n'} \exp(-z_{nn} - z_{n'n'}) \left\{ \left[I_0^2(z_{nn'}) \right. \right. \\
&- I_1^2(z_{nn'}) \left. \right] \int dp_z f_0(p_z) \left(1 + \frac{k_z^2 v_{th\perp}^2}{2(\omega - k_z v_z)^2} \right) + \sum_{q>0} \left[I_q^2(z_{nn'}) - I_{q+1}(z_{nn'}) I_{q-1}(z_{nn'}) \right] \\
&\times \int dp_z f_0(p_z) \left(\frac{2(\omega - k_z v_z)^2}{(\omega - k_z v_z)^2 - 4q^2 \omega_{\beta\perp}^2} + k_z^2 v_{th\perp}^2 \frac{(\omega - k_z v_z)^2 + 4q^2 \omega_{\beta\perp}^2}{[(\omega - k_z v_z)^2 - 4q^2 \omega_{\beta\perp}^2]^2} \right) \left. \right\}, \tag{B12}
\end{aligned}$$

where $r_b^2 = 2T_{b\perp}/m_b \omega_{\beta\perp}^2$, $v_{th\perp}^2 = 2T_{\perp b}/m_b$, and we have introduced $z_{n,n'} = (r_b/r_w)^2 \lambda_n \lambda_{n'}/4$. If the longitudinal distribution function $f_0(p_z)$ is Maxwellian with $f_0(p_z) = (2\pi m_b T_{\parallel b})^{-1/2} \exp(-p_z^2/2m_b T_{\parallel b})$, then the matrix elements in Eq. (B12) take the form

$$\begin{aligned}
D_{n,n'}(\omega) &= \frac{J_2^2(\lambda_n)}{2} (\lambda_n^2 + k_z^2 r_w^2 - r_w^2 \frac{\omega^2}{c^2}) \delta_{n,n'} + r_b^2 \frac{\hat{\omega}_{pb}^2}{c^2} z_{n,n'} \exp(-z_{nn} - z_{n'n'}) \\
&\times \left\{ \left[I_0^2(z_{nn'}) - I_1^2(z_{nn'}) \right] A_0(\omega) + \sum_{q>0} \left[I_q^2(z_{nn'}) - I_{q-1}(z_{nn'}) I_{q+1}(z_{nn'}) \right] \right. \\
&\quad \left. \times [A_{2q}(\omega) + A_{-2q}(\omega)] \right\}. \tag{B13}
\end{aligned}$$

Here, $A_n(\omega)$ is defined by

$$A_n(\omega) = \left(1 - \frac{v_{th\perp}^2}{v_{th\parallel}^2}\right) - Z\left(\frac{\omega - n\omega_{\beta\perp}}{k_z v_{th\parallel}}\right) \left[\frac{\omega}{k_z v_{th\parallel}} - \frac{\omega - n\omega_{\beta\perp}}{k_z v_{th\parallel}} \left(1 - \frac{v_{th\perp}^2}{v_{th\parallel}^2}\right) \right], \quad (\text{B14})$$

where $Z(\Omega)$ is the plasma dispersion function [1, 48], and $v_{th\parallel} = (2T_{\parallel b}/m_b)^{1/2}$.

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FIGURE CAPTIONS

Fig.1: Plot of the normalized growth rate $(Im\omega)/(\hat{\omega}_{pb}v_{\perp b}^{th}/c)$ versus $k_z r_b$ for normalized skin depth $c/r_b \hat{\omega}_{pb} = 100$ and $T_{\parallel b}/T_{\perp b} = 0$ obtained from Eq. (30). The four curves correspond to four different values of the dispersion matrix rank $N = 2, 3, 4, 5$.

Fig.2: Plots of the normalized eigenfunction versus r/r_w corresponding to $c/r_b \hat{\omega}_{pb} = 100$, $k_z r_w = 10$ and $T_{\parallel b}/T_{\perp b} = 0$ obtained from Eq. (30). The four curves correspond to four different values of the dispersion matrix rank $N = 2, 3, 4, 5$.

Fig.3: The longitudinal threshold temperature $T_{\parallel b}^{th}$ for the onset of instability normalized to the transverse temperature $T_{\perp b}$ is plotted versus the normalized skin depth $c/r_b \hat{\omega}_{pb}$ [Eq. (B13)].

Fig.4: Plot of maximum growth rate $(Im\omega)_{max}$ obtained from Eqs. (30) and (33) versus the average depressed tune $\bar{\nu}/\nu_0$.

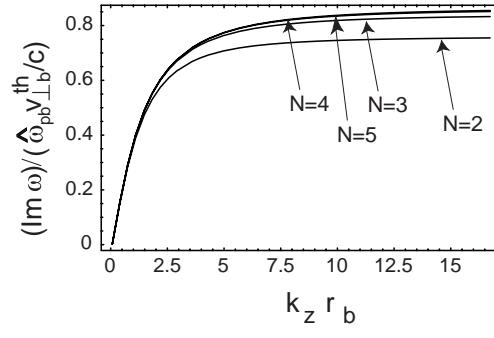


FIG. 1:

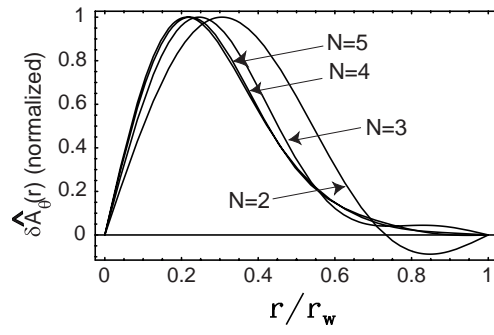


FIG. 2:

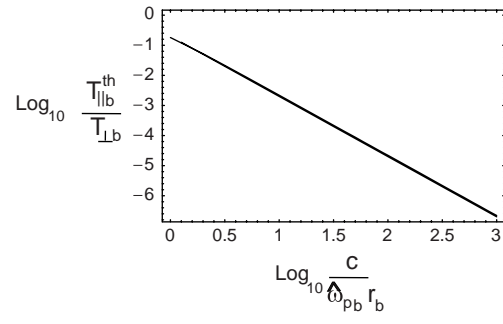


FIG. 3:

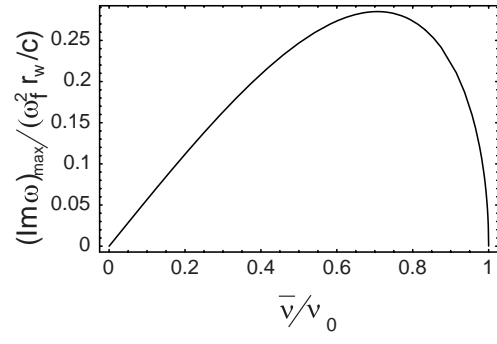


FIG. 4:

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