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ON THE BOUNDARY CONDITION BETWEEN TWO MULTIPLYING MEDIA

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Abstract

The transition region between two parts of a pile which have different compositions is investigated. In the case where the moderator is the same in both parts of the pile, it is found that the diffusion constant times thermal neutron density plus diffusion constant times fast neutron density satisfies the usual pile equations everywhere, right to the boundary. More complicated formulae apply in a more general case.

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ON THE BOUNDARY CONDITION BETWEEN TWO MULTIPLYING MEDIA

F. L. Friedman and E. P. Wigner

1. The pile equations of Fermi, as well as all the similar equations which describe the densities of neutrons with different energies in chain reacting units have many solutions. All the solutions can be written as superpositions of exponential functions of the position. The relaxation distances of these exponentials are different. Ordinarily, one relaxation distance is imaginary and only exponentials with this relaxation distance occur in the actual solution. We shall call these solutions the "regular" solutions. However, there are other solutions of the equations with real relaxation distances. Since these relaxation distances are all much smaller than the absolute value of the imaginary relaxation distance, they will be called transient solutions. In these, the ratio of the densities of neutrons with different energies is different from their ratio in the regular solution. Such solutions play a role if the ratio of the densities of neutrons with different energies cannot be constant throughout the pile. This is the case in the neighborhood of disturbing centers, such as e.g. control rods which absorb only thermal neutrons. Another example is that of a pile which contains two parts, with different compositions. In the latter case, e.g. the ratio of fast and thermal neutrons at the boundary between the two piles will be intermediate between the ratios in the "regular" solutions for both sides and go over into the regular ratio further away from the boundary. These facts have been brought out before on various occasions, most completely by Ibser and Wheeler in C-88.

2. We shall first treat a case in which the moderator is the same all over the pile so that the diffusion constant for both fast and thermal neutrons remains constant throughout the whole system. It has been stated in CP-455 that, for such a system, Fermi's equations can be solved and this solution will be given next. The same problem then will be solved with the two group theory as described in CP-1461 and CP-1554. It will be shown that in this case, the two methods of calculation give the same result. The general case of different moderators in the two parts of the pile will be treated finally by means of the two group theory.

We shall use Fermi's equations in the form

$$\frac{1}{3\sigma}\Delta q + \xi\frac{\partial}{\partial\tau}(\sigma q) + \sigma_a p_2 f(\tau)\, n = 0$$
$$\frac{1}{3\sigma_o}\Delta n - \sigma_a n + \xi\sigma_o p_1 q(0) = 0 \qquad (1)$$

Here, $\tau = \ln(E/E_{thermal})$ has a somewhat different definition from the usual one: it is zero for neutrons, the energy E of which is thermal, $\sigma(\tau)$ is the transport cross section which may depend on the energy but does not depend on the position; σ_o, the value of this quantity for $\tau = 0$, i.e. for thermal neutrons; ξ, the average logarithmic energy loss (independent of position); σ_a, the absorption cross section for thermal neutrons which depends on the position. $q(\tau)$ is the density of fast neutrons per unit τ (it is not Fermi's slowing down density Q), multiplied with the velocity, n the density of thermal neutrons/ times their velocity. $f(\tau)d\tau$ is the number of fission neutrons per slow neutron captured in U, for which τ is between τ and $\tau + d\tau$. Finally p_1 is the chance of escaping resonance absorption and p_2 the thermal utilization. The multiplication constant is, hence

$$k = p_1 p_2 \int_0^\infty f(\tau)\,d\tau \qquad (2)$$

The advantage of the above way of writing the ordinary pile equations is that it does not assume that the fission neutrons are monochromatic.

If σ_a depends on the position, the ratio of n and $q(\mathcal{T})$ will not be constant over the pile and no simple equation will hold for either of them separately. This is natural, since "transients" occur both in n and in q which make their behavior quite complicated. The method of solution to be given consists of finding a quantity γ in which the transients just cancel so that a simple equation shall hold for it.

One can solve the first of the above equations by writing

$$\xi\sigma(\mathcal{T})q(\mathcal{T}) = \left[\int_{\mathcal{T}}^{\infty} f(\mathcal{T}') \; e^{t(\mathcal{T}')\Delta} \, d\mathcal{T}' e^{-t(\mathcal{T})\Delta}\right] p_2 \, n\sigma_a \qquad (3)$$

In this

$$t(\mathcal{T}) = \int_0^{\mathcal{T}} \frac{d\mathcal{T}}{3\xi\sigma^2} \qquad (3a)$$

is Fermi's age for neutrons of an energy E characterized by $\ln(E/E_{thermal}) = \mathcal{T}$. The Δ is an operator, to operate on $p_2 n\sigma_a$ and the exponential of an operator is, as usual, the infinite series of operators obtained by expanding it. [$f(\mathcal{T})$ and $t(\mathcal{T})$ are independent of position.] One can convince oneself easily that the above σq indeed gives at least a formal solution of the first pile equation.

It also gives

$$\xi\sigma_o \, q(0) = \left[\int_0^{\infty} f(\mathcal{T}') \; e^{t(\mathcal{T}')\Delta} \; d\mathcal{T}'\right] p_2 n\,\sigma_a \qquad (4)$$

and this introduced into the second pile equation gives

$$\Delta\left(\frac{n}{3\sigma_o}\right) - \sigma_a n + p_1\left[\int_0^{\infty} f(\mathcal{T}') \; e^{t(\mathcal{T}')\Delta} d\mathcal{T}'\right] p_2 n\,\sigma_a = 0 \qquad (5)$$

This is the generalization of the usual equation connecting the multiplication constant and the Laplacian. In the present case, it is useful only if one can neglect the second and higher powers of Δ in the expansion of the exponential.

$$\Delta\left(\frac{n}{3\sigma_o}\right) + p_1 S \Delta(p_2 n \sigma_a) = \sigma_a n\left[1 - p_1 \int_0^\infty f(\tau)d\tau p_2\right] \tag{6}$$

$$= \sigma_a n(1 - k)$$

where

$$s = \int f(\tau') \, t(\tau')d\tau' \tag{6a}$$

is the average age of fission neutrons multiplied by their number. It is independent of position. If the same holds for p_1, we can put it behind the operator Δ and obtain

$$\Delta\left(\frac{n}{3\sigma_o} + p_1 p_2 S n \sigma_a\right) = \frac{1 - k}{M^2} \sigma_a n \left[\frac{1}{3\sigma_o\sigma_a} + p_1 p_2 S\right] \tag{7}$$

where

$$M^2 = \frac{1}{3\sigma_o\sigma_a} + p_1 p_2 S = \frac{1}{3\sigma_o\sigma_a} + k\bar{t} \tag{7a}$$

In this, $\bar{t}$ is the average age of fission neutrons as follows from (2). This shows that

$$\nu = n\sigma_a M^2 \tag{8}$$

satisfies the usual pile equation

$$\Delta\nu = \frac{1 - k}{M^2} \nu \tag{9}$$

and has no transients.

The above derivation presupposes that not only $\sigma(\tau)$ but also p_1 be constant throughout the pile. It further assumes that $|k - 1| \ll 1$, as a consequence of which one can assume that $t(\tau')\Delta \ll 1$. One can discard the second assumption if one is willing to assume that the Laplacian, Δ is constant throughout the pile. In this case

$$\nu = \frac{n}{\sigma_0} + p_1 \int_0^\infty q(\tau)\, e^{Bt(\tau)} \frac{d\tau}{\sigma(\tau)} \qquad (10)$$

and the calculation is quite elementary giving for B the implicit equation

$$B = 3\sigma_0\sigma_a \left[1 - p \int_0^\infty f(\tau)\, e^{Bt(\tau)} d\tau \right] \qquad (10a)$$

The equation for ν is

$$\Delta\nu = B\nu \qquad (10b)$$

However, the last assumptions are so specialized that they apply in the best case, if the two parts of the pile differ only in temperature which does not affect the Laplacian. The assumptions first made apply reasonably well for a pile which has the same moderator throughout although the amount of metal is different in different parts of it.

3. We now turn to the same problem, except that we shall assume a sharp boundary (at $x = 0$) between the two parts of the pile and use the two group theory described in reports CP-1461 and CP-1554. The notation will be the same as that adopted in CP-1461.

As in CP-1461 n_t is the sum of the velocities of all the thermal neutrons which are present in a cubic centimeter and n_f is the sum of the velocities of the fast neutrons present in a cubic centimeter. The pile equations then are

$$\Delta n_t - \kappa_t^2 n_t + \kappa_t^2 a n_f = 0 \qquad \text{(11a)}$$

$$\Delta n_f - \kappa_f^2 n_f + \kappa_f^2 (k/a) n_t = 0 \qquad \text{(11b)}$$

These equations already were used in the above mentioned C-83 by Ibser and Wheeler. In it, κ_t is the reciprocal diffusion length of thermal neutrons in the pile (that takes into account the presence of the metal), κ_f^2 is the reciprocal age, k is the multiplication constant, and

$$a = p_1 \sigma_{af}/\sigma_{at} \qquad \text{(11c)}$$

Herein σ_{at} is the absorption cross section for thermal neutrons per cubic centimeter, p_1 is the probability for a neutron to escape resonance absorption, and

$$\sigma_{af} = \frac{\sigma_f \, \xi}{\ln(E_f/E_t)} \; . \qquad \text{(11d)}$$

Herein again σ_f is the total cross section for a fast neutron per cubic centimeter, ξ is the average logarithmic energy loss of a neutron upon collision with an atom of the moderator, E_f and E_t are the energies of fission and thermal neutrons respectively.

The above notation holds for $x < 0$, i.e., on the left side of the boundary. On the right side of the boundary a similar notation will be used from equation (16) on, except that every small letter will be replaced by a capital.

On the left side of the pile the neutron densities are linear combinations of the four expressions

$$e^{\kappa_1 x} \, , \quad e^{-\kappa_1 x} \, ; \qquad e^{\kappa_2 x} \, , \quad e^{-\kappa_2 x} \; . \qquad \text{(12)}$$

It should be remarked that $\varkappa_1^2$ is smaller than 0 if the pile is to be chain reacting, it is the Laplacian of the neutron density far from the boundary. For the part of the neutron density which is composed of the first two exponentials of (12) the ratio of thermal and fast neutron densities is

$$n_t/n_f = a\varkappa_t^2/(\varkappa_t^2 - \varkappa_1^2) \approx a; \tag{12a}$$

The last part of the equation with the $\approx$ sign holds if $|k - 1| \ll 1$ in which case $\varkappa_1^2 \ll \varkappa_t^2$.

For the part of the density proportional to the last two exponentials of (12) the ratio is

$$n_t/n_f = -a\varkappa_t^2/(\varkappa_f^2 - \varkappa_1^2) \approx -\frac{a\varkappa_t^2}{\varkappa_f^2} \tag{12b}$$

We have

$$\varkappa_2^2 = \varkappa_t^2 + \varkappa_f^2 - \varkappa_1^2 \quad . \tag{12c}$$

4. Let us first consider a special case in which the ratio of the fast diffusion constants $\sigma_{af}/\varkappa_f^2$ of the two sides is the same as the ratio of the slow diffusion constants $\sigma_{at}/\varkappa_t^2$ of the two sides. In this case p_1 times the ratio of the two diffusion coefficients, $p_1\sigma_{af}\varkappa_t^2/\sigma_{at}\varkappa_f^2 = a\varkappa_t^2/\varkappa_f^2$, is also approximately the same on both sides, and this also holds if $k - 1 \ll 1$, for the ratio of the coefficients of the second two exponentials of (12) in the slow and fast densities:

$$a\,\frac{\varkappa_t^2}{\varkappa_f^2 - \varkappa_1^2} = p_1\,\frac{\sigma_{af}\varkappa_t^2}{\sigma_{at}(\varkappa_f^2 - \varkappa_1^2)} \tag{13}$$

The equality of this last quantity for both sides--whether or not it is a consequence of the first assumption--will be assumed in the following.

The second two exponentials of (12) drop out of the expression

$$\nu = n_t + \frac{a\varkappa_t^2}{\varkappa_f^2 - \varkappa_1^2} n_f \qquad (14a)$$

so that this satisfies the equation $\Delta\nu = \varkappa_1^2\nu$ and has, therefore, no transients. Since n_f and n_t are both continuous at the boundary, this holds also for ν. Furthermore, the condition for the equality of the fluxes means that the ratio of the derivatives of n_f shall be inversely proportional to the ratio of the fast diffusion coefficients and this holds, according to the second assumption, also for n_t. Hence

$$\frac{\sigma_{af}}{\varkappa_f^2}\frac{d\nu}{dx} \quad \text{or} \quad \frac{\sigma_{at}}{\varkappa_t^2}\frac{d\nu}{dx} \qquad (14b)$$

is also continuous.

The most usual case in which the above conditions are satisfied is that of a common moderator for both regions. In this case the thermal diffusion coefficient $\sigma_{at}/\varkappa_t^2$ is also the same on both sides so that not only ν but

$$\frac{\sigma_{at}}{\varkappa_t^2}\nu = \frac{\sigma_{at}}{\varkappa_t^2} n_t + \frac{a\sigma_{at}}{\varkappa_f^2 - \varkappa_1^2} n_f \qquad (15)$$

is also continuous. Furthermore, since with our first assumption the fast diffusion coefficients are also the same on both sides, the derivative of the above expression is also continuous. The present n_t was denoted by n in section 1 and $n_f = n_t/a$, so that if one neglects $\varkappa_1^2$ compared with $\varkappa_f^2$, this becomes

$$n\sigma_{at}\left(\frac{1}{\varkappa_t^2} + \frac{1}{\varkappa_f^2}\right)' . \qquad (15a)$$

The expression in the bracket is the migration area denoted by M^2. We see that (15a) is the same as (8). In (7a) $M^2 = \frac{1}{\varkappa_t^2} + \frac{k}{\varkappa_f^2}$ but $|k - 1| \ll 1$ is assumed in both derivations and (15a) is obtained from (15) by replacing k by 1.

The physical interpretation of (15) is quite simple. If the neutrons did not diffuse while in the thermal region but all the diffusion took place in the fast neutron region evidently n_f would be continuous and likewise the product of the derivative of n_f with the diffusion constant. On the other hand, if all the neutron diffusion took place while the neutrons are thermal, n_t would be continuous and so would the thermal diffusion constant times the derivative of n_t. If the diffusion takes place for both thermal and fast neutrons one must expect that a linear combination of the above quantities will be continuous with coefficients which are proportional to the amount of diffusion in the corresponding regions. This is exactly what the above equations show.

5. We now go over to the general case. It will be assumed, however, that $|k - 1| \ll 1$, i.e. $|\varkappa_1^2| \ll \varkappa_2^2$, $|K_1^2| \ll K_2^2$. In the neighborhood of the boundary we shall use for the neutron density the following expressions

$$n_f = \beta_o + \beta_1 x + \gamma \varkappa_f^2 e^{\varkappa_2 x} \qquad (16a)$$

$$n_t = a(\beta_o + \beta_1 x) - \gamma a \varkappa_t^2 \, e^{\varkappa_2 x} \qquad (16b)$$

$$N_f = B_o + B_1 x + \Gamma K_f^2 \, e^{-K_2 x} \qquad (16c)$$

$$N_t = A(B_o + B_1 x) - \Gamma A K_t^2 \, e^{-K_2 x} \qquad (16d)$$

The first two of these refer to $x < 0$; the latter two to $x > 0$, i.e. to the right side of the boundary. It would appear that replacing the first two exponentials of (12) by linear functions involves an approximation. This, however, is not the case as we shall use equations (13) only in the immediate neighborhood of $x = 0$. The reason for omitting the $e^{-\kappa_2 x}$ for $x < 0$ and $e^{K_2 x}$ in (13) is that the transient solutions must drop to zero far away from the boundary. The quantities β_o and B_o are the extrapolated neutron densities at the left and right side of the boundary if one neglects the transients when making the extrapolations, i.e., uses the equation $\Delta n = \kappa_1^2 n$ at the left side and $\Delta N = K_1^2 N$ at the right side of the boundary. β_1 and B_1 are the values of the derivatives of the extrapolated neutron densities.

The boundary conditions which are valid for the actual densities are

$$n_t(0) = N_t(0) \qquad\qquad n_f(0) = N_f(0) \tag{17a}$$

$$d_t\, n_t'(0) = D_t\, N_t'(0) \qquad\qquad d_f\, n_f'(0) = D_f\, N_f'(0) \tag{17b}$$

In this we have introduced the notation

$$\Gamma_{at}/\kappa_t^2 = d_t \qquad\qquad \Gamma_{af}/\kappa_f^2 = d_f \tag{18}$$

for the diffusion constant for thermal and fast neutrons in the left side of the pile and the notation D_t and D_f for the same quantities in the right side of the pile. Equations (17) expressed in terms of the β and γ are

$$\beta_o + \gamma\kappa_f^2 = B_o + \Gamma K_f^2 \tag{19a}$$

$$a\beta_o - a\gamma\kappa_t^2 = A\, B_o - A\Gamma K_t^2 \tag{19b}$$

$$d_f(\beta_1 + \gamma\kappa_f^2\kappa_2) = D_f(B_1 - \Gamma K_f^2 K_2) \tag{19c}$$

$$d_t a(\beta_1 - \gamma\kappa_t^2\kappa_2) = D_t A(B_1 + \Gamma K_t^2 K_2) \tag{19d}$$

If one wants the conditions for the extrapolated density one must write down the condition that these four equations for γ and Γ have a solution. The condition for this is that all three rowed determinants of

$$\begin{vmatrix} \varkappa_f^2 & -K^2 & \beta_o - B_o \\ -a\varkappa_t^2 & AK_t^2 & a\beta_o - AB_o \\ d_f\varkappa_f^2\varkappa_2 & D_fK_f^2K_2 & d_f\beta_1 - D_fB_1 \\ -d_ta\varkappa_t^2\varkappa_2 & -D_tAK_t^2K_2 & d_ta\beta_1 - D_tAB_1 \end{vmatrix} = 0 \qquad (20)$$

shall vanish. This condition is then the solution of our problem in the general case. It is contained, implicitly, already in C-88.

It may be worthwhile to calculate the magnitude of the transient solutions at the boundary. The sum of the two transients must be equal of course to the jump in the extrapolated neutron density. An easy calculation gives for the fast and thermal n at the boundary (x = 0)

$$n_f = \frac{d_f(\varkappa_2\beta_o - \beta_1) + D_f(K_2 B_o + B_1)}{d_f\varkappa_2 + D_f K_2} \qquad (21a)$$

$$n_t = \frac{d_ta(\varkappa_2\beta_o - \beta_1) + D_tA(K_2 B_o + B_1)}{d_t\varkappa_2 + D_t K_2} \qquad (21b)$$

These equations are quite symmetric. One must remember that β_o and B_o are the extrapolated fast neutron densities, $a\beta_o$ and AB_o the extrapolated thermal neutron densities, and the significance of β_1 and B_1 is similar. Except if the boundary between the two regions is quite close to the surface of the pile, the terms with β_1 and B_1 in (21) can be neglected because β_1/β_o is of the order of magnitude $\varkappa_1$. If one does this, the interpretation of (21) is very simple. It shows that the neutron densities at the

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boundary are weighted means between the extrapolated densities β_o and B_o for fast and a β_o and AB_o for thermal neutrons. The weighting factor is $d_f \varkappa_2$ and $D_f\ K_2$ in case of fast, $d_t \varkappa_2$ and $D_t\ K_2$ in case of thermal neutrons. The weighting factors for fast and thermal neutrons become equal in the special case treated in the fourth section.

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