Pseudomomentum and the Orthogonality of Modes in Shear Flows

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ABSTRACT

Linear modes on shear flows are not orthogonal in the sense of energy; if two modes are present, the eddy energy is not equal to the sum of the eddy energy in the separate modes. However, linear modes *are* orthogonal in the sense of pseudomomentum (or pseudoenergy). Two applications of this result to planetary waves in horizontal and vertical shear are discussed. 1) The qualitative character of the evolution of a disturbance to a stable meridional shear flow, as described by the barotropic vorticity equation, depends critically on whether the disturbance projects primarily onto discrete modes or onto continuum modes that cascade enstrophy to small meridional scales. It is demonstrated that the pseudomomentum and pseudoenergy orthogonality relations provide a natural framework for examining the relative excitation of discrete and continuum modes. 2) Using the quasi-geostrophic potential vorticity equation, it is shown that pseudomomentum orthogonality provides a simple explanation for how quasi-stationary neutral external modes of large amplitude can be excited by a small initial disturbance.

1. Introduction

The propagation of waves on shear flows and the resulting wave–mean flow interaction are conveniently analyzed in terms of wave pseudomomentum (or "wave activity") and the flux of pseudomomentum (or "Eliassen–Palm flux"), as elucidated by the work of Andrews and McIntyre (1976, 1978) and in numerous more recent papers. Pseudomomentum and the analogous concept of pseudoenergy are also very useful concepts when working with a modal decomposition. Whereas wave energy as usually defined cannot be decomposed into contributions from individual modes because these modes are not orthogonal in the appropriate sense, pseudomomentum and pseudoenergy can be so decomposed. Ripa (1981) utilizes pseudomomentum orthogonality for waves on a basic state at rest in a study of wave–wave interactions. Mohring (1980) discusses related results for acoustic modes in waveguides. We believe these orthogonality relations can play a useful role in a number of meteorological contexts. Following a discussion of the orthogonality of modes in shear in Section 2, two applications to models of large-scale waves in the atmosphere are described: the meridional dispersion of nondivergent Rossby waves on the sphere in the presence of a latitude-dependent zonal flow (Section 3); and the excitation of quasi-geostrophic external Rossby waves in vertical shear (Section 4).

2. Orthogonality of modes on a shear flow

Consider the nondivergent barotropic vorticity equation on a midlatitude beta-plane, linearized about the zonal flow $\bar{u}(y)$:

$$\zeta'_t + \bar{u}\zeta'_x + \gamma v' = 0,$$

$$\zeta' \equiv \psi'_{xx} + \psi'_{yy},$$

$$(u', v') \equiv (-\psi'_y, \psi'_x),$$

$$\gamma \equiv \beta - \bar{u}_{yy}. \tag{2.1}$$

For a particular zonal wavenumber, $\psi' = \psi e^{ikx} + \text{(c.c.)}$, we assume that the initial value problem can be solved by decomposing into modes:

$$\psi = \sum_n A_n \psi_n(y) e^{-ikc_n t} \tag{2.2}$$

where ψ_n and c_n satisfy the eigenvalue equation

$$(\bar{u}\nabla_k^2 + \gamma)\psi_n = c_n\nabla_k^2\psi_n,$$

$$\nabla_k^2 \equiv \partial_{yy} - k^2; \tag{2.3}$$

and where $\psi_n = 0$ at the channel walls, $y = 0, L$. If $\bar{u} = 0$, the modes are simple Rossby waves, $\psi_n = \sin(l_n y)$ with $l_n = n\pi/L$, and all phase speeds are negative, $c_n = -\beta/(k^2 + l_n^2)$. If $\bar{u}$ has meridional shear, the spectrum consists of a set of discrete neutral modes, possibly infinite in number, propagating westward with respect to the flow at all latitudes, a continuum of singular neutral modes with $\min[\bar{u}(y)] < c < \max[\bar{u}(y)]$ [for which the sum over n in (2.2) should be replaced by an integral], and, possibly, conjugate pairs of amplifying and decaying discrete modes with $\text{Im}(c) \neq 0$. [We assume that k is not located precisely at a stability boundary where discrete modes coalesce; at such a point, the solution to the initial value problem cannot in general be written as a sum over separable modes of the form (2.2).]

One of the difficulties that arises when thinking in terms of this modal decomposition is that the different modes are not orthogonal in the sense of energy or enstrophy. Using the notation

$$\{X, Y\} \equiv L^{-1} \int_0^L X^*Y dy \qquad (2.4)$$

the eddy enstrophy in wavenumber k averaged over the domain is proportional to

$$\{\zeta, \zeta\} = \sum_{m,n} A_n^* A_m \{\zeta_n, \zeta_m\} \exp[ik(c_n^* - c_m)t]. \qquad (2.5)$$

In general, it is not true that $\{\zeta_n, \zeta_m\} = 0$ if $n \neq m$, and one cannot decompose the eddy enstrophy into distinct parts contributed by individual modes. The same result holds for the eddy energy, $\{\psi, \zeta\}$. It makes no sense to ask how much energy resides in a given mode at a particular time, nor does it make sense to ask how the energy or enstrophy is partitioned between the discrete modes and the continuum. Interference effects of this sort are discussed by Lindzen, et al. (1982) and Farrel (1984) in the analogous problem of quasigeostrophic modes in vertical shear. If only two discrete neutral modes are excited (with frequencies ν_1 and ν_2), the wave energy and enstrophy will oscillate in time with frequency $\nu_1 - \nu_2$.

The nonorthogonality of modes in the sense of energy or enstrophy is a direct consequence of the fact that eddy energy and enstrophy (as opposed to total energy and enstrophy) are not conserved. A conservation law for a quantity that is quadratic in wave amplitude leads immediately to an inner product under which neutral modes are orthogonal. As an example, we have the familiar conservation law, easily obtained from (2.1),

$$(\gamma \overline{\eta'^2}/2)_t = (\overline{u'v'})_y \qquad (2.6)$$

or, for a particular zonal wavenumber,

$$(\gamma |\eta|^2)_t = (u^*v + uv^*)_y$$

where η' is the meridional particle displacement, $\zeta' = -\gamma \eta'$, and an overbar denotes the zonal mean. Integrating over the channel we have $dP/dt = 0$, where $P \equiv \{\eta, \gamma\eta\}$ is proportional to the pseudomomentum. (Following the terminology of Andrew and McIntyre, in fact P is the negative of the pseudomomentum.) If a mode η_1 exists with $\text{Im}(c_1) \neq 0$, it follows that $\{\eta_1, \gamma\eta_1\} = 0$, since

$$P \propto \{\eta_1, \gamma\eta_1\}e^{2k\text{Im}(c_1)t} \qquad (2.7)$$

for an initial condition consisting of this mode only, and dP/dt must vanish. The Rayleigh-Kuo necessary condition for instability follows in the usual way. Suppose now that the wave field consists of only two modes $\eta = b_1\eta_1 + b_2\eta_2$ at $t = 0$. Then

$$P = |b_1|^2\{\eta_1, \gamma\eta_1\} + |b_2|^2\{\eta_2, \gamma\eta_2\}$$

$$+ \{b_1^*b_2\{\eta_1, \gamma\eta_2\} \exp[ik(c_1^* - c_2)t] + (\text{c.c.})\}. \qquad (2.8)$$

[If $\text{Im}(c) \neq 0$ for one (or both) of the modes, then one (or both) of the first two terms on the rhs will not appear.] Assuming that $c_1 \neq c_2^*$, P can be conserved in time only if the modes are orthogonal in the sense $\{\eta_1, \gamma\eta_2\} = 0$, that is

$$0 = \int_0^L \gamma\eta_1^*\eta_2 dy = \int_0^L \gamma^{-1}\zeta_1^*\zeta_2 dy. \qquad (2.9)$$

If two or more neutral modes happen to have the same eigenvalue, one can always orthogonalize in the subspace spanned by the degenerate eigenfunctions. Therefore, the discrete neutral modes can be chosen to be orthogonal to each other, as well as to the unstable modes and the neutral continuum in this same sense. However, a growing mode is not, in general, orthogonal to its conjugate decaying mode: $\{\eta_1^*, \gamma\eta_1\} \neq 0$. Then P can be decomposed into separate contributions from individual neutral modes, plus one contribution from each amplifying-decaying pair. If

$$\eta = \sum A_n\eta_n + \sum (B_m\eta_m + C_m\eta_m^*). \qquad (2.10)$$

where the first sum is over the neutral modes and the second over the amplifying-decaying pairs, then

$$P = \sum |A_n|^2\{\eta_n, \gamma\eta_n\}$$

$$+ \sum [B_m^*C_m\{\eta_m, \gamma\eta_m^*\} + (\text{c.c.})]. \qquad (2.11)$$

If $\gamma > 0$, there are no modal instabilities and P is a sum over positive definite contributions from each neutral mode.

While one is tempted to refer to $-\gamma\overline{\eta'^2}/2$ as the pseudomomentum density, it differs from the pseudomomentum density as defined by Andrews and McIntyre (1978),

$$-\overline{(u' - (f - \bar{u}_y)\eta')\xi_x'} - \overline{v'\eta_x'} \qquad (2.12)$$

where ξ' is the zonal particle displacement satisfying

$$\xi_t' + \bar{u}\xi_x' = u' + \bar{u}_y\eta'. \qquad (2.13)$$

However, the integrals of the two quantities over the channel are identical, since one can show from (2.1) and the definitions of the particle displacements that

$$\overline{(u' - (f - \bar{u}_y)\eta')\xi_x'} + \overline{v'\eta_x'} - \gamma\overline{\eta'^2}/2$$

$$= -(\overline{u'\eta'} - (f - \bar{u}_y)\overline{\eta'^2}/2)_y. \qquad (2.14)$$

The rhs of (2.14) is proportional to $M_E - M_L$, where M_E is the absolute zonal momentum of the fluid $(\bar{u} - fy)$ averaged over the infinitesimal Eulerian volume between the latitude circles y and $y + \epsilon$, and M_L is the absolute zonal momentum averaged over the Lagrangian volume between the two material lines whose zonally averaged latitudes are y and $y + \epsilon$. One can prove that an alternative way of writing the pseudomomentum orthogonality relation is

$$\{u_1 - (f - \bar{u}_y)\eta_1, \xi_1^*\} + \{u_2 - (f - \bar{u}_y)\eta_2, \xi_1^*\}$$
$$+ \{v_1, \eta_2^*\} + \{v_2, \eta_1^*\} = 0 \quad (2.15)$$

if $c_1 \neq c_2^*$.

A more direct proof of orthogonality is also available. The eigenvalue equation (2.3) can be rewritten in the form $N\eta = c\gamma\eta$, where $N \equiv \gamma\bar{u} + \gamma\nabla_k^{-2}\gamma$ and where ∇_k^{-2} is the inverse of ∇_k^2. N is self-adjoint for any two eigenfunctions η_1 and η_2, with eigenvalues c_1 and c_2, $\{N\eta_1, \eta_2\} = \{\eta_1, N\eta_2\} \Rightarrow \{c_1^* - c_2\}\{\eta_1, \gamma\eta_2\} = 0$. One again has the orthogonality condition $(\eta_1, \gamma\eta_2) = 0$ when $c_1 \neq c_2^*$.

Pseudoenergy is another conserved quantity that is quadratic in wave amplitude. For the linear barotropic model the pseudoenergy E is

$$E = (2L)^{-1} \int_0^L (\overline{u'^2} + \overline{v'^2} - \bar{u}_y\overline{\gamma\eta'^2})dy \quad (2.16)$$

or, for solutions of the form $\eta' = \eta e^{ikx} + $ (c.c.), $E = -\{\eta, N\eta\}$ where N is defined as in the previous paragraph. The related orthogonality relation is $\{\eta_1, N\eta_2\} = 0$ when $c_1 \neq c_2^*$. The same result can be obtained by rewriting the eigenvalue equation (2.3) in the form $Q\eta = cN\eta$, with

$$Q = \gamma\bar{u}^2 + \bar{u}\gamma\nabla_k^{-2}\gamma + \gamma\nabla_k^{-2}\bar{u}\gamma + \gamma\nabla_k^{-2}\gamma\nabla_k^{-2}\gamma \quad (2.17)$$

and with N as above; Q is also self-adjoint.

In this barotropic model, it is easy to show for a neutral mode with phase speed c that $E = -cP$. It follows that the pseudoenergy in a neutral mode is identically zero in a frame of reference moving with the phase speed of the mode, a result that holds more generally. For a disturbance consisting of a superposition of many neutral modes, one has

$$E = -\sum_n c_n P_n. \quad (2.18)$$

Andrews and McIntyre have shown that such pseudomomentum and pseudoenergy conservation laws hold for a wide class of waves on shear flows, as they are a direct result of the translational and time invariance of the basic state. It follow that analogous orthogonality relations hold in these other cases. An analogous pseudoenergy conservation law also holds for linear waves on a time-independent basic state that varies with longitude as well as latitude, as long as this basic state satisfies the unforced nonlinear equations of motion (Andrews, 1983a). Therefore, the corresponding orthogonality relation for normal modes carries over to this more general case.

As one further illustration, consider the shallow water equations on an equatorial beta-plane, linearized about a basic state with zonal flow $\bar{u}(y)$ and height $\bar{h}(y)$:

$$Du' = (f + \bar{\zeta})v' - gh'_x,$$
$$Dv' = -fu' - gh'_y,$$
$$Dh' = -v'\bar{h}_y - \bar{h}(u'_x + v'_y), \quad (2.19)$$

where $D \equiv \partial_t + \bar{u}\partial_x$. Manipulation of these equations, included for completeness in Appendix A, results in the following pseudomomentum conservation law (cf. Andrews, 1983b; and Ripa, 1981):

$$[\bar{h}^2\overline{Q_y\eta'^2}/2 - \overline{u'h'}]_t = (\overline{hu'v'})_y \quad (2.20)$$

where $Q \equiv (f + \bar{\zeta})/h$ is the potential vorticity and $Q' \equiv -\bar{Q}_y\eta'$. This conservation law immediately yields the orthogonality relation for two modes satisfying $c_1 \neq c_2^*$:

$$\{\bar{h}\eta_1, \bar{Q}_y\bar{h}\eta_2\} - \{u_1, h_2\} - \{u_2, h_1\} = 0 \quad (2.21)$$

where braces again denote integration over the domain. As in the nondivergent case, a direct proof of orthogonality can be obtained by writing the eigenvalue problem for normal modes in the form $\mathbf{N}\Psi = c\mathbf{P}\Psi$, with $\mathbf{N}$ and $\mathbf{P}$ both self-adjoint, and with $\{\Psi, \mathbf{P}\Psi\}$ proportional to the pseudomomentum in the mode. This form of the eigenvalue problem is displayed explicitly in Appendix A.

The generalization to the primitive equations, linearized about a zonal flow with arbitrary horizontal and vertical shear, is straightforward if we work in isentropic coordinates and if the lower boundary is an isentropic surface. The appropriate orthogonality relation takes the same form as (2.21) if we identify h with $-\partial p/\partial\Theta$ and redefine the inner product to include a vertical as well as a latitudinal integral. If the lower boundary is not an isentropic surface, in addition to the vertical integral one has a contribution from the lower boundary analogous to that in quasi-geostrophic theory discussed to Sec. 4 (D. G. Andrews, personal communication, 1984).

3. Barotropic decay on a sphere

Consider the barotropic vorticity equation on a sphere of radius a linearized about the zonal flow $\bar{u}(\theta) = a\mu\bar{\omega}(\theta)$ [where $\mu \equiv \cos(\theta)$]

$$\zeta'_t + \bar{\omega}\zeta'_\lambda = -\gamma(a\mu)^{-1}\psi'_\lambda,$$
$$\gamma \equiv a^{-1}(f + \bar{\zeta})_\theta. \quad (3.1)$$

The domain integrals, $\iint (\cdots)\mu d\theta d\lambda$, of $p \equiv \mu\overline{\zeta'^2}/(2\gamma)$ and $(\overline{u'^2} + \overline{v'^2})/2 - \bar{\omega}p$, are conserved. Setting $\zeta' = \zeta e^{im(\lambda - ct)} + $ (c.c.), the corresponding eigenvalue problem is

$$[\bar{\omega}\nabla_m^2 + \gamma(a\mu)^{-1}]\psi = c\nabla_m^2\psi,$$
$$\nabla_m^2 \equiv (a\mu)^{-2}[\mu\partial_\theta(\mu\partial_\theta) - m^2]. \quad (3.2)$$

We take the zonal flow to be of the form

$$\bar{u}(\theta) = A \cos(\theta) - B \cos^3(\theta) + C \sin^2(\theta) \cos^6(\theta). \quad (3.3)$$

The values $A = 25$, $B = 30$, and $C = 300$ m s^{-1} yield winds that are broadly similar to those in the upper troposphere. For this choice of parameters, $\bar{u}(\theta)$ and the corresponding γ are plotted in Fig. 1. Since $\gamma > 0$, all modes are neutral.

The relative excitation of the discrete modes and the continuum is critical for the character of the evolution of the linear perturbation as well as for the resulting mean flow modification. To the extent that the continuum dominates, the evolution will be qualitatively similar to that of a perturbation on a linear shear flow (e.g., Yamagata, 1976; Tung, 1983); the vorticity field will be sheared by the flow, eventually creating ever smaller scales in the vorticity and larger vorticity gradients. To the extent that the discrete modes dominate, the evolution will be similar to that in the absence of shear; a number of modes of different scales might be excited, but there would be no continual cascade of enstrophy to smaller scales.

The qualitative distinction between discrete and continuum modes is clear if one can decompose a localized initial disturbance into wavepackets with different dominant meridional wavenumbers and, therefore, different angular phase speeds. Those packets with phase speeds greater than the minimum value of $\bar{\omega}$ in the domain make up the continuum; they propagate meridionally, and their meridional wavenumbers become infinite as $t \to \infty$, as the packets asymptotically approach the latitude at which their phase speeds match the local $\bar{\omega}$. Those packets with phase speeds less than the minimum value of $\bar{\omega}$ propagate freely at all latitudes
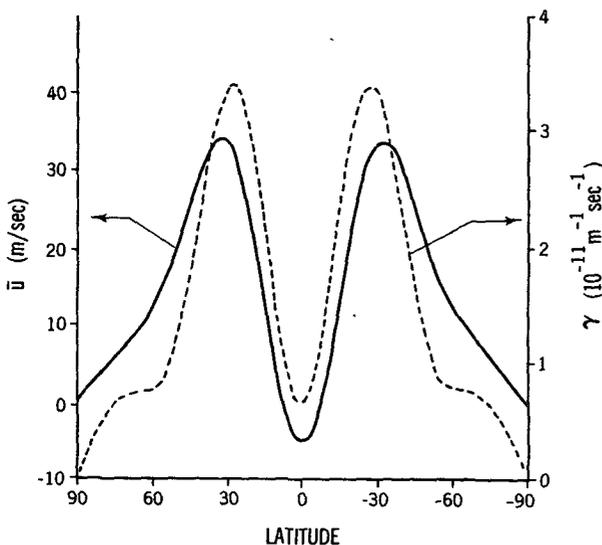
(between the two polar turning points) and excite the discrete modes. If we add a small scale-selective damping of vorticity to the problem, then the continuous spectrum will decay rapidly once the vorticity has been sheared to the point that meridional scales in the dissipation range have been generated (see below). The associated mean flow acceleration will be largest where the waves with the dominant meridional scales in the initial condition have their critical latitudes. In contrast, the decay of the discrete modes with large meridional scales will be extremely slow.

Consider as an example the initial condition

$$\zeta' = \cos(\theta)e^{-[(\theta-\theta_0)/\theta_w]^2}e^{im\lambda} + \text{c.c.}, \quad (3.4)$$

with $\theta_0 = 45°$, $\theta_w = 10°$, and $m = 3$. For this initial condition and the zonal flow (3.3), the projection onto the first mode is found to account for 5% of the pseudomomentum, the projection onto the second mode for 10%, and third mode for 7%. The contribution of the other discrete modes is negligible, so the remaining 78% of the pseudomomentum resides in the continuum. We add a small linear diffusion of vorticity, $\nu\nabla^2\zeta'$ with $\nu = 1.0 \times 10^4$ m^2 s^{-1}, to the vorticity equation (3.1) and then integrate forward in time. This value of the diffusion coefficient is sufficiently small that significant dissipation does not occur on the scale of the initial condition. The resulting time evolution of the global mean eddy enstrophy and pseudomomentum are plotted in Fig. 2a, and the global mean eddy energy and pseudoenergy in 2b. The enstrophy, pseudomomentum and energy are all normalized to unity at $t = 0$. The pseudoenergy is computed with the eddies normalized so that the energy equals unity at $t = 0$; the difference between the two curves in 2b is then the integral of p weighted by the mean angular velocity $\bar{\omega}$.

Following a period of rapid decay in eddy energy, the energy and enstrophy oscillate in time with a period of 3.3 days as they continue to decay slowly. The corresponding frequency is the difference between the frequencies of the first and third modes. (Although the second mode is also excited, it happens not to interfere with the other two modes in these global integrals as it is antisymmetric about the equator while the other two are symmetric.) The pseudomomentum decays very slowly initially, for it is conserved until small enough scales are generated that the dissipation becomes active. The continuum has begun to decay significantly before day 10 and its dissipation is nearing completion by day 40, after which the bulk of the pseudomomentum is accounted for by the three excited discrete modes. There is no oscillation similar to that in the enstrophy or energy because of the orthogonality relation discussed in Section 2.

The pseudoenergy E is free of oscillations for the same reason. The change in sign of E can be understood by recalling that $E = -cP$ for a mode with angular phase speed c. Initially, the disturbance is dominated

FIG. 1. The basic state zonal flow $\bar{u}$ and the corresponding absolute vorticity gradient γ used in the numerical example of barotropic decay on the sphere.
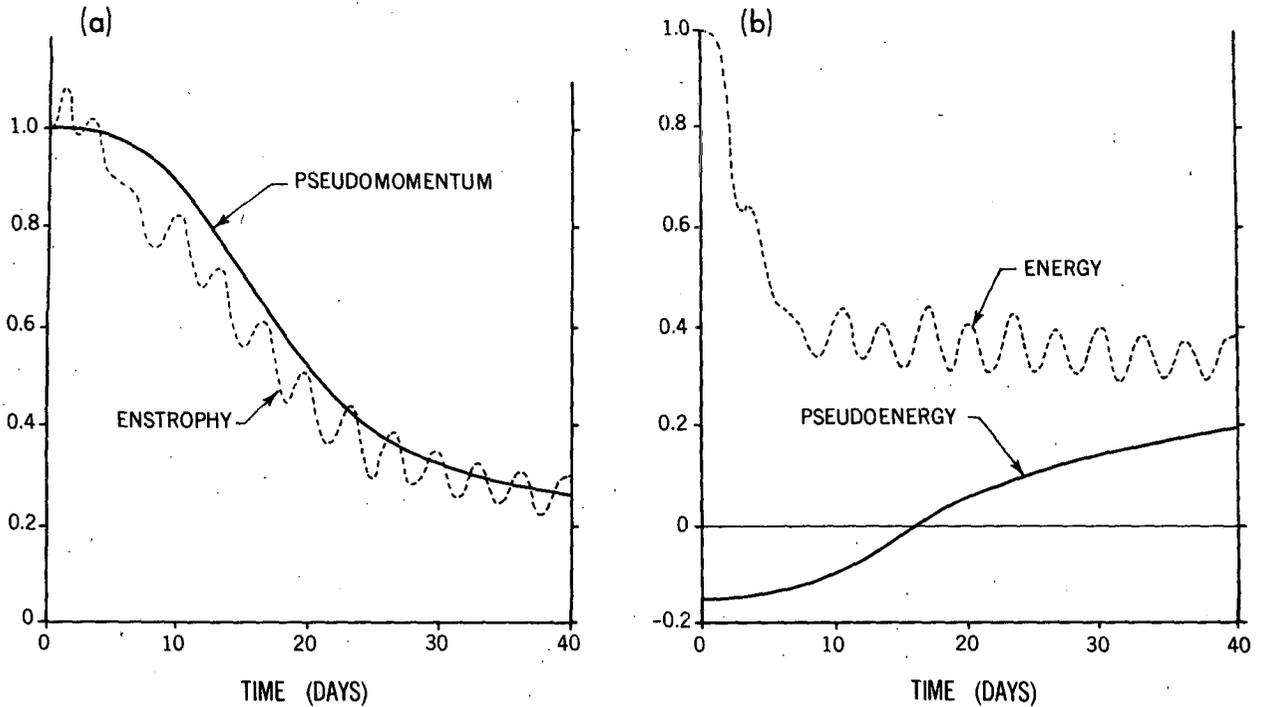
FIG. 2. (a) The time evolution of the globally integrated eddy enstrophy and pseudomomentum, both normalized to unity at $t = 0$, for the initial condition given in the text; (b) the time evolution of the integrated eddy energy and pseudoenergy, with the eddy energy normalized to unity at $t = 0$.

by modes in the continuum with $c > 0$; as the continuum is dissipated away, the discrete modes with $c < 0$ dominate and E turns positive. The very rapid decay of eddy energy in the first few days can be understood by noting that E is nearly conserved in this initial stage, while the bulk of the pseudomomentum is found to drift rapidly equatorward, into regions of smaller $\bar{\omega}$, due to Rossby wave radiation. Equivalently, one can argue that equatorward propagation is necessarily associated with poleward momentum flux, and propagation into regions of smaller angular velocity then implies that $\overline{u'v'} \,\omega_\theta > 0$, which in turn implies a conversion of eddy energy into zonal energy.

The fraction of the pseudomomentum that projects onto the discrete modes of the flow (3.3) is shown in Fig. 3 for various initial conditions of the form (3.4). The zonal wavenumber m ranges from 1 to 6 and the point of maximum eddy vorticity, θ_0, is chosen to be either 45°N or 30°N. The critical difference between the long and short zonal wavelength disturbances is apparent; the long waves have a substantial projection onto westward propagating discrete modes, while the shorter waves project primarily onto the continuum. For given m, a disturbance at 30° excites the discrete modes more strongly than one with the same meridional structure at 45°. While one can understand this result by examining the structure of the discrete modes, one can also attribute it to the fact that γ reaches its maximum value near 30°. Think of the initial distur-

bance as exciting a wavepacket with a particular dominant meridional wavenumber. Because of the larger γ, a packet excited at 30° will have more rapid westward phase propagation than one at 45°. As a result, it more easily excites the westward propagating discrete modes.

4. Excitation of the external mode

Consider a quasi-geostrophic flow on a β-plane channel, linearized about a zonal wind $\bar{u}(y, z)$. In $\ln(p)$ coordinates (and ignoring the small non-Doppler term in the lower boundary condition)

$$(\partial_t + \bar{u}\partial_x)q' = -\bar{q}_y\psi'_x \quad \text{for} \quad z \geq 0,$$

$$(\partial_t + \bar{u}\partial_x)\psi'_z = \bar{u}_z\psi'_x \quad \text{at} \quad z = 0,$$

$$q' \equiv \rho_0^{-1}(\rho_0\epsilon\psi'_z)_z + \psi'_{xx} + \psi'_{yy},$$

$$\bar{q}_y \equiv \beta - \bar{u}_{yy} - \rho_0^{-1}(\rho_0\epsilon\bar{u}_z)_z,$$

$$\rho_0 \equiv e^{-z/H}, \quad \epsilon \equiv f^2 N^{-2}. \tag{4.1}$$

The vertically integrated pseudomomentum conservation law for this system reads

$$\partial_t \left[\int_0^\infty (\rho_0\overline{\eta'^2}\bar{q}_y/2)dz - \epsilon(0)\overline{\eta'^2}(0)\bar{u}_z(0)/2 \right]$$

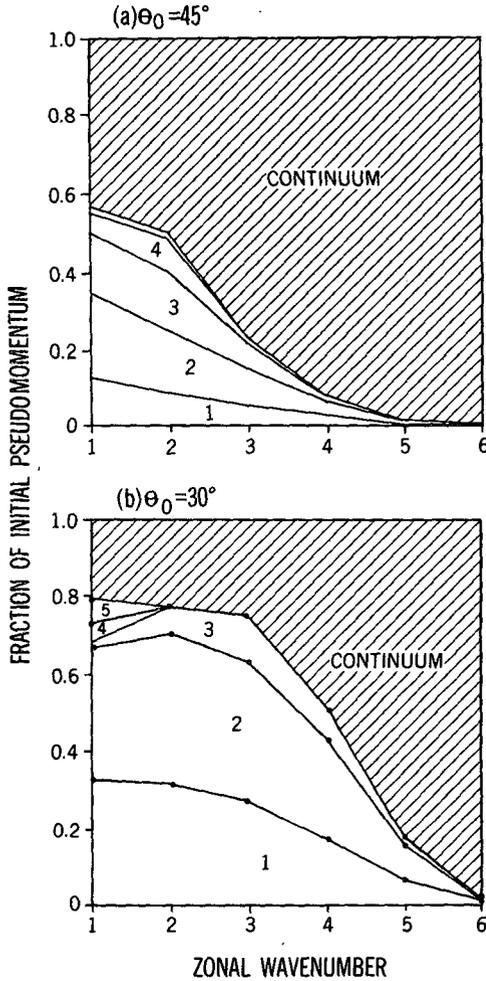$$= \partial_y \int_0^\infty \rho_0(\overline{u'v'})dz. \tag{4.2}$$

FIG. 3. The fraction of the initial pseudomomentum contained in the individual discrete modes for an initial condition of the form (3.4), as a function of zonal wavenumber m. In (a) the latitude of the initial eddy vorticity maximum, θ_0, is 45°; in (b) it is 30°.

Here $q' = -\bar{q}_y\eta'$ for $z \geqslant 0$ while $\psi'_z = \bar{u}_z\eta'$ at $z = 0$. Restricting consideration to the separable case, $\bar{u} = \bar{u}(z)$, and setting $\psi' = \psi(z)e^{i(kx+ly)} + $ (c.c.), the vertical modal structures ψ satisfy the eigenvalue problem

$$\bar{u}\mathbf{R}\psi + \bar{q}_y\psi = c\mathbf{R}\psi, \quad z > 0,$$

$$\bar{u}\psi_z - \bar{u}_z\psi = c\psi_z, \quad z = 0,$$

$$\mathbf{R} \equiv \rho_0^{-1}\partial_z(\rho_0\epsilon\partial_z) - K^2, \quad (4.3)$$

where $K^2 \equiv k^2 + l^2$. The orthogonality relation that corresponds to the conservation law (4.2) once again takes the form $\{\eta_1, \mathbf{P}\eta_2\} = 0$ if $c_1^* \neq c_2$. Here $\mathbf{P} \equiv \bar{q}_y - \epsilon\bar{u}_z\delta(z+)$, where $\delta(z+)$ is a Dirac delta-function located slightly above the surface, and the brackets again correspond to an average over the domain:

$$\{A, B\} = H^{-1}\int_0^\infty \rho_0 A^*B dz. \quad (4.4)$$

The structure of the spectrum in Charney's model ($\bar{u} = \Lambda z$, $\Lambda = $ constant, $N^2 = $ constant) is in many ways similar to that found for other vertical wind profiles of relevance for the troposphere (e.g., Geisler and Dickinson, 1975). In the following discussion of this model, we nondimensionalize with the radius of deformation NH/f, H, and ΛH as horizontal scale, vertical scale, and velocity scale, respectively, retaining the same notation for the nondimensional variables. The sole remaining parameter characterizing the basic state is $r \equiv \beta N^2 H/(f^2\Lambda)$. Define $K_n^2 \equiv [(1 + r)^2/n^2 - 1]/4$. From Burger's (1962) analysis, we know that if $K^2 > K_1^2 \equiv r(2 + r)/4$, there are no discrete neutral modes, but there is one unstable mode (the "Charney" mode) and its conjugate, as well as a continuum of neutral modes. If $\max(0, K_2^2) < K^2 < K_1^2$, there is again one unstable mode (the "Burger" or "Green" mode) and its conjugate, and the neutral continuum with $c > 0$, but there is also a neutral mode with $c < 0$, the external Rossby wave. If $\max(0, K_3^2) < K^2 < K_2^2$ (which can only occur if $r > 1$), there are two neutral modes in addition to an amplifying-decaying pair and the continuum. Typically, $r \leqslant 1$ in the extratropical troposphere, so that the external mode is often the only neutral mode. With a more realistic vertical wind profile, trapping a second neutral mode within the tropospheric wave guide remains difficult (Held, et al., 1985), although other modes can be trapped by stratospheric or mesospheric winds.

In these nondimensional units $\mathbf{P}$ reduces to $1 + r - \delta(z+)$ for Charney's model. Although $\{\eta, \mathbf{P}\eta\}$ can be negative, because of the δ-function resulting from the surface temperature gradient, the contribution from each neutral discrete mode in Charney's model can be shown to be positive. However, unless the waves are propagating very rapidly westward, one finds a large cancellation between the interior and boundary contributions to $\{\eta, \mathbf{P}\eta\}$.

It is of particular interest to consider the external mode excited by that component of the initial condition with K^2 slightly smaller than $K_1^2 \equiv r(2 + r)/4$. Such a mode propagates slowly westward with respect to the mean surface wind. Indeed, by manipulating the explicit solutions to Charney's model (e.g., Pedlosky, 1979), one can show that

$$K^2 - K_1^2 = -r(1 + r)^3c^2/4 + O(c^3), \quad (4.5)$$

where c is the (negative) phase speed of the mode with respect to the surface wind. Since mean surface winds are typically small and positive in midlatitudes, waves that are quasi-stationary with respect to the ground and, thus, contribute to the low frequency variability of the atmosphere, will have total horizontal wavenumbers in this range. The component of the initial disturbance with K slightly less than K_1 can be decomposed into contributions from the external mode, the amplifying and decaying Burger mode, and the continuum:

$$\eta_I = a_e \eta_e + \text{(Burger mode)} + \text{(continuum).} \quad (4.6)$$

Using pseudomomentum orthogonality,

$$a_e = \{\eta_e, \mathbf{P}\eta_I\}/\{\eta_e, \mathbf{P}\eta_e\}. \quad (4.7)$$

One can also show that the vertical structure of the external mode's streamfunction, ψ_e, approaches $z \exp(-rz/2) + O(c)$ as $K \to K_1$, while the corresponding particle displacement, η_e, approaches $\exp(-rz/2) + O(c)$ (e.g., Held et al., 1985). Defining $\eta_0 = \exp(-rz/2)$, it is important to observe that $\{\eta_0, \mathbf{P}\eta_0\} = 0$, as can be confirmed by performing the integration. For given streamfunction amplitude, the pseudomomentum in the mode vanishes as it approaches its point of bifurcation with the Charney and Green modes. As described in Appendix B, one must consider the $O(c)$ terms in the eigenfunction to evaluate the external mode's pseudomomentum. The result is

$$\{\eta_e, \mathbf{P}\eta_e\} = r|c| = O(K_1^2 - K^2)^{1/2}. \quad (4.8)$$

Assuming that η_I is substantially different from η_e, so that $\{\eta_e, \mathbf{P}\eta_I\}$ is not $O(c)$, the external mode component of the initial condition is $O(c^{-1})$. Therefore, the amplitude of the excited neutral mode will be very large if K is close to K_1. The unbounded algebraic growth one finds when K is precisely equal to K_1 can be thought of as the limiting case. Farrell (1984) has made essentially the same point by explicitly solving the initial value problem for Eady's model. Pseudomomentum orthogonality provides a useful way of understanding this behavior. This result is invalid in the exceptional case that the initial condition consists of nearly a pure external mode, $\eta_I \approx \eta_e$, for then $\{\eta_e, \mathbf{P}\eta_I\} \approx 0$. In this case, $a_e \eta_e \approx \eta_I$; there can be no amplification if only a single neutral vertical mode is excited. The counterintuitive result is that the vertical structure of the initial condition should not be too close to that of the neutral mode we wish to excite.

Energy is being extracted from the zonal flow as this large amplitude neutral external mode evolves out of the initial condition, just as in the growth of an unstable mode. Indeed, the lesson to be learned is that there is little physical distinction between the neutral modes with K slightly smaller than K_1 and the unstable modes. The unstable modes, having zero pseudomomentum, can grow to arbitrarily large amplitude as far as linear theory is concerned, while the neutral modes with small pseudomomentum grow to a large but finite amplitude that depends on the initial conditions. In reality, nonlinearity limits the amplitude of the neutral modes if this amplitude is sufficiently large, just as it must in the case of unstable waves.

In Appendix B we discuss the fact that $\int \rho_0 \psi^2 dz$ equals the pseudomomentum in a mode multiplied by $\partial c/\partial K^2$. The large streamfunction amplitude consistent with a small amount of pseudomomentum when c is small in Charney's model is, therefore, intimately related to the large group velocities of these modes (see

Held et al., 1985). In the more realistic model atmospheres with bounded winds considered by Held et al., it is found that the group velocity of the external mode does become large as the phase speed of the wave approaches the surface wind, but it does not increase without bound as it does in Charney's model (compare Figs. 7d and 9c in that paper). Therefore, in more realistic atmospheres one expects behavior qualitatively similar to that described above, but not quite as dramatic.

5. Concluding remarks

The conservation laws for pseudomomentum and pseudoenergy for linear waves on shear flows immediately yield orthogonality relations for the modes of the system. These relations are useful whenever one is concerned with the excitation of neutral modes. We have described two instances from planetary wave theory. The first of these concerns the barotropic decay ory. The first of these concerns the barotropic decay of a perturbation on the sphere, where the orthogonality relations are useful in clarifying "how much" of an initial disturbance projects onto discrete neutral modes as opposed to the continuum. Analogous calculations can be performed for the shallow water equations or for the primitive equations on the sphere, and may also be useful in analyzing the excitation of equatorially trapped ocean waves in the presence of strong equatorial currents.

The distinction between discrete and continuum modes is an important one, even if the discrete mode resonances in more realistic atmospheric models have significant width (see Salby, 1981) so that the distinction is not as sharp as in our idealized model. A disturbance that projects primarily onto discrete modes interacts with the zonal mean flow in a fundamentally different way from a disturbance which projects primarily onto the continuum. In the latter case, the linear dynamics itself produces an enstrophy cascade to small meridional scales, so that the mean flow modification that ultimately results from the dissipation of the wave is relatively insensitive to the details of the dissipation mechanism. In contrast, the mean flow modification due to the decay of discrete modes is entirely dependent on the dissipation mechanism.

The second example is the excitation of external Rossby waves in vertical shear. Pseudomomentum orthogonality provides a simple way of understanding the ease with which one can excite large amplitude waves that propagate slowly westward with respect to the surface wind. The key observation is that such external Rossby waves have very little pseudomomentum and are therefore physically similar to unstable waves. A somewhat counterintuitive aspect of this result is that in order to excite a large amplitude external mode, the initial disturbance should not have a vertical structure similar to that of the mode.

APPENDIX A

The Shallow Water Equations

The linearized shallow water equations are given by Eqs. (2.19). Setting, $Q = (f + \bar{\zeta})/\bar{h}$, we also have $DQ' + v'\bar{Q}_y = 0$, where $D \equiv \partial_t + \bar{u}\partial_x$ and

$$\bar{h}^2 Q' = \bar{h}\zeta' - (f + \bar{\zeta})h'. \qquad (A1)$$

From (A1) we have

$$\bar{h}^2\overline{v'Q'} = -\bar{h}(\overline{u'v'})_y + \bar{h}\overline{v'_y u'} - (f + \bar{\zeta})\overline{h'v'}. \qquad (A2)$$

Also, from conservation of Q',

$$[\bar{Q}_y\overline{\eta'^2}/2]_t = -\overline{v'Q'}, \qquad (A3)$$

where $Q' \equiv -\bar{Q}_y\eta'$. Further, one obtains directly from the linearized zonal momentum and height equations that

$$(\overline{u'h'})_t = \overline{u'h'_t} + \overline{u'_t h'} = (f + \bar{\zeta})\overline{h'v'} - \overline{u'v'}\bar{h}_y - \bar{h}\overline{u'v'_y}. \qquad (A4)$$

Substituting (A2) into (A3) and then subtracting (A4), one finds the pseudomomentum conservation law discussed in the text,

$$[\bar{h}^2\bar{Q}_y\overline{\eta'^2}/2 - \overline{u'h'}]_t = (\bar{h}\overline{u'v'})_y. \qquad (A5)$$

We now demonstrate that the eigenvalue problem obtained from these shallow water equations can be written in the form $\mathbf{N}\Psi = c\mathbf{P}\Psi$, with $\{\Psi, \mathbf{P}\Psi\}$ proportional to the pseudomomentum in the wave field described by the state vector Ψ. Define an auxiliary variable χ' such that $v' \equiv \chi'_x$. Setting $D = (\bar{u} - c)\partial_x$, the conservation of Q' reduces to $(\bar{u} - c)Q = -\bar{Q}_y\chi$, or

$$\bar{u}\bar{Q}_y(\bar{h}\eta) - \bar{h}\bar{Q}_y\chi = c\bar{Q}_y(\bar{h}\eta), \qquad (A6a)$$

where variables without primes or overbars refer to complex wave amplitudes, as in the text. Similarly, the zonal momentum and height equations reduce to

$$\bar{u}u - (f + \bar{\zeta})\chi + gh = cu \qquad (A6b)$$

$$\bar{u}h + (\bar{h}\chi)_y + \bar{h}u = ch. \qquad (A6c)$$

Finally, the expression (A1) for Q' can be rewritten as

$$\bar{h}\bar{Q}_y(\bar{h}\eta) - (f + \bar{\zeta})h + \bar{h}\chi_{xx} - \bar{h}u_y = 0. \qquad (A6d)$$

The four equations (A6a–d) can be combined into the matrix equation

$$\begin{bmatrix} \bar{u}\bar{Q}_y & 0 & 0 & -\bar{h}\bar{Q}_y \\ 0 & -g & -\bar{u} & (f + \bar{\zeta}) \\ 0 & -\bar{u} & -\bar{h} & -\partial_y\bar{h} \\ -\bar{h}\bar{Q}_y & (f + \bar{\zeta}) & \bar{h}\partial_y & -k^2\bar{h} \end{bmatrix} \begin{bmatrix} \bar{h}\eta \\ h \\ u \\ \chi \end{bmatrix}$$

$$= c \begin{bmatrix} \bar{Q}_y & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{h}\eta \\ h \\ u \\ \chi \end{bmatrix}. \qquad (A7)$$

The two matrices in (A7) are self-adjoint. Denoting the matrix on the rhs as $\mathbf{P}$ and setting Ψ equal to the vector $[\bar{h}\eta, h, u, \chi]$, one can immediately prove that $(c_1^* - c_2)\{\Psi_1, \mathbf{P}\Psi_2\} = 0$, which is precisely the orthogonality relation obtained from pseudomomentum conservation.

APPENDIX B

External Mode Pseudomomentum

The eigenfunction for the neutral external mode with small negative phase speed c that is sufficiently accurate to allow computation of the pseudomomentum is

$$\psi = (z - c)(1 - rc^2/[2(z - c)])e^{-rz/2}. \qquad (B1)$$

The corresponding particle displacement, η, is obtained by dividing by $(z - c)$. The underlined term is $O(c)$ only in a small layer near the ground; elsewhere it is $O(c^2)$ and negligible. $P \equiv \{\eta, \mathbf{P}\eta\}$ equals

$$\int_0^\infty (1 + r)\eta^2 e^{-z}dz - \eta^2(0). \qquad (B2)$$

The integral equals $[1 + O(c^2)]$, while $\eta^2(0) = [1 + rc + O(c^2)]$. Therefore, $P = r|c|$.

An alternative derivation of this result, based on the relation between pseudomomentum and group velocity, is illuminating. Assuming that the basic state is independent of y, the vertically integrated meridional flux of pseudomomentum equals the meridional group velocity of the mode G_y times the vertically integrated pseudomomentum:

$$-H^{-1}\int_0^\infty \rho_0\overline{u'v'}dz = G_y P \qquad (B3)$$

(for example, see Grimshaw, 1984). Noting that $G_y = 2kl\partial c/\partial K^2$ and that $\overline{u'v'} = -2kl|\psi|^2$, and nondimensionalizing as in the discussion of Charney's model in Section 4, we have

$$\int_0^\infty e^{-z}|\psi|^2dz = P\partial c/\partial K^2. \qquad (B4)$$

This last equation is equivalent to one derived by other means in the Appendix of Held *et al.* (1985) [although it should be noted that the normalization used in that paper is such that the lhs of (B4) equals unity]. Setting

$\psi = z \exp(-rz/2)$ to lowest order, the lhs $= 2(1 + r)^{-3}$, while from (4.5) $\partial K^2/\partial c = r(1 + r)^3|c|/2$ to lowest order. The result $P = r|c|$ is once again obtained.

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