Friction Term Formulation and Convective Instability in a Shallow Atmosphere

GARETH P. WILLIAMS

Geophysical Fluid Dynamics Laboratory, NOAA, Princeton University, Princeton, N. J. 08540

(Manuscript received 15 October 1971, in revised form 25 February 1972)

ABSTRACT

The form of the friction terms for a shallow layer of fluid on a sphere is discussed for isotropic and transversely-isotropic fluids. We then examine the nature of convection in a transversely-isotropic fluid and find that long flat convection cells with a width to height ratio of $2(\nu_H/\nu_V)^{\frac{1}{2}}$ are produced, where ν_H, ν_V are the horizontal and vertical diffusion coefficients. The critical Rayleigh number is given by $R^c=4\pi^4\nu_H/\nu_V$, but another Rayleigh number $P=\beta g\Delta T d^5/(\nu_V\kappa_V L^2)$, with a constant critical value $P^c=4\pi^4$ is shown to be a more relevant parameter. Results for convection in a rotating system are also given.

1. Introduction

The consistency of the so-called traditional approximation to the dynamical equations describing motion in a shallow atmosphere has been discussed by Phillips (1966) and Veronis (1968). Their discussion concerned the form of the rotation and inertia terms when the approximation $r\rightarrow a$ is to be made. Phillips showed that advantages can be realized by introducing the shallow approximation as a geometric approximation via the curvilinear scale parameters. The limitations of this approach and the non-generality of the resulting equations, however, were pointed out by Veronis.

In this paper we would like to discuss a related problem, namely that of choosing the most appropriate form of the friction terms for a shallow atmosphere. Such a formulation is needed for use in the numerical integration of equations describing laboratory and geophysical flows. In the derivation we will apply Phillips' geometric approximation to the *tensor* (2nd and 4th order) form of the stress and rate of strain relationships. (For the rotation and inertia terms Phillips introduces the approximation into the *vector*, i.e., the 1st order tensor forms). This approximation is chosen on the basis that it leads to the most consistent set of simplified equations without excluding any terms which maintain known physical processes (unlike the case with the rotation and inertia terms). However, the possibility of there being phenomena needing other approximate forms or the exact form should be remembered.

To establish the approximation procedure most simply we first derive the approximate form to the (isotropic) Navier-Stokes friction terms for a shallow shell. The resulting expressions are relevant to laboratory-scale flows of the type discussed, for example, by Greenspan (1968) or Baker and Robinson (1970).

For planetary-scale flow the form of the friction terms is less well defined, being dependent upon assumptions about the turbulence, and there is no unique exact form to which the geometric approximation can be applied. As a first approximation to this problem we derive the friction terms for a transversely-isotropic fluid and apply the geometric approximation. However, the assumption of transverse-isotropy itself implies *a priori* the shallowness of the system and it is thus an intrinsic element in a shallow atmosphere approximation.

To examine some of the properties of a transversely-isotropic fluid, an analytical solution describing the convective instability of such a fluid is obtained. The convection problem is a suitable one for analysis as the complex transversely-isotropic equations can be solved for this problem. Physically the solutions may give an idea of how large-scale convection can organize itself out of smaller scale mixing.

In summary, we consider three different but related problems concerning friction term behavior in a shallow atmosphere: 1) in Section 2 the geometric approximation is applied to the Navier-Stokes friction terms, 2) in Section 3 we consider the form of friction terms in a transversely-isotropic (and therefore shallow) fluid and the form when the geometric approximation is made, and 3) in Section 4 convective instability in a transversely-isotropic fluid is analyzed.

2. Geometric approximation of Navier-Stokes friction

The exact Navier-Stokes friction terms in spherical coordinates are

$$F_r\equiv\nu\left[\nabla^2 w-\frac{2w}{r^2}-\frac{2(v\sin\theta)_\theta}{r^2\sin\theta}-\frac{2u_\varphi}{r^2\sin\theta}\right],\tag{1a}$$

$$F_\theta\equiv\nu\left[\nabla^2 v-\frac{v}{r^2\sin^2\theta}+\frac{2w_\theta}{r^2}-\frac{2\cos\theta}{r^2\sin^2\theta}u_\varphi\right],\tag{1b}$$

$$F_\varphi \equiv \nu\left[\nabla^2 u - \frac{u}{r^2\sin^2\theta} + \frac{2w_\varphi}{r^2\sin\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}v_\varphi\right], \quad (1c)$$

where

$$\nabla^2 q \equiv \frac{1}{r^2}(r^2 q_r)_r + \frac{1}{r^2\sin\theta}(q_\theta\sin\theta)_\theta + \frac{1}{r^2\sin^2\theta}q_{\varphi\varphi}, \quad (2)$$

and

$$\frac{1}{r^2}(r^2 w)_r + \frac{1}{r\sin\theta}(v\sin\theta)_\theta + \frac{1}{r\sin\theta}u_\varphi = 0 \quad (3)$$

is the equation of mass conservation. An incompressible fluid and constant coefficient of viscosity ν are assumed for convenience. The expressions are given by Batchelor (1967) and we will use the classical physical coordinate system rather than the meteorological one. Thus, we take coordinates (r,θ,φ) to denote radius, co-latitude and longitude, respectively, and (w,v,u) to be the related velocities. Eqs. (1) are derived from stresses τ_{ij} and rate of strain functions e_{ij} which are taken to be related by the expression $\tau_{ij}=2\nu e_{ij}$. In spherical coordinates the exact rate of strain expressions are

$$\left.\begin{aligned}
&e_{rr}=w_r, \quad e_{\theta\varphi}=\frac{\sin\theta}{2r}\left(\frac{u}{\sin\theta}\right)_\theta + \frac{v_\varphi}{2r\sin\theta}\\
&e_{\theta\theta}=\frac{v_\theta}{r}+\frac{w}{r}, \quad e_{\varphi r}=\frac{w_\varphi}{2r\sin\theta}+\frac{r}{2}\left(\frac{u}{r}\right)_r\\
&e_{\varphi\varphi}=\frac{u_\varphi}{r\sin\theta}+\frac{w}{r}+\frac{v\cot\theta}{r}, \quad e_{r\theta}=\frac{r}{2}\left(\frac{v}{r}\right)_r+\frac{w_\theta}{2r}
\end{aligned}\right\}, \quad (4)$$

and the friction-to-stress relationships are

$$\left.\begin{aligned}
F_r&=\frac{\partial}{\partial r}\tau_{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}\tau_{r\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\tau_{r\varphi}\\
&\qquad+\frac{1}{r}(2\tau_{rr}-\tau_{\theta\theta}-\tau_{\varphi\varphi}+\tau_{r\theta}\cot\theta)\\
F_\theta&=\frac{\partial}{\partial r}\tau_{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\tau_{\theta\varphi}\\
&\qquad+\frac{1}{r}[(\tau_{\theta\theta}-\tau_{\varphi\varphi})\cot\theta+3\tau_{r\theta}]\\
F_\varphi&=\frac{\partial}{\partial r}\tau_{r\varphi}+\frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta\varphi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\tau_{\varphi\varphi}\\
&\qquad+\frac{1}{r}(3\tau_{r\varphi}+2\tau_{\theta\varphi}\cot\theta)
\end{aligned}\right\}. \quad (5)$$

Eqs. (4) and (5) yield those given in (1).

The question arises as to the form of (1)–(5) when the shallow layer approximation of $r=a+z$, where $z\ll a$, is to be made. We will follow Phillips suggestion of approximating the scale factors h_i from

$$\left.\begin{aligned}
&h_r=1;\quad h_\theta=r;\quad h_\varphi=r\sin\theta\\
&\qquad\text{to}\\
&h_z=1;\quad h_\theta=a;\quad h_\varphi=a\sin\theta
\end{aligned}\right\}. \quad (6)$$

For the friction terms to have a consistent stress-strain behavior the approximations are introduced first into the stress and rate-of-strain relationships and then the friction terms are derived from them. To do this we need the curvilinear form of the equations (e.g., Batchelor, 1968, p. 600; Love, 1944, p. 90):

$$\left.\begin{aligned}
e_{11}&=\frac{1}{h_1}\frac{\partial u_1}{\partial\xi_1}+\frac{u_2}{h_1h_2}\frac{\partial h_1}{\partial\xi_2}+\frac{u_3}{h_3h_1}\frac{\partial h_1}{\partial\xi_3}\\
e_{23}&=\frac{h_3}{2h_2}\frac{\partial}{\partial\xi_2}\left(\frac{u_3}{h_3}\right)+\frac{h_2}{2h_3}\frac{\partial}{\partial\xi_3}\left(\frac{u_2}{h_2}\right)
\end{aligned}\right\}, \quad (7)$$

$$\begin{aligned}
F_{\xi_1}=\frac{1}{h_1h_2h_3}\Big[&\frac{\partial}{\partial\xi_1}(h_2h_3\tau_{11})+\frac{\partial}{\partial\xi_2}(h_1h_3\tau_{12})+\frac{\partial}{\partial\xi_3}(h_1h_2\tau_{13})\Big]\\
&+\frac{\tau_{12}}{h_1h_2}\frac{\partial h_1}{\partial\xi_2}+\frac{\tau_{13}}{h_1h_3}\frac{\partial h_1}{\partial\xi_3}-\frac{\tau_{22}}{h_1h_2}\frac{\partial h_2}{\partial\xi_1}-\frac{\tau_{33}}{h_1h_3}\frac{\partial h_3}{\partial\xi_1}\Big].
\end{aligned} \quad (8)$$

Introducing the scale factor approximation into (7) and (8) leads to the following approximate rate of strain and friction-stress expressions:

$$\left.\begin{aligned}
&e_{zz}=w_z, \quad e_{\theta\varphi}=\frac{u_\theta}{2a}-\frac{u\cot\theta}{2a}+\frac{v_\varphi}{2a\sin\theta}\\
&e_{\theta\theta}=\frac{v_\theta}{a}, \quad e_{\varphi z}=\frac{w_\varphi}{2a\sin\theta}+\frac{u_z}{2}\\
&e_{\varphi\varphi}=\frac{u_\varphi}{a\sin\theta}+\frac{v\cot\theta}{a}, \quad e_{z\theta}=\frac{v_z}{2}+\frac{w_\theta}{2a}
\end{aligned}\right\}, \quad (9)$$

$$\left.\begin{aligned}
F_z&=\frac{\partial}{\partial z}\tau_{zz}+\frac{\partial}{a\partial\theta}\tau_{z\theta}+\frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi}\tau_{z\varphi}+\frac{\tau_{z\theta}\cot\theta}{a}\\
F_\theta&=\frac{\partial}{\partial z}\tau_{z\theta}+\frac{\partial}{a\partial\theta}\tau_{\theta\theta}+\frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi}\tau_{\theta\varphi}\\
&\qquad+\frac{1}{a}(\tau_{\theta\theta}-\tau_{\varphi\varphi})\cot\theta\\
F_\varphi&=\frac{\partial}{\partial z}\tau_{z\varphi}+\frac{\partial}{a\partial\theta}\tau_{\theta\varphi}+\frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi}\tau_{\varphi\varphi}+\frac{2}{a}\tau_{\theta\varphi}\cot\theta
\end{aligned}\right\}, \quad (10)$$

with the equation of mass conservation becoming

$$w_z+\frac{u_\varphi}{a\sin\theta}+\frac{v_\theta}{a}+\frac{v}{a}\cot\theta=0. \quad (11)$$

Substitution of (9) into (10) using $\tau_{ij}=2\nu e_{ij}$ gives the following friction terms for a shallow layer:

$$F_z=\nu\nabla_a^2 w, \tag{12a}$$

$$F_\theta=\nu\left[\nabla_a^2 v-\frac{v\cos 2\theta}{a^2\sin^2\theta}-2\frac{u_\varphi\cot\theta}{a^2\sin\theta}\right], \tag{12b}$$

$$F_\varphi=\nu\left[\nabla_a^2 u-\frac{u\cos 2\theta}{a^2\sin^2\theta}+2\frac{v_\varphi\cot\theta}{a^2\sin\theta}\right], \tag{12c}$$

where

$$\nabla_a^2 q\equiv q_{zz}+\frac{1}{a^2}\left[\frac{1}{\sin\theta}(q_\theta\sin\theta)_\theta+\frac{q_{\varphi\varphi}}{\sin^2\theta}\right]. \tag{13}$$

Eqs. (12b,c) can also be written

$$F_\theta=\nu\left\{v_{zz}+\frac{1}{a^2\sin^2\theta}\left[\sin^3\theta\left(\frac{v}{\sin\theta}\right)_\theta\right]_\theta+\frac{1}{a^2\sin^2\theta}(v_{\varphi\varphi}-2u_\varphi\cos\theta)\right\}, \tag{14a}$$

$$F_\varphi=\nu\left\{u_{zz}+\frac{1}{a^2\sin^2\theta}\left[\sin^3\theta\left(\frac{u}{\sin\theta}\right)_\theta\right]_\theta+\frac{1}{a^2\sin^2\theta}(u_{\varphi\varphi}+2v_\varphi\cos\theta)\right\}, \tag{14b}$$

and the latter shows that for solid rotation, $u=a\sin\theta$, the fluid is stress free. This corresponds to the case of $u=r\sin\theta$ in the exact equations, set (1). Furthermore, the form of the second term of (14b) indicates that relative angular momentum can only be changed by torques exerted on the boundaries. We will show at the end of Section 3, for a general case that includes Eqs. (12), that the dissipation of kinetic energy associated with (12) is positive definite and give by $\Phi=2\nu e_{ij}e_{ij}$.

The equation set (12) is similar to that for a cylindrical system of equations as might be expected. The unusual term involving $\cos 2\theta$ makes possible the transformation to the form (14) and thence the angular momentum properties. The system of equations (9), (10) and (12) derived by approximating (7) and (8) is not the same as that which would be produced by letting $r\rightarrow a$ in (1), (4) and (5). Neither would this system be obtained from the *vector* invariant form

$$\mathbf{F}=-\nu\nabla_a\wedge(\nabla_a\wedge\mathbf{V}).$$

For these latter approximations the solid rotation must be defined as $u=r\sin\theta$ or $(r^2/a)\sin\theta$ to give approximate zero dissipation to within $(r-a)/a$ order. The stress-strain relationship of those systems would also only be consistent to within this order.

Thus, the system of equations (9)–(12) appears to be the most consistent set of approximated equations. Although there are limitations to this formalistic approach it is believed that this equation system does not exclude any physical process. Also it is to be noted that the hydrostatic assumption has not been made and that w-related contributions to friction have been retained.

3. Geometric approximation of transversely-isotropic friction

The set of approximated Navier-Stokes equations, (12), is appropriate to laminar flows on a laboratory scale. When turbulent flows are to be discussed it is customary to use similar equations, replacing the molecular viscosity by an eddy coefficient K. This coefficient represents a parameterization of the turbulence stresses which are assumed to behave in an analogous way to the molecular stresses and in the simplest closing procedure are taken to be proportional to the deformation, i.e.,

$$\bar{\tau}_{ij}=-\overline{u_i'u_j'}+\tfrac{1}{3}\delta_{ij}\overline{u_k'u_k'}=\frac{K}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right), \tag{15}$$

where $\bar{\tau}_{ij}$ represents the deviatoric turbulence stress tensor. The term $\frac{1}{3}\delta_{ij}\overline{u_k'u_k'}$ is included in the stress tensor and turbulent pressure term to ensure a correct relation (15) when the stress tensor is contracted. Hinze (1959, p. 21) and Monin and Yaglom (1971, Section 6.3) provide a complete discussion of the formulation.

On a planetary scale the large difference between the horizontal and vertical length scales means that the large-scale turbulence is non-isotropic and that the eddy coefficient K becomes a fourth-order tensor. The non-isotropic stress formulation is similar to that adopted in studies of the elastic properties of crystals or materials such as wood which have preferred directions or "grains," as was noted by Richardson (1922, p. 222). Complete non-isotropy has a complex stress-strain behavior involving 21 viscosity coefficients (see, e.g., Love, 1944, p. 159). For planetary-scale flows it can, however, be assumed that the fluid is isotropic in the horizontal plane.[1] This provides a substantial reduction in complexity and forms so called "transverse-isotropy" (Love, p. 161).

The formulation of transversely-isotropic stress for planetary-scale flow has been discussed by Saint-Guily (1956), Smagorinsky (1963), Kamenkovich (1967) and Kirwan (1969). Although these derivations are consistent the authors ignore the form of the w-stress term which we require for completeness and application to the convection problem in Section 4. Thus, we will provide an alternative derivation of the transversely-isotropic stresses, initially in a Cartesian framework for simplicity, and then discuss the form in spherical coordinates under the geometric approximation.

[1] Except close to any lateral boundary where the problem is more complex.

a. The Cartesian transverse-isotropy relations

In this section we use the Cartesian coordinates x_1, x_2, x_3 with velocities u_1, u_2, u_3. These are interchanged with x, y, z and u, v, w, respectively, when simplification is possible. The stress-strain relationship for an elastic body that is transversely-isotropic with respect to the x_3 direction can be written upon obtaining the symmetry properties as (after Green and Zerna, 1954)

$$\begin{bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}c_{11}{}^{11} & c_{22}{}^{11} & c_{33}{}^{11} & 0 & 0 & 0\\ c_{22}{}^{11} & c_{11}{}^{11} & c_{33}{}^{11} & 0 & 0 & 0\\ c_{33}{}^{11} & c_{33}{}^{11} & c_{33}{}^{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{13}{}^{13} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{13}{}^{13} & 0\\ 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2}(c_{11}{}^{11}-c_{22}{}^{11})\end{bmatrix}\begin{bmatrix}e_{11}\\ e_{22}\\ e_{33}\\ 2e_{23}\\ 2e_{13}\\ 2e_{12}\end{bmatrix}, \tag{16}$$

where

$$e_{rs}=\frac{1}{2}\left(\frac{\partial u_r}{\partial x_s}+\frac{\partial u_s}{\partial x_r}\right).$$

From the stress-strain matrix (16), we obtain the simplified stress forms:

$$\left.\begin{aligned}\tau_{xx}&=A_1u_x+A_2v_y+A_3w_z,\\ \tau_{yy}&=A_2u_x+A_1v_y+A_3w_z,\\ \tau_{zz}&=A_3(u_x+v_y)+A_4w_z,\\ \tau_{yz}&=A_5(v_z+w_y)\\ \tau_{zx}&=A_5(w_x+u_z)\\ \tau_{xy}&=\tfrac{1}{2}(A_1-A_2)(u_y+v_x)\end{aligned}\right\}. \tag{17}$$

As the friction equations deal with deviatoric stress the constraint $\tau_{11}+\tau_{22}+\tau_{33}=0$ must also be added. [Note that this constraint also holds for the turbulence stresses in Eq. (15).] This constraint gives $A_3+A_4=A_1+A_2$ for an incompressible fluid. Anticipating the final form of the equations, we define $\nu_H\equiv\frac{1}{2}(A_1-A_2)$, $\nu_V\equiv A_5$ and $\epsilon\equiv A_3-A_2$. Then Eqs. (17) may be rewritten

$$\left.\begin{aligned}\tau_{xx}&=2\nu_Hu_x+\epsilon w_z, & \tau_{yz}&=\nu_V(v_z+w_y)\\ \tau_{yy}&=2\nu_Hv_y+\epsilon w_z, & \tau_{xz}&=\nu_V(w_x+u_z)\\ \tau_{zz}&=2(\nu_H-\epsilon)w_z, & \tau_{xy}&=\nu_H(u_y+v_x)\end{aligned}\right\}, \tag{18}$$

which upon substitution into $F_i=\partial\tau_{ij}/\partial x_j$ give the friction terms:

$$\left.\begin{aligned}F_x&=\nu_H(u_{xx}+u_{yy})+\nu_Vu_{zz}-(\nu_H-\nu_V-\epsilon)w_{zx}\\ F_y&=\nu_H(v_{xx}+v_{yy})+\nu_Vv_{zz}-(\nu_H-\nu_V-\epsilon)w_{zy}\\ F_z&=\nu_V(w_{xx}+w_{yy})+(2\nu_H-\nu_V-2\epsilon)w_{zz}\end{aligned}\right\}. \tag{19}$$

Alternatively we could define $\nu_*=A_1+A_2-A_5-2A_3$ and derive the friction equations in the form

$$\left.\begin{aligned}F_x&=\nu_H(u_{xx}+u_{yy})+\nu_Vu_{zz}-\tfrac{1}{2}(\nu_*-\nu_V)w_{zx}\\ F_y&=\nu_H(v_{xx}+v_{yy})+\nu_Vv_{zz}-\tfrac{1}{2}(\nu_*-\nu_V)w_{zy}\\ F_z&=\nu_V(w_{xx}+w_{yy})+\nu_*w_{zz}\end{aligned}\right\}. \tag{20}$$

The equation set (20) reduces to the isotropic Navier-Stokes equations if $\nu_H=\nu_V=\nu_*=\nu$ and must therefore be a form appropriate to a limiting case of transverse isotropy in which the ratio of the vertical length scale d to the horizontal L approaches unity. However, the form (19) is more appropriate for a shallow atmosphere and our discussion will concentrate on it. The set (19) reduces to the Navier-Stokes form if $\nu_H=\nu_V=\nu$ and $\epsilon=0$.

An interesting result in (19) is that the horizontal coefficient of the F_z component is ν_V not ν_H as might be assumed in a simple splitting procedure. The interdependence among the coefficients of the three equations (19) makes it impossible to make separate assumptions in approximating this system.

The unknown quantity in (19) is the coefficient ϵ. The magnitude of this coefficient probably lies between the two limiting values of $\epsilon\approx 0$ and $\epsilon\approx\nu_H$, depending on the type of atmospheric motion under consideration. Comparing the stress formulation (18) with the turbulence stresses (15) we see that ϵ is related to $\overline{w'^2}$, and thus its magnitude is determined by the degree of vertical turbulence. For an atmosphere such as Jupiter's that is highly convective in the vertical, it will probably be large so that $\epsilon\approx\nu_H$ can be expected. For a less convective or stable atmosphere where $\overline{w'^2}\ll\overline{v'^2}$, such as the earth's atmosphere or ocean, then $\epsilon\approx 0$ can be expected.

The energy dissipation Φ produced by the stress forms (18) can be obtained from $\Phi=\tau_{ij}e_{ij}$ as

$$\Phi=\nu_H[(e_{xx}-e_{yy})^2+4e_{xy}{}^2]+3(\nu_H-\epsilon)e_{zz}{}^2+4\nu_V(e_{xz}{}^2+e_{yz}{}^2), \tag{21}$$

which is positive definite if $\nu_H>0$, $\nu_V>0$, $\nu_H>\epsilon$. This expression also reduces to the Navier-Stokes equivalent.

b. The spherical transverse-isotropy relations

We can obtain the geometric approximated stresses in a transversely-isotropic shallow atmosphere by substituting the rate of strain relations (9) into the stress-strain matrix (16) to give

$$\left.\begin{aligned}\tau_{\varphi\varphi}&=\nu_H\left(\frac{u_\varphi}{a\sin\theta}+\frac{v\cot\theta}{a}\right)+\epsilon w_z, & \tau_{\theta z}&=\nu_V\left(v_z+\frac{w_\theta}{a}\right)\\ \tau_{\theta\theta}&=\nu_H\left(\frac{v_\theta}{a}\right)+\epsilon w_z, & \tau_{\varphi z}&=\nu_V\left(u_z+\frac{w_\varphi}{a\sin\theta}\right)\\ \tau_{zz}&=2(\nu_H-\epsilon)w_z, & \tau_{\theta\varphi}&=\nu_H\left(\frac{u_\theta}{a}-\frac{u\cot\theta}{a}+\frac{v_\varphi}{a\sin\theta}\right)\end{aligned}\right\}. \tag{22}$$

Applying (22) to (10) gives the required friction terms

$$\left.\begin{aligned} F_z &= \frac{\nu_V}{a^2}\nabla_H{}^2 w + (2\nu_H - \nu_V - 2\epsilon) w_{zz} \\ F_\theta &= \frac{\nu_H}{a^2}\left[\nabla_H{}^2 v - v\frac{\cos 2\theta}{\sin^2\theta} - 2u_\varphi \frac{\cot\theta}{\sin\theta}\right] \\ &\qquad + \nu_V v_{zz} - (\nu_H - \nu_V - \epsilon)\frac{w_{z\theta}}{a} \\ F_\varphi &= \frac{\nu_H}{a^2}\left[\nabla_H{}^2 u - u\frac{\cos 2\theta}{\sin^2\theta} + 2v_\varphi \frac{\cot\theta}{\sin\theta}\right] \\ &\qquad + \nu_V u_{zz} - (\nu_H - \nu_V - \epsilon)\frac{w_{z\varphi}}{a\sin\theta} \end{aligned}\right\}, \quad (23)$$

where

$$\nabla_H{}^2 q \equiv q_{\theta\theta} + q_\theta \cot\theta + \frac{q_{\varphi\varphi}}{\sin^2\theta}.$$

To obtain the kinetic energy dissipation associated with (23), Eqs. (10) are multiplied by the velocity components. This leads upon regrouping to

$$wF_z + vF_\theta + uF_\varphi = \Delta - \Phi, \quad (24)$$

where

$$\begin{aligned} \Delta \equiv & (w\tau_{zz} + v\tau_{z\theta} + u\tau_{z\varphi})_z \\ & + \frac{1}{a\sin\theta}[\sin\theta(w\tau_{z\theta} + v\tau_{\theta\theta} + u\tau_{\theta\varphi})]_\theta \\ & + \frac{1}{a\sin\theta}(w\tau_{z\varphi} + v\tau_{\theta\varphi} + u\tau_{\varphi\varphi})_\varphi, \end{aligned} \quad (25)$$

$$\begin{aligned} \Phi \equiv & \tau_{zz} w_z + \tau_{\theta\theta}\frac{v_\theta}{a} + \tau_{\varphi\varphi}\left(\frac{u_\varphi}{a\sin\theta} + \frac{v}{a}\cot\theta\right) + \tau_{z\theta}\left(\frac{w_\theta}{a} + v_z\right) \\ & + \tau_{z\varphi}\left(\frac{w_\varphi}{a\sin\theta} + u_z\right) + \tau_{\theta\varphi}\left(\frac{v_\varphi}{a\sin\theta} + \frac{u_\theta}{a} - \frac{u}{a}\cot\theta\right). \end{aligned} \quad (26)$$

The quantity Δ represents the energy diffusion (zero volume integral) and Φ is the dissipation. Upon rewriting the various terms in (26) in terms of the rate of strain functions using definitions in (9) and (22) and rearranging, we obtain

$$\begin{aligned} \Phi = & \nu_H[(e_{\theta\theta} - e_{\varphi\varphi})^2 + 4e_{\theta\varphi}{}^2] \\ & + 3(\nu_H - \epsilon)e_{zz}{}^2 + 4\nu_V(e_{z\theta}{}^2 + e_{z\varphi}{}^2). \end{aligned} \quad (27)$$

Thus, the dissipation is positive definite and given by $\Phi = \tau_{ij}e_{ij}$ as comparison with (21) shows.

4. Convection in a transversely-isotropic fluid

It is possible to obtain analytical solutions to the equation set (19) for the convective instability problem. This provides a comparison of the alternative stress formulations for this particular problem. The nature of convection in a turbulent flow is not well understood so although our solutions for convection in a transverse-isotropic fluid may give some idea of how large-scale convection can organize itself out of small scale mixing, we cannot as yet establish the physical reality of such convection. However, experiments on turbulent convection between two large horizontal plates with a narrow gap, a system resembling a shallow atmosphere, indicate that elongated convection cells of the type to be discussed do exist (Deardorff and Willis, 1968). This lends some credence to the eddy viscosities. The analysis may be more relevant to the problem of understanding the type of convection that could occur in a mathematical system designed for studies of planetary-scale flow. For simplicity the convection is examined in a Cartesian framework. Convection on a sphere has been examined by numerical integration and will be discussed in a separate paper.

The standard Bénard cells of classical convection theory have a fixed cell-width-to-depth ratio of about 3, whereas for some observed geophysical and astrophysical cells this ratio is much larger. Ray (1965) has shown that the non-isotropy of eddy mixing can alter the shape of the convection cells and perhaps explain the observations. Ray's analysis was for friction terms which were of the form

$$\mathbf{F} = \nu_H \nabla_1{}^2 \mathbf{V} + \nu_V \mathbf{V}_{zz} \quad (28)$$

in all three velocity components. This so-called *split* form of the friction terms is clearly different to that for a transversely-isotropic fluid [(19)]. We will compare the solutions obtained for the two sets, (19) and (28). The complex form of Ray's numerical solutions obscures the physical results so we will also derive simplified expressions for the cell properties.

a. *Solutions of the perturbation equations*

The equations of motion for a Boussinesq fluid are

$$\left.\begin{aligned} \frac{Du}{Dt} &= -\pi_x + \nu_H\nabla_1{}^2 u + \nu_V u_{zz} - \nu_1 w_{xz} \\ \frac{Dv}{Dt} &= -\pi_y + \nu_H\nabla_1{}^2 v + \nu_V v_{zz} - \nu_1 w_{yz} \\ \frac{Dw}{Dt} &= -\pi_z + \beta g T + \nu_V\nabla_1{}^2 w + \nu_2 w_{zz} \\ \frac{DT}{Dt} &= \kappa_H\nabla_1{}^2 T + \kappa_V T_{zz} \\ & u_x + v_y + w_z = 0 \\ & \nabla_1{}^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 \end{aligned}\right\}, \quad (29)$$

where $\pi=p/\rho_0$ is the hydrostatic pressure deviation, T the temperature difference from the value T_0 of the lower boundary, κ_H, κ_V are thermal conductivities, β the expansion coefficient, and $\nu_1\equiv\nu_H-\nu_V-\epsilon$, $\nu_2\equiv 2\nu_H-\nu_V-2\epsilon$. For an unbounded horizontal layer of fluid of depth d heated from below by a temperature differential ΔT, the perturbation equations are

$$D_3T\equiv\left(\frac{\partial}{\partial t}-K_H\nabla_1^2-K_V\frac{\partial^2}{\partial z^2}\right)T=w\frac{\Delta T}{d}, \tag{30a}$$

$$D_1u\equiv\left(\frac{\partial}{\partial t}-\nu_H\nabla_1^2-\nu_V\frac{\partial^2}{\partial z^2}\right)u=-\pi_x-\nu_1 w_{xz}, \tag{30b}$$

$$D_1v=-\pi_y-\nu_1 w_{yz}, \tag{30c}$$

$$D_2w\equiv\left(\frac{\partial}{\partial t}-\nu_V\nabla_1^2-\nu_2\frac{\partial^2}{\partial z^2}\right)w=-\pi_z+\beta gT, \tag{30d}$$

where the static equilibrium state is given by

$$T^*=T_0-\Delta T\frac{z}{d},\quad \pi_z{}^*=\beta gT^*, \tag{31}$$

and D_1, D_2, D_3 are differential operators.

Eliminating pressure and temperature in (30) gives an equation for w:

$$D_3[(D_1-\nu_1\nabla_1^2)w_{zz}+D_2\nabla_1^2w]=\beta g\frac{\Delta T}{d}\nabla_1^2w. \tag{32}$$

Following Chandrasekhar (1961) we substitute

$$w=W(z)\exp[i(k_xx+k_yy)+\omega t]$$

and then set $\omega=0$ to obtain the equation for the marginal state. Following Chandrasekhar it can be shown that ω is real. This gives

$$[(D^2-ma^2)(D^4+a^4-n^*a^2D^2)+\mathrm{R}a^2]W=0, \tag{33}$$

where we have made the definitions

$$\mathrm{R}=\frac{\beta g\Delta Td^3}{\kappa_V\nu_V},\quad a^2=d^2(k_x{}^2+k_y{}^2),\quad \zeta=\frac{z}{d},\quad D=\frac{d}{d\zeta},$$
$$\kappa_H=m\kappa_V,\quad \nu_H=n\nu_V,\quad \epsilon=l\nu_V$$

so that $n^*=4n-2-3l$.

For simplicity, the case of two free boundaries is taken and as in Chandrasekhar it can be shown that the boundary conditions are $W=D^{2M}W=0$ so that $W=W_0\sin r\pi\zeta$ is the form of the solution. This gives for $r=1$ the critical Rayleigh number as

$$\mathrm{R}^c=\left(ma+\frac{\pi^2}{a}\right)\left(a^3+\frac{\pi^4}{a}+a\pi^2n^*\right), \tag{34}$$

where a is given by $\partial\mathrm{R}/\partial a=0$, i.e.,

$$4mb^3+2\pi^2(mn^*+1)b^2-2\pi^6=0, \tag{35}$$

where

$$b=a^2.$$

If the above analysis is repeated with the "split" friction form (28), we have (Ray, 1965).

$$\mathrm{R}^c=\left(ma+\frac{\pi^2}{a}\right)\left(na+\frac{\pi^2}{a}\right)(\pi^2+a^2), \tag{36}$$

$$4mb^3+2\pi^2(m+n+mn)b^2-2\pi^6=0. \tag{37}$$

Although the cubic equations (35) and (37) can be solved numerically it is more informative to derive their simple asymptotic solutions for large m. The two extreme values of ϵ give the range of values for a and R^c in (34) and (35) as

$$\left.\begin{array}{l}\text{(i)}\quad \epsilon=\nu_H \quad\text{and}\quad l=n=m,\\ \quad a=\pi/(m)^{\frac{1}{4}},\quad \gamma=2(m)^{\frac{1}{4}},\quad \mathrm{R}^c=4m\pi^4\end{array}\right\} \tag{38}$$

$$\left.\begin{array}{l}\text{(ii)}\quad \epsilon=0 \quad\text{and}\quad m=n,\quad l=0,\\ \quad a=\pi/(2m)^{\frac{1}{4}},\quad \gamma=(8m)^{\frac{1}{4}},\quad \mathrm{R}^c=9m\pi^4\end{array}\right\} \tag{39}$$

where $\gamma\equiv 2\pi/a$ is the cell wavelength width-to-depth ratio. The cubic (37) has the same asymptotic solution as (38). Clearly there is significant qualitative difference between the two solutions (38) and (39). The lower R^c in (38) is consistent with the earlier argument that $\epsilon\approx\nu_H$ is likely in a convective atmosphere and that $\epsilon\approx 0$ represents a more stable atmosphere. The asymptotic solutions correspond to solving equations (30) with $D_2=0$, i.e., quasi-hydrostatic.

b. *Dimensional analysis*

To interpret the analytical results further it is useful to make a dimensional analysis of the governing equations to extract the basic parameters. Consider for convenience the case of two-dimensional motion and introduce a streamfunction ψ and vorticity η as

$$v=\psi_z,\quad w=-\psi_y,\quad \eta=\psi_{zz}+\psi_{yy}. \tag{40}$$

The governing equations may then be written

$$\frac{\partial\eta}{\partial t}+\frac{\partial(\psi,\eta)}{\partial(z,y)}=-\beta gT_y+\nu_H\eta_{yy}+\nu_V\eta_{zz}, \tag{41}$$

$$\frac{\partial T}{\partial t}+\frac{\partial(\psi,T)}{\partial(z,y)}=\kappa_HT_{yy}+\kappa_VT_{zz}. \tag{42}$$

Taking L and d as horizontal and vertical length scales where $L\gg d$, ψ^* as the scale of the streamfunction, ψ^*d^{-2} for the vorticity and τ for time, we can nondimensionalize (41) and (42) as

$$\frac{\psi^*}{d^2\tau}\frac{\partial\eta}{\partial t}+\frac{\psi^{*2}}{Ld^3}\frac{\partial(\psi,\eta)}{\partial(z,y)}=-\frac{\beta g\Delta T}{L}T_y$$
$$+\frac{\nu_V\psi^*}{d^4}\left(\eta_{zz}+\frac{\nu_Hd^2}{\nu_VL^2}\eta_{yy}\right), \tag{43}$$

$$\frac{\Delta T}{\tau}\frac{\partial T}{\partial t}+\frac{\Delta T\psi^*}{Ld}\frac{\partial(\psi,T)}{\partial(z,y)}=\frac{\kappa_V\Delta T}{d^2}\left(T_{zz}+\frac{\kappa_H d^2}{\kappa_V L^2}T_{yy}\right), \quad (44)$$

where the variables now represent dimensionless forms.

Comparing the second and third terms of (44) gives $\psi^*=\kappa_V(L/d)$. And then comparing the two right-hand terms of (43) gives a parameter

$$\mathrm{P}=\frac{\beta g\Delta T d^5}{\nu_V\kappa_V L^2}, \quad (45)$$

the Rayleigh number for this problem of elongated cells. The nonlinear vorticity term is of the order of κ_V/ν_V in relation to the other terms. We have assumed that the viscosity parameter $\nu_H d^2/\nu_V L^2$ is of the order of unity.

5. Conclusions

The isotropic and transverse isotropic friction equations have been derived for a shallow layer of fluid on a sphere [(12) and (23)]. The non-isotropy introduces additional terms with interdependent viscosity coefficients whose values depend on the nature of the atmosphere under examination.

The analysis of Section 4 has shown that long flat convection cells can be produced in a shallow convective atmosphere and that the cell shape parameter and critical Rayleigh number have simple forms $\gamma=2m^{\frac{1}{2}}$ and $\mathrm{R}^c=4m\pi^4$ when $\epsilon\sim\nu_H$. The parameter m may be called the relative mixing ratio.

Dimensional analysis indicates that $\mathrm{P}=\mathrm{R}d^2/L^2$ is the relevant parameter for elongated convection not R. If the length scale L is taken to be the cell width then we have that $L=m^{\frac{1}{2}}d$ and this gives $\mathrm{P}=\mathrm{R}/m$ as the basic parameter. Thus, the onset of convection depends upon $P=4\pi^4$ which is constant.

It is a straightforward extension to include rotation into the convection analysis for the case of large $l=m=n$, i.e., $\epsilon\sim\nu_H$. When this is done we obtain

$$a^2=\frac{\alpha}{m}, \quad \gamma=2\pi\left(\frac{m}{\alpha}\right)^{\frac{1}{2}}, \quad \mathrm{R}^c=2\pi^2 m(\alpha+\pi^2), \quad (46)$$

where $\alpha^2\equiv\pi^4+T_a$, $T_a\equiv4\Omega^2d^4/\nu_V^2$. For the less interesting case of $\epsilon=0$ we obtain the result

$$a^2=\frac{\alpha}{2m}, \quad \gamma=2\pi(2m/\alpha)^{\frac{1}{2}}, \quad \mathrm{R}^c=m\pi^2(4\alpha+5\pi^2). \quad (47)$$

The analysis of elongated convection cells also implies that the use of eddy viscosities in numerical models prevents the growth of small-scale convection with the limiting horizontal length scale being defined as $d(\nu_H/\nu_V)^{\frac{1}{2}}$.

REFERENCES

Baker, D. J., and A. R. Robinson, 1970: A laboratory model for the general ocean circulation. *Phil. Trans. Roy. Soc., London*, **A267**, 533–566.

Batchelor, G. K., 1967: *An Introduction to Fluid Dynamics*. The Cambridge University Press, 270 pp.

Chandrasekhar, S., 1961: *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, 652 pp.

Deardorff, J. W., and G. E. Willis, 1968: Investigation of turbulent thermal convection between horizontal plates. *J. Fluid Mech.*, **28**, 675–704.

Green, A. E., and W. Zerna, 1954: *Theoretical Elasticity*. Oxford University Press, 442 pp.

Greenspan, H. P., 1968: *The Theory of Rotating Fluids*. Cambridge University Press, 327 pp.

Hinze, J. O., 1959: *Turbulence*. New York, McGraw-Hill, 586 pp.

Kamenkovich, V. M., 1967: On the coefficients of eddy diffusion and eddy viscosity in large-scale oceanic and atmospheric motions. *Izv. Atmos. Oceanic Phys.*, **3**, 1326–1333.

Kirwan, A. D., 1969: Formulation of constitutive equations for large-scale turbulent mixing. *J. Geophys. Res.*, **74**, 6953–6959.

Love, A. E. H., 1944: *Mathematical Theory of Elasticity*, 4th ed. Dover Publ., 643 pp.

Monin, A. S., and A. N. Yaglom, 1971: *Statistical Hydromechanics*, Vol. 1. M.I.T. Press.

Phillips, N. A., 1966: The equations of motion for a shallow rotating atmosphere and the "traditional approximation." *J. Atmos. Sci.*, **23**, 626–628.

Ray, D., 1965: Cellular convection with nonisotropic eddies. *Tellus*, **17**, 434–439.

Richardson, L. F., 1922: *Weather Prediction by Numerical Process*. Cambridge University Press, 236 pp.

Saint-Guily, B., 1956: Sur la thèorie des courants marins induits par le vent. *Ann. L'Inst. Oceanogr.*, **33**, 1–64.

Smagorinsky, J., 1963: A formulation of eddy transport processes for the quasi-static inertial subrange of atmospheric motions (unpublished manuscript).

Veronis, G., 1968: Comments on Phillips' proposed simplification of the equations of motion for a shallow rotating atmosphere. *J. Atmos. Sci.*, **25**, 1154–1155.