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A TIME AVERAGE MODEL OF THE GENERAL CIRCULATION

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In this paper we attempt to construct equations describing the spatial distribution on a spherical earth of the mean flow of momentum and heat, averaged over a long period of time such as a year and around latitude circles.

We begin by writing down the hydrodynamical equations in spherical polar coordinates (r — radial distance, and θ — colatitude), using axes fixed at the centre of the earth, and we omit terms involving viscosity because the Reynolds stresses are generally much greater than viscous stresses. We then assume that in our model the surface is uniformly rough, so that when a long period time average is taken, terms depending on longitude can be neglected. Equations of motion relative to the surface are next obtained by subtracting terms describing a uniform solid rotation at constant temperature T_0 and earth's angular velocity Ω (details are given by Davies and Oakes [1]). Denoting the zonal velocity by u, the meridional velocity by v, the vertical velocity by w, the temperature by T, and following the Reynolds technique we write $u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$, $T = \bar{T} + T'$, where the bar indicates an average over a long time period. The non-linear inertial terms in the equations of motion then lead to the three shearing Reynolds stresses; $\rho\overline{u'v'}$ describing the average meridional flow of zonal momentum, $\rho\overline{u'w'}$ the vertical flow of zonal momentum, and $\rho\overline{v'w'}$ the vertical flow of meridional momentum: normal stresses such as $\rho\overline{u'u'}$ are generally considered to be small compared to static pressures and are neglected and so are ρ' terms.

To close the problem mathematically we now have to decide on a physical premise. The background physical picture on which our model is based consists of large scale turbulence in which angular momentum is directed (on average) along meridians from low to high latitudes; this is activated by the mean meridional temperature gradient; the eddy energy is then broken down by the effects of surface friction and small scale turbulence and a mean vertical transfer of momentum ensues. For this large scale turbulence we write (in the troposphere only)

$$\overline{u'v'} = -k_{zm}(\zeta)\,\Omega\partial T/a\partial\theta, \qquad (1)$$

where a — mean radius of the earth, Ω — angular velocity of the earth, $\zeta = (r-a)/h$, h — height of the model troposphere so that $0<\zeta<1$, and $k_{zm}(\zeta)$ is taken to be a quadratic function of ζ to represent the known increase of $\overline{u'v'}$ with ζ. We have assumed as a «working» hypothesis a linear dependence of $\overline{u'v'}$ on $\partial T/\partial\theta$, but the flow must of course be strongly influenced by Coriolis force and we assume a linear dependence on Ω also. The relation (1) then gives us a mean northward eddy transport of angular momentum except at high latitudes and there is no need to introduce the artificial device of a negative eddy coefficient. A more detailed discussion of (1) is given in [2].

For vertical transfer by smaller eddies we follow classical theory and write

$$\overline{u'w'} = -K_{zv}\partial\bar{u}/\partial r, \qquad (2)$$

and

$$\overline{v'w'} = -K_{mv}\partial\bar{v}/\partial r. \qquad (3)$$

The first is generally true with K_{zv} positive (see e. g. [3]) except in the mean jet region where thermal gradients must become primary factors; there is evidence [4] that K_{zv} depends on the static stability, but in a first approach we assume K_{zv} and other turbulence parameters to be independent of T and of θ. The evidence concerning $\overline{v'w'}$ is not so clear cut and it is only possible to make approximate estimates of K_{mv} from an analysis given by Molla and Loisel [5]. However, (1), (2), (3) are consistent with the basic physical model if not with the real atmosphere at all points. We also need an expression for mean meridional transfer of sensible heat and we take

$$\overline{v'T'} = -K_{mT}\partial T/a\partial\theta; \qquad (4)$$

this is not of course applicable in the lower stratosphere. Various analyses of observations suggest *approximate* values of the turbulence parameters to be of the following orders of magnitude: $\Omega\, k_{zm} = 10^{14}$ *cm·sec*, $K_{zv} = 10^6$ *cm²·sec*, $K_{mv} = 10^8$ *cm·sec*, $K_{mT} = 10^{10}$ *cm·sec*.

We now substitute (1), (2), (3) into the equation of flow, carry out some lengthy manipulation using boundary layer approximations, assume time derivatives can be neglected in a first approximation, on the grounds that the *main* climatological characteristics change only very slow from year to year, and take h to be independent of θ. We then derive a set of differential equations describing the mean flow. We express these in terms of non-dimensional velocities (u, v) by multiplying $(\bar{u}, \bar{v})$ by K_{mv}/gh^2 writing $\Omega\, h^2/K_{mv} = P$, and retaining only the dominant terms (the details are given by Williams and Davies [2] and by Williams [6]).

So we obtain the set

$$\left(\frac{\partial}{\partial\zeta} - \frac{gh}{RT_0}\right)\left(\frac{\partial^2 v}{\partial\zeta^2} + \frac{gh^4}{aK^2_{mv}}\, u^2 \operatorname{ctg}\theta + 2Pu\cos\theta\right) =$$

$$= -\frac{h}{a}\frac{\partial}{\partial\theta}\left(\frac{\rho_*}{\rho_0}\right) - \left(\frac{aK^2_{mv}}{gh^4}\right) P^2 \sin\theta\cos\theta \frac{\partial}{\partial\zeta}\left(\frac{\rho_*}{\rho_0}\right), \quad (5)$$

where ρ_* is a deviation from a reference value ρ_0 in the uniform rotation, ρ_*/ρ_0 has been neglected by comparison with unity in the inertial terms, and

$$\frac{\partial^2 u}{\partial\zeta^2} = 2\left(\frac{K_{mv}}{K_{zv}}\right) Pv\cos\theta -$$

$$- \frac{k_{zm}(\zeta)\, K^2{}_{mv} P}{ga^2h^2 K_{zv}}\left(\frac{\partial^2 T}{\partial\theta^2} + 2\operatorname{ctg}\theta\frac{\partial T}{\partial\theta}\right). \quad (6)$$

We clearly need an analytically convenient relation between ρ_* and T; using hydrostatic equilibrium and the gas law we obtain (the details being given by Williams and Davies [2]) by integration that

$$\frac{\rho_*}{\rho_0} + 1 = \frac{T_0}{T}\exp\left\{\frac{g}{RT_0}\int_h^\zeta\left(1 - \frac{T_0}{T}\right)d\zeta\right\}. \quad (7)$$

Finally by considering the meridional flux of heat through a thin zonal ring, a thermodynamic equation can be constructed in the form

$$\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\left[\rho vT - \rho K_{mT}\frac{\partial T}{a\partial\theta}\right]\right\} = aQ\,(\theta, \zeta), \quad (8)$$

where Q is a total heating function.

The main calculations were based on the *total* heating function given by Smagorinsky [7] as a function of latitude, together with a linearized variation in the vertical. However, an attempt was also made to consider Q as composed of two factors, the first consisting of total heating (without a contribution due to the flux of latent and sensible heat) and the second consisting of this latent and sensible heat flux. The latter was controlled by the model variable themselves at the surface, the surface flux being given by the form

$$(Q_{s+L})_0 = c\,|\mathbf{V}_s|\,(T_g - T_s),$$

where T_g — mean surface temperature (assumed known empirically), T_s — model surface air temperature, V_s — model surface velocity: $(Q_{s+L})_0$ is then multiplied by an empirical function $J(\zeta)$ to give $Q_{s+L} = (Q_{s+L})_0 J(\zeta)$. Our mathematical method of solution is applicable, but choice of c value is difficult, as it has to be chosen so that the balance condition of the atmosphere $\int Q\sin\theta = 0$ is satisfied. This difficulty has not yet been overcome but an estimate of this feed-back effect was made by considering a total Q with a small feedback c value.

Boundary conditions at $\zeta = 0$ were chosen in the form $(K_{zv})_0\partial u/\partial\zeta = ku$, $(K_{mv})_0\partial v/\partial\zeta = kv$ with k (depending on the viscosity of air) known from climatological values and the surface values of K_0 being taken at 10^3 *cm.* units. Then at the upper surface, $\zeta = 1$, we assumed $\partial u/\partial\zeta = \partial v/\partial\zeta = 0$ in an attempt to allow for the damping effect of the stratosphere. We also need to use the condition of zero net annual mass flux across a latitude circle, i. e. $\int_0^1 \rho v\, d\zeta = 0$. Then at $\zeta = 0$ we use the temperature form $T_g/T_0 = 1 + (\Delta T/T_0)F_g(\theta)$, where ΔT denotes the mean temperature contrast between pole and equator so that if $\Delta T = 0$, $T_g = T_0$; $F_g(\theta)$ is matched to available observations on the sea surface. It was also necessary to prescribe the temperature distribution in the vertical at the equator.

We now obtain a solution of equations (5), (6), (7), (8) as double expansions in powers of the fundamental parameters P and $(\Delta T/T_0)$. We write:

$$T/T_0 = 1 + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(\Delta T/T_0)^i P^j T_{ij}(\theta, \zeta), \quad (9)$$

$$v = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(\Delta T/T_0)^i P^j V_{ij}(\theta, \zeta), \quad (10)$$

and

$$u = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(\Delta T/T_0)^i P^j U_{ij}(\theta, \zeta). \quad (11)$$

Substituting (9) into (7) we obtain

$$\rho_*/\rho_0 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(\Delta T/T_0)^i P^j \times$$

$$\times\left[-T_{ij} + b\int_1^\zeta T_{ij}\, d\zeta + S_{ij}\right], \quad (12)$$

where $b = gh/RT_0$ and S_{ij} are products of velocity and temperature but depend on lower order coefficients of T_{ij}. Substitutions of (9), (10), (11), (12) into (5), (6), (8) and comparison of like powered coefficients leads to a system of linear differential equations (the details are given by Williams and Davies [2]). If we write $d/d\zeta = D$, a characteristic equation of the system is

$$(D-b)\,D^2V_{ij} - c\left[V_{ij} - b\int_1^{\zeta} V_{ij}\,d\zeta\right] = R(\zeta, \theta), \quad (13)$$

where $c = gh^3/K_{mT}K_{mv}$ and $R(\zeta, \theta)$ is a forcing function known numerically at grid-points in (ζ, θ) from the Q function and a solution of the differential equations of order lower than (j, i). Differentiating we obtain

$$(D-b)(D^3-c)V_{ij} = DR \quad (14)$$

and, writing $c = \gamma^3$ the solution is then

$$V_{ij} = A_1e^{b\zeta} + A_2e^{\gamma\zeta} + e^{-\frac{1}{2}\gamma\zeta} \times$$
$$\times \{A_3\cos(\sqrt{3}\zeta\gamma/2) + A_4\sin(\sqrt{3}\zeta\gamma/2)\} +$$
$$+\frac{DR}{(D-b)(D^3-c)}. \quad (15)$$

The four constants are determined from the three boundary conditions and by substituting (15) into (13). We used a 15 × 11 grid in the region $\zeta = 0$ to 1 with the step 0,1 and $\theta = 0$ to 70° with the step 5° and the complementary function in (15) was computed following an inversion of a 4 × 4 matrix for each (c, b, K) data set. At each stage of the computation the R function was represented in ζ, for each θ, by a Fourier series (with 9 terms) and the method of trigonometric nterpolation (see e. g. [8]) used; we write

$$R = d_1 + d_2\zeta + \sum_{s=1}^{s=9} b_s\sin(\pi s\zeta). \quad (16)$$

The particular integral of (14) is then easy to evaluate formally and (16) is convenient to handle numerically. The U_{ij} and T_{ij} coefficients in (9) and (11) are easy to compute once V_{ij} are known; the details are given in [6]. We have $D^2U_{ij+1} = S(\zeta, \theta)$, $\frac{\partial}{\partial\theta}T_{ij} = T(\zeta, \theta)$, S and T being known at each stage. Five terms were needed in the P expansions and three in the $(\Delta T/T_0)$ expansions to obtain convergence, but the successive terms become extremely complicated and quite a large storage computer (an I. B. M. 7090) was required for the calculation.

A number of computations were carried out over the range from 20° to 90° in θ using several distributions of turbulence parameters. Using a total heating function and e. g. the set $K_{mv} = 3{,}4\cdot10^8$, $K_{zv} = 7{,}5\cdot10^5$, $K_{mT} = 2{,}7\cdot10^{10}$, $k_{zm}\Omega = 1{,}2\cdot10^{14}$, we obtain the calculated u, v, T distributions shown in Figs 1, 2, 3; these are quite realistic. We also attempted a feed-back calculation using a total Q function plus a $(Q_{s+L})_0T(\zeta)$ term with a small c value. Its effect was to produce higher temperature in midtroposphere, as expected. Hence it has been shown that it is possible to reproduce realistic (long time average) mean

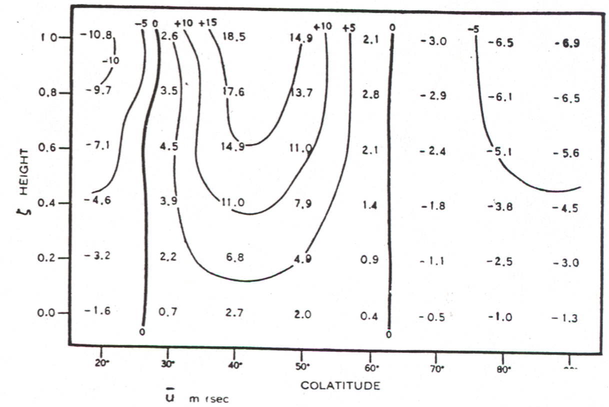

Fig. 1. Calculated values of $\bar{u}$; + denote westerlies

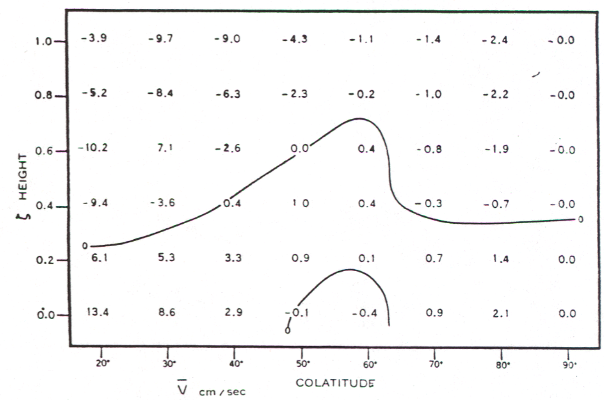

Fig. 2. Calculated values of $\bar{v}$; + denote northerlies

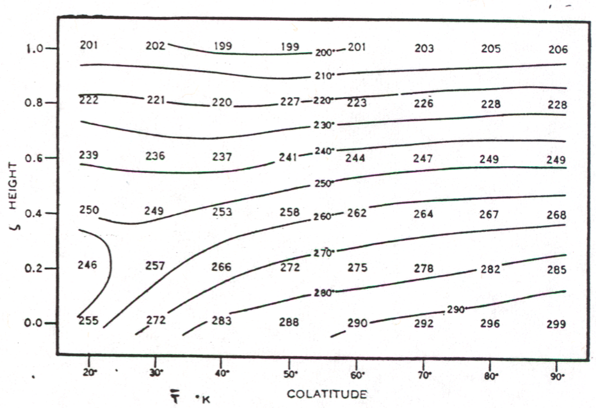

Fig. 3. Calculated values of $\bar{T}$.

flows on the basis of quite simple physical assumptions, using numerical values of turbulence parameters, suggested by various analysis of observations. Finally, it is clear that there are many ways in which the model can be improved; the work is only a stepping stone (a small one) to a more complete statistical theory.

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Discussion

J. Van Isacker took part in the discussion.

ОСРЕДНЕННАЯ ПО ВРЕМЕНИ МОДЕЛЬ ОБЩЕЙ ЦИРКУЛЯЦИИ

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Для того чтобы представить квазистационарный осредненный по времени поток момента и тепла в сферической системе координат, сконструированы уравнения, осредненные по кругу широты и по такому большому промежутку времени, как год. Круговые рейнольдовские напряжения сдвига, связанные с меридиональным переносом зональной скорости, предполагаются линейно зависящими от произведения угловой скорости вращения Земли Ω и меридионального градиента средней температуры. Напряжения, связанные с вертикальным переносом зональной и меридиональной скорости, полагаются линейно зависящими от вертикальных градиентов зональной и меридиональной скоростей соответственно, а средний турбулентный перенос тепла вдоль меридиана — от среднего меридионального градиента температуры. Все коэффициенты пропорциональности, определяющие крупномасштабную турбулентность, выбраны независящими от широты. Изучены две различные формы функции неадиабатических источников тепла в атмосфере Q, используемой в термодинамическом уравнении: во-первых, Q принималось заданной функцией высоты и широты и, во-вторых, содержало термический член, зависящий от принимаемой модели и представляющий некоторую характеристику обмена явной и скрытой теплотой с поверхностью.

В обоих случаях получено решение, дающее распределение температуры и скорости. Решение искалось в форме двойного разложения в ряд по степеням двух основных параметров (один зависел от Ω, а другой от ΔT, средней годовой разности температур между экватором и полюсом); вывод уравнений для численного решения проводился по методу Фурье.

Найдено, что эти разложения достаточно хорошо сходятся для отвечающих действительности величин различных исследуемых параметров. (При этом оказывается достаточным в разложении по ΔT оставлять лишь три члена, а в разложении по Ω — пять членов.) Тем не менее потребовались довольно большие численные вычисления на ЭВМ. Рассматриваемая область ограничивалась тропопаузой и лежала между экватором и 70° с. ш. Подсчитанная зональная скорость имеет реальную величину с характерными изменениями направления: восток — запад — восток, меридиональная скорость имеет характерную трехъячейковую структуру, и распределение температуры вполне реально.