NOTES AND CORRESPONDENCE

Another Look at the Scale Analysis for Deep Moist Convection

FRANK B. LIPPS AND RICHARD S. HEMLER

Geophysical Fluid Dynamics Laboratory/NOAA, Princeton University, Princeton, NJ 08542

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ABSTRACT

In this note, a more rational approach is given to specify the parameters G and B in the scale analysis of Lipps and Hemler. The thermodynamic equation is written in a different form so that a closed expression for B can be derived. The present values of G and B are very similar to those in the previous scale analysis. A new result is that the time scale τ is expressed in terms of the moist convective instability rather than the inverse of the Brunt–Väisälä frequency.

The ratio of volume integrated kinetic energy to volume integrated first-order sensible heat is also discussed in more detail. It is found that for an accurate estimate of sensible heat the region of compensating downward motion between the active clouds must be taken into account. As indicated by earlier authors, the amount of sensible heat produced inside the clouds is relatively small.

1. Introduction

In the scale analysis for deep moist convection (Lipps and Hemler, 1982) the nondimensional parameters G and B had to be specified in order to determine the characteristic length scale l and velocity scale W. The values $G = 1$, $B = 4$ were chosen in that study (hereafter referred to as LH82) in order to give results which were compatible with the associated numerical calculations. Then and now the authors consider this method of obtaining G and B as arbitrary. A major purpose of the present note is to give a more rational approach to specify these parameters. It is found that the new values of G and B are close to the previous values so that the primary conclusions of the scale analysis are unchanged.

The second purpose of this note is to discuss in greater detail the ratio of the volume-integrated kinetic energy PK to the volume-integrated first-order sensible heat SH. A more careful look at the numerical data in LH82 indicates that the magnitude of SH was underestimated in the scale analysis. For an accurate estimate of SH the region of compensating downward motion between the active clouds and the associated adiabatic warming must be taken into account. As indicated by earlier authors (Gray, 1973; Yanai *et al.*, 1973; and others) the amount of sensible heat produced inside the clouds is relatively small.

2. Determination of the parameters G and B

a. Review of scale analyses

For clarity of this presentation we give a brief review of the scale analysis in LH82. The set of basic equations were first nondimensionalized using the length scale l and vertical velocity W. Values for these characteristic scales of the convection were not specified until the assumptions of the scale analysis were discussed.

The first assumption is the existence of two small parameters of which the first is the most important and is defined by

$$\epsilon = \Delta\theta_c/\theta_{00} \tag{1}$$

where $\Delta\theta_c$ is a characteristic value of the potential temperature excess in the clouds and θ_{00} is the base state potential temperature at the ground. The second parameter δ gives a measure of the water vapor present in the cloud environment. A final result of the scale analysis is the set of basic equations to leading order in ϵ and δ.

For the determination of l and W, two other primary assumptions of LH82 need to be discussed. It is these assumptions that introduce the parameters G and B.

1) The first-order buoyancy and vertical acceleration are required to be the same order of magnitude in the vertical momentum equation. Since the first-order buoyancy $\sim g\epsilon$ and the vertical acceleration $\sim W^2/l$, this balance is expressed by

$$G = g\epsilon l/W^2 \sim 1. \tag{2}$$

2) The base state potential temperature $\theta_0(z)$ is a slowly varying function of z. Specifically, the vertical advection of base state potential temperature θ_0 is assumed to be the same order of magnitude as the total time derivative of first-order potential temperature θ_1. As indicated by LH82, this assumption leads to

$$\frac{\Delta\theta_T}{\theta_{00}}\frac{l}{d} = B\epsilon, \quad B \sim 1 \tag{3}$$

where $\Delta\theta_T$ is the total change in base state potential temperature θ_0 through the depth of the troposphere and d is the depth of the troposphere. Equations (2) and (3) correspond to Eqs. (16) and (17) in LH82.

The scale analysis cannot give explicit values to G and B. Equations (2) and (3) indicate only that these parameters are the order of unity. As discussed before, however, the values of l and W have not yet been specified. When values for l and W are chosen and these values are substituted into (2) and (3), explicit values will be obtained for G and B. For this reason (2) and (3) can be looked upon as defining equations for l and W when G and B are given explicit values. This is the approach taken in LH82.

In that study the values $G = 1$, $B = 4$ were specified so that the resulting values of l and W would be compatible with the associated numerical calculations. Here we use dynamical considerations for the selection of G and B.

b. Discussion of the thermodynamic equation

In order to specify the parameter B, it is necessary to write the thermodynamic equation in a different form than given previously. All equations in the present study are dimensional, following the notation in Section 7 of LH82. Also, as in the earlier study, viscous and diffusive effects are not included. Thus, the thermodynamic equation from the scale analysis can be written as

$$c_p\pi_0\left(\frac{d\theta_1}{dt} + w\frac{d\theta_0}{dz}\right) = L(C_d - E) \tag{4}$$

where π_0 is the base state Exner pressure function, θ_0 the base state potential temperature, θ_1 the first-order potential temperature, C_d the condensation/evaporation of cloud water, E the evaporation of rain water, L the (constant) latent heat of vaporization, and c_p the specific heat of dry air at constant pressure. All dimensional constants have the values given in Appendix B of LH82.

For the present discussion we consider a saturated in-cloud parcel so that the evaporation of rainwater E vanishes in Eq. (4). Following Mason (1971) it can be shown that to a high degree of accuracy the condensation C_d is proportional to the vertical velocity w. Setting the supersaturation σ to zero in Mason's Eq. (3.24), it follows that:

$$C_d = \frac{A_1}{A_2}w \tag{5}$$

where A_1 and A_2 are given by

$$A_1 = \frac{g}{c_p}\frac{1}{T}\left(\frac{L}{R_vT} - \frac{c_p}{R_d}\right), \quad A_2 = \frac{L^2}{c_pR_vT}2 - \frac{1}{q_{vs}} \tag{6}$$

with q_{vs} being the saturated water vapor mixing ratio. Combining (5) and (6):

$$C_d = \frac{g}{c_p}\frac{q_{vs}}{T}\left(\frac{L}{R_vT} - \frac{c_p}{R_d}\right)\left(1 + \frac{L^2q_{vs}}{c_pR_vT^2}\right)^{-1} w. \tag{7}$$

To obtain the alternate form of the thermodynamic equation, (7) is used to define C_d and (4) is divided by $c_p\pi_0$. Thus we find

$$\frac{d\theta_1}{dt} + \left[\frac{d\theta_0}{dz} - \left.\frac{\partial\theta}{\partial z}\right|_m\right]w = 0 \tag{8}$$

where $\partial\theta/\partial z|_m$ is defined by

$$\left.\frac{\partial\theta}{\partial z}\right|_m = \frac{g}{c_p}\frac{Lq_{vs}}{c_pT}\frac{1}{\pi_0}\left(\frac{L}{R_vT} - \frac{c_p}{R_d}\right)\left(1 + \frac{L^2q_{vs}}{c_pR_vT^2}\right)^{-1}. \tag{9}$$

From (8) it is seen that $\partial\theta/\partial z|_m$ is the moist adiabatic lapse rate of potential temperature. Equation (9) is compatible with the moist adiabatic lapse rate of temperature as given by Hess [1959, in his Eq. (7.3)] and by others.

For the discussion below it is convenient to define the parameter $g\Gamma_m$ as a measure of moist convective instability:

$$g\Gamma_m = -\frac{g}{\theta_0}\left[\frac{d\theta_0}{dz} - \left.\frac{\partial\theta}{\partial z}\right|_m\right]. \tag{10}$$

This parameter will be compared with N^2 which is a measure of dry static stability

$$N^2 = \frac{g}{\theta_0}\frac{d\theta_0}{dz} \tag{11}$$

where N is the Brunt–Väisälä frequency. A simple comparison can be made between N and $(g\Gamma_m)^{1/2}$ when moisture and water loading terms are neglected in the buoyancy term in the vertical momentum equation. In that case N represents the frequency of very short gravity waves whereas $(g\Gamma_m)^{1/2}$ represents the growth rate of very short waves in a moist convectively unstable atmosphere.

c. Specification of the parameters G and B

Dynamical considerations are now used to suggest an appropriate specification for the parameters G and B in LH82. From Eq. (2) it is evident that G is defined as the order of magnitude ratio of the first-order buoyancy to the vertical acceleration. As in LH82, the choice of $G = 1$ appears reasonable for moist convection on the scales of meso-γ or smaller. For these scales the buoyancy term is expected to be the same general magnitude as the vertical acceleration. There is also another consideration which favors the choice of $G = 1$. When the vorticity equation is considered, it is seen that the solenoidal term is the primary source term for horizontal vorticity. Setting the time derivative of horizontal vorticity and the solenoidal term to the same order of magnitude is compatible with setting $G = 1$ in Eq. (2). This follows[1] since the time derivative of horizontal vorticity $\sim 1/\tau^2$, the solenoidal term $\sim g\epsilon/l$ and $\tau = lW^{-1}$.

The parameter B is defined by Eq. (3). It can be shown that B is the order of magnitude ratio of the vertical advection of base state potential temperature $(wd\theta_0/dz)$ to the total time derivative of first-order potential temperature $(d\theta_1/dt)$. This result can be derived using Eqs. (1) and (3) and the order of magnitude relations $wd\theta_0/dz \sim W\Delta\theta_T/d$ and $d\theta_1/dt \sim W\Delta\theta_c/l$. Thus we obtain for B

$$B \sim \left|\frac{wd\theta_0/dz}{d\theta_1/dt}\right| \sim \frac{N^2}{g\Gamma_m} \tag{12}$$

where the second part of this expression is obtained using Eqs. (8), (10) and (11). In the above relation N^2 and $g\Gamma_m$ are functions of z. Thus to specify B, appropriate mean values must be chosen for N^2 and $g\Gamma_m$. As in LH82, the vertical average

$$\bar{N}^2 = \frac{g}{\theta_{00}}\frac{\Delta\theta_T}{d} \tag{13}$$

is assigned to N^2. The value $g\bar{\Gamma}_m$ is assigned to $g\Gamma_m$ where $\bar{\Gamma}_m$ is defined as

$$\bar{\Gamma}_m = -\frac{1}{\theta_{00}}\overline{\left[\frac{d\theta_0}{dz} - \frac{\partial\theta}{\partial z}\bigg|_m\right]}. \tag{14}$$

The bar on the right represents a vertical average over the levels of strongest moist instability. Typically these levels are in the lower part of the deep clouds. Thus the parameters G and B are specified by

$$G = 1, \quad B = \bar{N}^2/g\bar{\Gamma}_m. \tag{15}$$

A numerical value for B must be obtained by using (13)–(15) above. For this purpose we consider the base state potential temperature $\theta_0(z)$ and the associated dashed curve for $\theta_m(z)$ shown in Fig. 1 of LH82. The potential temperature gradient difference on the right in (14) can be estimated from the difference of the slopes of $\theta_0(z)$ and $\theta_m(z)$ shown in LH82. From the level of cloud base to the level of the maximum difference between these curves we find the mean value of 1.125 K km^{-1}. From LH82 the gradient $\Delta\theta_T/d \approx 5$ K km^{-1} is also given. Comparing (13)–(15) it is seen that B is equal to the ratio of these two potential temperature gradients. Thus G and B have the values

$$G = 1, \quad B = 4.44. \tag{16}$$

Using (2), (3) and (16) we obtain $l \approx 2.7$ km and $W \approx 16$ m s^{-1}. Thus the parameters G and B and the characteristic length and velocity scales l and W have values very similar to those in LH82. For this reason the primary conclusions of the scale analysis remain unchanged. It is considered, however, that the present method of obtaining B is much less arbitrary than that given previously.

The present analysis with B defined as in (15) gives closed expressions for the characteristic scales l, W and τ. Combining (2), (3), (13) and $\tau = lW^{-1}$ gives[2] Eq. (18) in LH82:

$$\tau = lW^{-1} = (GB)^{1/2}(\bar{N})^{-1}. \tag{17}$$

Now using the definition of B from (15) gives

$$\tau = (g\bar{\Gamma}_m)^{-1/2}. \tag{18}$$

Thus the time scale τ is given in terms of the moist convective instability. Using the potential temperature gradient difference of 1.125 K km^{-1} discussed above and $\theta_{00} \approx 300$ K we find $\bar{\Gamma}_m = 0.375 \times 10^{-5}$ m^{-1} from (14). Thus $\tau = 165$ s from (18). Although this value may seem rather small, it is compatible with previously given values of l and W so that $\tau = lW^{-1}$.

Expressions for l and W can be obtained from (2), (15), (18) and $\tau = lW^{-1}$.

$$l = (\bar{\Gamma}_m)^{-1}\epsilon \tag{19a}$$

$$W = (g/\bar{\Gamma}_m)^{1/2}\epsilon. \tag{19b}$$

Finally it follows that

$$\frac{l}{H} = (H\bar{\Gamma}_m)^{-1}\epsilon \tag{20}$$

where $H = c_p\theta_{00}/g$ and is thus the height of an isentropic atmosphere. From the definition of B in (15) it can be shown that (20) is compatible with the expression (21) for l/H given in LH82. Thus in the present analysis the parameter $\bar{\Gamma}_m$ is seen to play a dominant role in the expressions for τ, l, W and l/H.

3. Production of sensible heat by moist convection

The kinetic energy PK and the first order sensible heat SH are defined in LH82 by the vertical integrals

$$\left.\begin{aligned} PK &= \frac{1}{2}\int_0^d \rho_0\langle\mathbf{V}^2\rangle dz \\ SH &= c_p\int_0^d \rho_0\pi_0\langle\theta_1\rangle dz \end{aligned}\right\}. \tag{21}$$

where angle bracket denotes a horizontal average and $\mathbf{V}$ is the vector velocity. Since $\langle\theta_1\rangle = 0$ at $t = 0$, both PK and SH are zero at the beginning of each calculation. In the previous scale analysis the ratio of PK to SH was estimated by taking the ratio of the maximum kinetic energy to the first-order sensible heat deter-

[1] The authors obtained this argument from discussions with I. Orlanski.

[2] Equation (18) in LH82 has a typographical error.

mined by the maximum temperature excess in the active clouds. Since $\pi_0 \sim 1$, this ratio can be obtained from Eq. (47) in LH82.

$$\frac{1}{2}\frac{W^2}{c_p\Delta\theta_c}=\frac{1}{2}\frac{1}{G}\frac{l}{H}. \tag{22}$$

With $G = 1$ and $l/H \sim 8\epsilon$ it is seen that this ratio $\sim 4\epsilon$. Looking at Table 1 in LH82, however, indicates that numerically calculated values of $PK/SH \sim 0.25\epsilon$ with $\epsilon \approx 10^{-2}$. This represents a factor of 16 difference between the scale analysis estimate and the numerical results in LH82.

The authors now believe that this factor is primarily due to an underestimate of SH in the scale analysis. While $c_p\Delta\theta_c$ may be a valid estimate of the first order sensible heat in the clouds, it is not a valid estimate for determining the first order sensible heat in the total volume. Neglecting a term the order of l/H, the production of SH can be obtained from Eq. (45) in LH82.

$$\frac{\partial}{\partial t}SH = L\int_0^d \rho_0\langle(C_d - E)\rangle dz. \tag{23}$$

Thus to this accuracy, the production of SH is equal to the net latent heat release in the total volume.

Inside the clouds the latent heat release is nearly cancelled by the adiabatic cooling associated with upward motion. That this is indeed the case can be seen from an approximate parcel calculation using (4). Setting $E = 0$, $C_d = -dq_{vs}/dt$ and $\pi_0 \sim 1$ we find for the potential temperature increase for a parcel moving from cloud base to cloud top

$$(\Delta\theta)_{\text{parcel}} \approx \frac{L}{c_p}(q_{vs})_{\text{cloud base}} \sim 40°\text{K} \tag{24}$$

using the values of physical quantities in LH82. The actual observed increase of potential temperature $\Delta\theta_c$ over that of the environment is only 3-4 K. Thus, as the parcel rises, its increase in potential temperature with height is nearly balanced by the increase in potential temperature with height in the environment. Stated in another way, the increase of disturbance potential temperature in the rising parcel due to latent heat release is nearly compensated for by adiabatic cooling.

The above discussion suggests that $c_p(\Delta\theta)_{\text{parcel}}$ would represent a better value of the first-order sensible heat to be substituted into Eq. (22) than the scale analysis estimate $c_p\Delta\theta_c$. Making this change in (22) gives

$$\frac{1}{2}\frac{W^2}{c_p(\Delta\theta)_{\text{parcel}}}=\frac{1}{2}\frac{1}{G}\frac{l}{H}\frac{\Delta\theta_c}{(\Delta\theta)_{\text{parcel}}}. \tag{25}$$

Using $\Delta\theta_c \sim 3$ K and $(\Delta\theta)_{\text{parcel}} \sim 40$ K gives the ratio $\Delta\theta_c/(\Delta\theta)_{\text{parcel}} \sim 3/40$. Since the ratio of PK to SH indicated by (22) was $\sim 4\epsilon$, it is seen that the additional factor in (25) gives a ratio of $\sim 0.3\epsilon$. This new value is in satisfactory agreement with the numerically calculated values of $PK/SH \sim 0.25\epsilon$.

Hence the production of sensible heat through latent heat release is accomplished primarily by an indirect mechanism: The compensating downward subsidence in the cloud free air results in heating the cloud environment due to adiabatic warming. This argument has been given by previous authors (Gray, 1973; Yanai *et al.*, 1973; Fritsch, 1975). Not taking account of this indirect mechanism in the scale analysis was the primary reason the predicted ratio of PK/SH was much larger than seen in the numerical calculations.

4. Summary

In Section 2 a less arbitrary approach is given to specify the parameters G and B in the scale analysis of LH82. As in the earlier study we set $G = 1$. This choice is made on the basis that the buoyancy term and the vertical acceleration term in the w-momentum equation are the same order of magnitude. An equally valid alternative argument for this choice is that the total time derivative of horizontal vorticity and the solenoidal term in the vorticity equation are also the same order of magnitude.

In LH82 the parameter B is defined as the order of magnitude ratio of the vertical advection of base state potential temperature ($wd\theta_0/dz$) to the total time derivative of disturbance potential temperature ($d\theta_1/dt$). To obtain a value for B, an alternative form of the thermodynamic equation is derived as given by (8). Using this equation and the definition of B, a closed expression (15) is found for B. This expression gives B as the ratio of the square of the Brunt–Väisälä frequency $\bar{N}^2$ to the parameter $g\bar{\Gamma}_m$ representing the moist convective instability. Using data from the base state in LH82, the value $B = 4.44$ is found. In the previous scale analysis the value $B = 4$ was chosen to give values of l and W compatible with the numerical calculations.

Since the present values of G and B are very similar to those in LH82, the primary conclusions of the scale analysis remain unchanged. A new perspective, however, is provided by this study through the expressions for τ, l, W and l/H given in (18)–(20). In particular, $\tau = (g\bar{\Gamma}_m)^{-1/2}$ so that the time scale is given in terms of the moist convective instability rather than the inverse Brunt–Väisälä frequency $(\bar{N})^{-1}$ as previously.

In Section 3 the ratio of volume-integrated kinetic energy PK to volume-integrated first-order sensible heat SH is examined in greater detail. This ratio was estimated in the scale analysis of LH82 by taking the ratio of the maximum kinetic energy to the first-order sensible heat determined by the maximum temperature excess in the clouds. It is seen that this estimate is a factor of 16 larger than the ratio of PK to SH found in the numerical calculations. The reason for this discrepancy is that SH was underestimated in the scale analysis. For an accurate estimate of *SH* the compensating downdrafts outside the clouds and the resultant adiabatic warming must be taken into account. As revealed by the simple considerations given in Section 3, the relative production of sensible heat inside the clouds is small. A new estimate of the ratio of *PK* to *SH* is given by (25). This estimate is in satisfactory agreement with the numerical calculations.

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