Nonlinear Axially Symmetric Circulations in a Nearly Inviscid Atmosphere

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ABSTRACT

The structure of certain axially symmetric circulations in a stably stratified, differentially heated, rotating Boussinesq fluid on a sphere is analyzed. A simple approximate theory [similar to that introduced by Schneider (1977)] is developed for the case in which the fluid is sufficiently inviscid that the poleward flow in the Hadley cell is nearly angular momentum conserving. The theory predicts the width of the Hadley cell, the total poleward heat flux, the latitude of the upper level jet in the zonal wind, and the distribution of surface easterlies and westerlies. Fundamental differences between such nearly inviscid circulations and the more commonly studied viscous axisymmetric flows are emphasized. The theory is checked against numerical solutions to the model equations.

1. Introduction

The importance of mixing induced by large-scale baroclinic or barotropic instabilities for the general circulation of the atmosphere can best be appreciated by artificially suppressing these instabilities and examining the circulation which develops in their absence. This is most easily accomplished in the idealized case for which radiative forcing and the lower boundary condition are both axially symmetric (independent of longitude). The flow of interest in this case is the large-scale axisymmetric flow consistent with radiative forcing and whatever small-scale mixing is still present in the atmosphere after the large-scale instabilities have been suppressed.

Such axisymmetric circulations have not received as much attention in the meteorological literature as one might expect, given what would appear to be their natural position as first approximations to the general circulation. Reasons for this neglect are not hard to find. It is the accepted wisdom that large-scale zonally asymmetric baroclinic instabilities are the prime determinant of the structure of the tropospheric circulation. If this view is correct and there is, in fact, little resemblance between the axisymmetric circulation remaining after these eddies have been suppressed and the observed zonally averaged flow, then the study of such circulations cannot be expected to arouse much interest in the meteorological community. Furthermore, in the linear viscous models often utilized in studies of axisymmetric flows in rapidly rotating atmospheres (e.g., Dickinson, 1971; Leovy, 1964), the meridional circulation is effectively determined by the parameterized small-scale frictional stresses in the zonal momentum equation. Detailed analyses of such models do not promise to be very fruitful as long as theories for small-scale momentum mixing are themselves not very well developed.

Schneider and Lindzen have recently computed some axisymmetric flows forced by small-scale fluxes of heat and momentum that do bear some resemblance to the observed circulation (Schneider and Lindzen, 1977; Schneider, 1977). Using simple theories for moist convective as well as boundary and radiative fluxes, Schneider obtains a Hadley cell which terminates abruptly at more or less the right latitude, a very strong subtropical jet at the poleward boundary of the Hadley cell, strong trade winds in the tropics, and a shallow Ferrel cell and surface westerlies poleward of the trades. Nakamura (1978) describes an effectively axisymmetric calculation (with heating and frictional formulations differing considerably from Schneider's) which also yields a Hadley cell of well-defined meridional extent. Evidently, certain features of the tropospheric circulation, notably the width of the Hadley cell and the position of the subtropical jet, can be understood in some crude first approximation within an axisymmetric framework.

Schneider (1977) also argues that the meridional extent and the heat transported by the Hadley cell become independent of the small-scale vertical

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mixing of momentum when this mixing is sufficiently small, in contradistinction to results from linear viscous models. To the extent that this nonlinear nearly inviscid limit is the relevant one, certain features of the circulation are not sensitive to one's necessarily tentative and uncertain frictional formulation. Analysis of this limiting case, therefore, is of particular interest.

In this work we analyze an axisymmetric model with highly idealized diabatic heating and frictional forcing. Our goal is to clarify which parameters control the width and strength of the Hadley cell, the location and strength of the subtropical jet and surface winds, and the presence or absence of a Ferrel cell in such a simplified axisymmetric model, particularly in the nonlinear, nearly inviscid limit. We restrict ourselves in this paper to flows which are symmetric with respect to the equator, in which, therefore, there is no cross-equatorial flow.

Our model equations are presented in Section 2. Some comments on the limited range of validity of certain linear, viscous approximate solutions are included in Section 3. A qualitative discussion of the nonlinear, nearly inviscid limit, from which one can derive approximate expressions for the Hadley cell's width, its heat and momentum transports, and the latitudinal profile of the surface wind stress, is provided in Section 4. In Section 5 numerical solutions to the model equations are described which illustrate how the transition from a linear viscous to nonlinear, nearly inviscid balance occurs as the small-scale, linear diffusive vertical mixing of momentum diminishes in importance. A brief attempt is made in Section 6 to argue that the behavior of moist axisymmetric model atmospheres should be similar in certain respects to that of the highly simplified dry stably stratified axisymmetric model analyzed in this paper.

2. The model equations

We consider the primitive equations for a dry Boussinesq fluid on a hemisphere, confined between the surface and a rigid lid at height H above the surface. The flow is forced by radiative heating proportional to the difference between the fluid temperature and a specified "radiative equilibrium" temperature. Linear vertical diffusion of heat and momentum (with Prandtl number unity, unless otherwise noted) is the only small-scale mixing in the fluid interior. The diffusivity is chosen to be a constant, independent of height and latitude. A zero stress boundary condition is imposed at the top surface, and the stress at the ground is taken to be proportional to surface wind. Zero vertical heat flux is imposed at both upper and lower boundaries.

The equations for steady flow are

$$\left.\begin{aligned}
0 &= -\nabla\cdot(\mathbf{v}u) + fv + \frac{uv\tan\theta}{a} + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)\\
0 &= -\nabla\cdot(\mathbf{v}v) - fu - \frac{u^2\tan\theta}{a} - \frac{1}{a}\frac{\partial\Phi}{\partial\theta} + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right)\\
0 &= -\nabla\cdot(\mathbf{v}\Theta) - (\Theta-\Theta_E)\tau^{-1} + \frac{\partial}{\partial z}\left(\nu\frac{\partial\Theta}{\partial z}\right)\\
0 &= -\nabla\cdot\mathbf{v}\\
\frac{\partial\Phi}{\partial z} &= g\Theta/\Theta_0
\end{aligned}\right\}, \quad (1)$$

with boundary conditions

$$\left.\begin{aligned}
&\text{at } z=H: \quad w=0; \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial\Theta}{\partial z} = 0\\
&\text{at } z=0: \quad w=0; \quad \frac{\partial\Theta}{\partial z} = 0;\\
&\qquad \nu\frac{\partial u}{\partial z} = Cu; \quad \nu\frac{\partial v}{\partial z} = Cv
\end{aligned}\right\}. \quad (1a)$$

Here $\mathbf{v} = (v,w)$ is the velocity and $\nabla = [(a\cos\theta)^{-1} \times \partial(\cos\theta\ )/\partial\theta,\ \partial/\partial z]$ the gradient operator in the meridional-vertical plane. τ is a constant radiative damping time, and C a constant drag coefficient. $g\Theta/\Theta_0$ is the buoyancy in the Boussinesq approximation. The notation is otherwise standard. Θ_E is given the form

$$\frac{\Theta_E(\theta,z)}{\Theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\theta) + \Delta_v\left(\frac{z}{H} - \frac{1}{2}\right), \quad (2)$$

where Θ_0 is the global mean of Θ_E, Δ_H and Δ_v are both nondimensional constants, and P_2 is the second Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Δ_H and Δ_v are, respectively, the fractional change in potential temperature from equator to pole and from the top to the bottom of the fluid in radiative equilibrium. We restrict consideration to the case of statically stable radiative equilibrium, $\Delta_v > 0$, thereby removing the burden of maintaining the stability of the atmosphere from the large-scale circulation. One could try to justify this choice by arguing that moist convection would maintain a statically stable profile in the absence of large-scale flow, but it is not legitimate to think of a fixed function $\Theta_E(\theta,z)$ as incorporating the effects of moist convection, since water vapor is transported and latent heating modified by the circulation. The question of the extent to which the behavior of the system (1) with $\Delta_v > 0$ is similar to the behavior of moist atmospheres is addressed in Section 6.

When $\nu = 0$, Eq. (1) has an exact solution: a flow with no meridional circulation ($v = w = 0$), temperatures in radiative equilibrium ($\Theta = \Theta_E$), and a balanced zonal wind (denoted by u_E) satisfying

$$\frac{\partial}{\partial z}\left(fu_E + \frac{u_E^2 \tan\theta}{a}\right) = -\frac{g}{a\Theta_0}\frac{\partial\Theta_E}{\partial\theta}. \qquad (3)$$

From the zonal momentum and continuity equations and the stress boundary conditions, it follows that

$$\frac{1}{a\cos^2(\theta)}\frac{\partial}{\partial\theta}\left(\cos^2\theta\int_0^H uvdz\right) = -Cu\big|_{z=0}. \qquad (4)$$

If $v = 0$ the zonal wind must therefore be identically zero at the ground, and

$$fu_E + \frac{u_E^2}{a}\tan\theta = -\frac{g}{a\Theta_0}\frac{\partial\Theta_E}{\partial\theta}z$$

$$= \frac{2\Delta_H}{a}gz\sin\theta\cos\theta.$$

The solution to this equation for which $u(z = 0) = 0$ is

$$\frac{u_E}{\Omega a} = [(1 + 2Rz/H)^{1/2} - 1]\cos\theta, \qquad (5)$$

where

$$R = \frac{gH\Delta_H}{\Omega^2 a^2}. \qquad (6)$$

If $R \ll 1$, then $u_E/\Omega a \approx R\cos(\theta)z/H$. In this limit R can be thought of as a thermal Rossby number. Note that this inviscid solution has westerly winds at the equator.

It is tempting to look for solutions to (1) for small but nonzero ν by perturbing this inviscid solution. But this procedure cannot work since the solution to (1), in fact, does not approach this inviscid solution as $\nu \to 0$. This can easily be demonstrated after rewriting the zonal momentum equation as

$$0 = -\nabla\cdot(\mathbf{v}M) + \frac{\partial}{\partial z}\left(\nu\frac{\partial M}{\partial z}\right), \qquad (7)$$

where

$$M = \Omega a^2\cos^2\theta + ua\cos\theta$$

is the fluid's angular momentum per unit mass. From the well-known argument for conservation equations of the form (7), M cannot have a local maximum in the interior of the fluid. (For those unfamiliar with the argument: if there were a local maximum in the interior, with $M = M_0$, say, then for sufficiently small ϵ one could draw a closed curve surrounding this point, on which $M = M_0 - \epsilon$; the advective flux out of the region within this curve is identically zero, by conservation of mass, and cannot balance the nonzero outward diffusive flux; therefore such a point cannot exist.) Using the stress boundary conditions at $z = 0$ and $z = H$, one can easily show that M must, in fact, attain its maximum value at a point on the lower boundary where $u \leq 0$. Only at such a point can the surface stress opposing the easterlies balance the diffusive loss of westerly momentum into the interior. It follows that M must be less than or equal to Ωa^2 everywhere in the fluid and that u must be everywhere less than or equal to

$$u_M \equiv \Omega a\sin^2(\theta)/\cos(\theta). \qquad (8)$$

In particular, the winds at the equator cannot be westerly and the solution cannot approach the inviscid solution described above as $\nu \to 0$. Hide (1969) has emphasized the importance of this constraint on axisymmetric flows forced by *down-gradient* angular momentum fluxes.

If $R \ll 1$, then $u_E(\theta,H) \approx \Omega aR$ close to the equator, whereas $u_M(\theta) \approx \Omega a\theta^2$. Thus, u_E is larger than u_M for $\theta < R^{1/2}$. More generally, $u_E(\theta,H) > u_M(\theta)$ if $\theta < \tan^{-1}\{[(1 + 2R)^{1/2} - 1]^{1/2}\}$. No matter what the values of the other parameters in the problem, a circulation must exist at least within this region in order to maintain winds and an associated temperature field consistent with angular momentum conservation. If one estimates the equator-to-pole temperature difference on the earth when forced with annual mean solar fluxes as ~100 K, then $\Delta_H \approx 1/3$. Using $H = 1.5 \times 10^4$ m as the height of the poleward flow in the Hadley cell, one finds $R \approx 0.2$, and, therefore, $u_E(H) > u_M$ for $\theta \lesssim 27°$. It is of interest that this region, within which this model is constrained to depart from radiative equilibrium, corresponds roughly to the region occupied by the earth's Hadley cell.

3. Approximate linear viscous solutions

Linear viscous models of axisymmetric flows in rapidly rotating atmospheres are often obtained by ignoring the advection of relative angular momentum,

$$fv = -\nu\frac{\partial^2 u}{\partial z^2}, \qquad (9a)$$

and assuming thermal wind balance except, perhaps, for viscous drag

$$f\frac{\partial u}{\partial z} = -\frac{g}{a\Theta_0}\frac{\partial\Theta}{\partial\theta} + \nu\frac{\partial^3 v}{\partial z^3}. \qquad (9b)$$

If one now sets $\Theta = \Theta_E$ (e.g., as in Charney, 1973), one obtains perhaps the simplest model of a "Hadley cell," a direct circulation confined to two Ekman layers at the top and bottom boundaries, each with a mass flux proportional to the viscous stress in the interior. (The stress at the surface must be zero since the horizontal advection of angular momentum has been neglected.) As $\nu \to 0$ this Hadley cell disappears. Because of the angular momentum constraint, we know that this cannot be an adequate description of this limit for the system (1).

A linear model of more interest in low latitudes is obtained if one allows temperatures to change by approximating the thermodynamic equation by

$$-w\frac{\partial\Theta}{\partial z} = (\Theta - \Theta_E)\tau^{-1} \tag{9c}$$

and assuming that $\partial\Theta/\partial z$ is a constant, independent of the flow. Schneider and Lindzen (1976) discuss the resulting set of equations in some detail. Defining a meridional streamfunction by

$$v = -\frac{1}{\cos\theta}\frac{\partial\psi}{\partial z}, \quad w = \frac{1}{a\cos\theta}\frac{\partial\psi}{\partial\theta},$$

the set (9a)–(9c) reduces to

$$E^2\frac{\partial^4\tilde{\psi}}{\partial\zeta^4} + \sin^2\theta\tilde{\psi} - R^*\cos\theta\frac{\partial}{\partial\theta}\left(\cos^{-1}\theta\frac{\partial\tilde{\psi}}{\partial\theta}\right) = R^*\sin\theta\cos^2\theta, \tag{10}$$

for

$$\tilde{\psi} = \psi\left(\frac{\tau\Delta_v}{2aH\Delta_H}\right),$$

where

$$E \equiv \nu/2\Omega H^2, \quad R^* \equiv \left(\frac{\tau\nu}{4H^2}\right)\left(\frac{gH}{\Omega^2a^2}\right)\Delta_v$$

and

$$\zeta \equiv z/H.$$

For sufficiently small Ekman number E, Eq. (10) predicts that temperatures will be modified significantly only within an equatorial boundary layer extending from $\theta = 0$ to $\theta \approx (R^*)^{1/4}$. The width of this equatorial boundary layer shrinks to zero as $\nu \to 0$. Since temperatures and zonal winds in the true solution to (1) must be modified by the circulation out to at least $\theta \approx R^{1/2}$ (restricting consideration to the limit $R \ll 1$ for simplicity), this approximate set of equations cannot be adequate unless $(R^*)^{1/4} \gtrsim R^{1/2}$, i.e., unless

$$\nu \gtrsim \nu_m \equiv \frac{H^2}{\tau}\frac{gH}{\Omega^2a^2}\frac{\Delta_H^2}{\Delta_v}. \tag{11}$$

Leovy (1964) and Dickinson (1971) consider very similar axisymmetric models of the upper atmosphere and tropical troposphere, respectively. Vertical momentum diffusion is replaced with linear drag in the zonal momentum equation ($fv = \kappa u$) and ignored in the meridional momentum equation. Essentially the same equatorial boundary layer appears for these equations, of width

$$\sim\left(\kappa\tau\frac{gH}{\Omega^2a^2}\Delta_v\right)^{1/4},$$

although both Dickinson and Leovy choose values of κ sufficiently large that their solutions do not, in fact, have a boundary-layer character. [See, however, Schoeberl and Strobel (1978), who display in their Fig. 4 solutions to a modified version of Leovy's model for successively smaller values of κ.] Pedlosky (1969) describes a similar equatorial boundary layer in a model in which vertical conduction of heat, rather than radiation, balances adiabatic heating or cooling in the interior.

The key terms neglected in such linear models when ν (or κ) is small are those responsible for the advection of relative angular momentum. In these linear solutions, $u(\theta,H)$ rises from zero at the equator to $\sim u_E$ outside of the equatorial boundary layer. If the width of this layer is $<R^{1/2}$, u must increase faster than u_M with increasing latitude. The approximation of retaining $fv \equiv \nabla\cdot(\mathbf{v}u_M\cos\theta)/\cos\theta$ while neglecting

$$\nabla\cdot(\mathbf{v}u) - \frac{uv\tan\theta}{a} \equiv \nabla\cdot(\mathbf{v}u\cos\theta)/\cos\theta$$

is then clearly inappropriate. Angular momentum conservation, not surprisingly, is the key to the character of the solution as $\nu \to 0$.

4. The inviscid limit

a. Description of circulation

If the exact inviscid solution described in Section 2 is not the limiting solution, and if the linear models described in Section 3 also fail in this limit, then what does happen as $\nu \to 0$? One can gain some insight into this limit with an argument based on potential temperature and angular momentum conservation. The essence of the argument is due to Schneider (1977). The region of the model's parameter space within which the required approximations are self-consistent is most easily determined *a posteriori*. The question of self-consistency is therefore postponed until Section 4b. Some comments on the symmetric instability of the circulation in this limit are included in Section 4c.

A steady Hadley cell is assumed to exist near the equator, with fluid rising in the neighborhood of the equator and spreading poleward just beneath the upper boundary. It is further assumed that viscous stresses are small enough in the interior that angular momentum is nearly conserved within this poleward flow, so that $u(\theta,H) \approx u_M(\theta)$ but that surface drag is sufficiently strong that $u(\theta,0) \approx 0$.

This momentum conserving flow clearly cannot continue to the pole. It is assumed, therefore, that it continues only up to some latitude θ_H. Poleward of θ_H the meridional circulation is assumed to be identically zero, so that $\Theta = \Theta_E$ and $u = u_E$. θ_H can be determined by requiring continuity of temperature at $\theta = \theta_H$ and assuming a balanced zonal wind:

$$fu + \frac{u^2\tan\theta}{a} = -\frac{1}{a}\frac{\partial\Phi}{\partial\theta}.$$

Evaluating this expression at $z = 0$ and $z = H$, and using the vertically integrated hydrostatic equation, one obtains

$$f[u(H) - u(0)] + \frac{\tan\theta}{a}[u^2(H) - u^2(0)] = -\frac{gH}{a\Theta_0}\frac{\partial\bar{\Theta}}{\partial\theta},$$

where an overbar denotes a vertical average. Substituting $u(H) \approx u_M$ and $u(0) \approx 0$ and integrating, one finds

$$\frac{\bar{\Theta}(0) - \bar{\Theta}(\theta)}{\Theta_0} = \frac{\Omega^2 a^2}{gH}\frac{\sin^4\theta}{2\cos^2\theta}. \quad (12)$$

Continuity of potential temperature requires

$$\bar{\Theta}(\theta_H) = \bar{\Theta}_E(\theta_H), \quad (13a)$$

while conservation of potential temperature requires

$$\int_0^{\theta_H} \bar{\Theta}\cos\theta d\theta = \int_0^{\theta_H} \bar{\Theta}_E\cos\theta d\theta. \quad (13b)$$

After substituting from (2) and (12) for $\bar{\Theta}_E$ and $\bar{\Theta}$, Eqs. (13a)–(13b) may be regarded as two equations in the two unknowns $\bar{\Theta}(0)$ and θ_H. An "equal-area" geometric construction equivalent to solving these simultaneous equations is illustrated in Fig. 1.

If the small-angle approximation is permissible, then

$$\frac{\bar{\Theta}}{\Theta_0} = \frac{\bar{\Theta}(0)}{\Theta_0} - \frac{1}{2}\frac{\Omega^2 a^2}{gH}\theta^4, \quad (14a)$$

$$\frac{\bar{\Theta}_E}{\Theta_0} = \frac{\bar{\Theta}_E(0)}{\Theta_0} - \Delta_H\theta^2. \quad (14b)$$

Substituting into (13a) and (13b), one finds that

$$\frac{\bar{\Theta}(0)}{\Theta_0} = \frac{\bar{\Theta}_E(0)}{\Theta_0} - \frac{5}{18}R\Delta_H, \quad (15)$$

$$\theta_H = (5/3 R)^{1/2}, \quad (16)$$

R being defined by (6). If $R \ll 1$ then θ_H is also much less than unity and the small-angle approximation is, in fact, appropriate. The width of the Hadley cell in this limit is directly proportional to the square root of the horizontal temperature gradient, directly proportional to the square root of the height of the cell, inversely proportional to the rotation rate and independent of static stability. [These parameter dependencies are different from those obtained by Schneider (1977), who analyzes a model with fixed surface temperature and requires the atmosphere to adjust its static stability in order to achieve balance between radiative cooling and a given heating distribution. The result is a Hadley cell width dependent on static stability.]

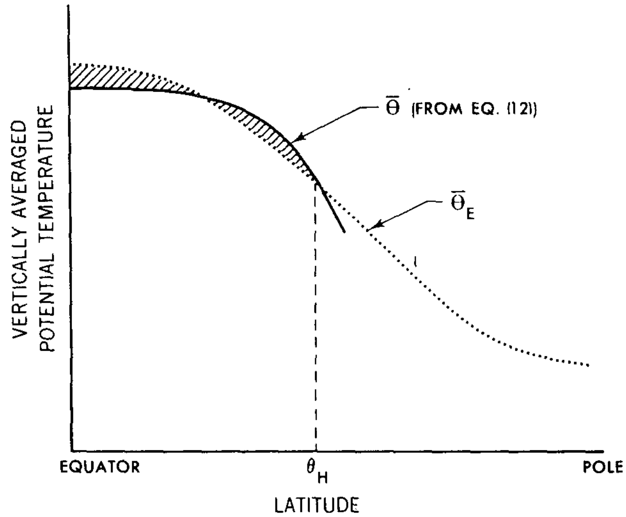

FIG. 1. The equal-area geometric construction equivalent to the argument of Section 4a. The two shaded areas are equal.

In the general case, Eqs. (12) and (13a,b) reduce to

$$\tfrac{1}{3}(4R - 1)y_H^3 - \frac{y_H^5}{1 - y_H^2} - y_H + \tfrac{1}{2}\ln\left(\frac{1 + y_H}{1 - y_H}\right) = 0 \quad (17)$$

for $y_H \equiv \sin\theta_H$. The solution to this equation as a function of R is compared with the small-angle approximation $(5/3 R)^{1/2}$ in Fig. 2.

Since $\bar{\Theta}_E - \bar{\Theta}$ is determined by (14) and (15) within our idealized Hadley cell, one can compute the vertically integrated flux of potential temperature from the thermodynamic equation

$$\frac{1}{H}\int_0^H \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}(v\Theta\cos\theta)dz = (\bar{\Theta}_E - \bar{\Theta})\tau^{-1}.$$

In the limit $R \ll 1$, one finds

$$\frac{1}{\Theta_0}\int_0^H v\Theta dz = \frac{5}{18}\left(\frac{5}{3}\right)^{1/2}\frac{Ha\Delta_H}{\tau}R^{3/2} \times\left[\frac{\theta}{\theta_H} - 2\left(\frac{\theta}{\theta_H}\right)^3 + \left(\frac{\theta}{\theta_H}\right)^5\right]. \quad (18)$$

For the sake of obtaining a qualitative estimate of momentum fluxes, suppose that the profiles of u and Θ are self-similar in the sense that

$$\frac{u(z) - u(0)}{u(H) - u(0)} \approx \frac{\Theta(z) - \Theta(0)}{\Theta(H) - \Theta(0)}.$$

Suppose further that the static stability is not affected appreciably by either the meridional circulation or by the vertical diffusion of heat, so that $[\Theta(H) - \Theta(0)]/\Theta_0 \approx \Delta_v$ and

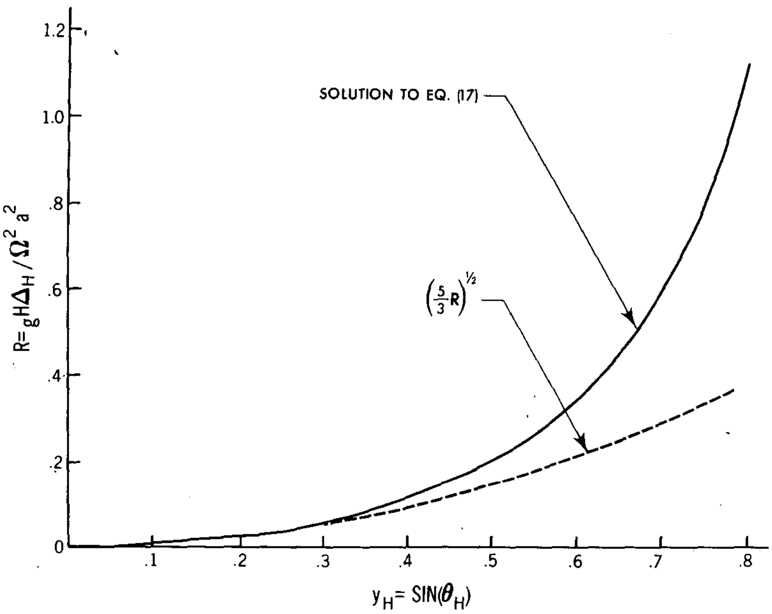

FIG. 2. The poleward boundary of the Hadley cell as given by (17) and the approximate solution $\Theta_H^2 = \frac{5}{3}R$ for $R \ll 1$.

$$\int_0^H vudz \approx \frac{u(H) - u(0)}{\Theta(H) - \Theta(0)} \int_0^H v\Theta dz$$

$$\approx \frac{\Omega a \theta^2}{\Delta_v} \frac{1}{\Theta_0} \int_0^H v\Theta dz.$$

[One obtains the same result without the self-similarity assumption if, instead, one assumes that the meridional circulation is confined to boundary layers at $z = 0$ and $z = H$, in addition to assuming that $u(H) \approx u_M$, $u(0) \approx 0$, and $\Theta(H) - \Theta(0) \approx \Theta_0\Delta_v$.]

Using (4) one obtains the following expression for the surface stress:

$$Cu(0) \approx -\frac{25}{18}\frac{\Omega a H \Delta_H}{\tau \Delta_v} R^2 \left[\left(\frac{\theta}{\theta_H}\right)^2 - \frac{10}{3}\left(\frac{\theta}{\theta_H}\right)^4 + \frac{7}{3}\left(\frac{\theta}{\theta_H}\right)^6\right]. \quad (19)$$

Surface easterlies are predicted for $\theta < (3/7)^{1/2}\theta_H$ and surface westerlies for $(3/7)^{1/2}\theta_H < \theta < \theta_H$. The magnitude of the stress is proportional to the cube of the imposed horizontal temperature gradient. Plots of $u(0)$, $u(H)$, and the heat and momentum fluxes are displayed in Fig. 3. The analogous expressions when R is not small are easily obtained.

Ignoring the advective terms near the ground and integrating through the surface stress boundary layer, one finds

$$fV_G \approx Cu(0) - \nu \frac{\partial u}{\partial z}\bigg|_\delta, \quad (20)$$

where $V_G \equiv \int_0^\delta vdz$, δ being the depth of this boundary layer. If the interior is sufficiently inviscid, therefore, one expects poleward flow near the ground in the region of surface westerlies. The associated indirect or Ferrel cell cannot be expected to penetrate to the top of the atmosphere; by interrupting the poleward flow near $z = H$ it would destroy the momentum transport responsible for its own existence.

Eqs. (19) and (20) predict that the mass flux in the surface boundary layer is of the order

$$\frac{aH}{\tau}\frac{\Delta_H}{\Delta_v}R^{3/2}.$$

One obtains the same scaling for the mass flux in the circulation as a whole if one divides the vertically averaged heat flux [Eq. (18)] by Δ_v. If the regions of upward and downward motion are of comparable size, the magnitude of the vertical velocities must be $\sim H\Delta_H R/\tau\Delta_v$. (The same estimate for vertical velocities results from setting

$$0 \approx -w\frac{\partial\bar{\Theta}}{\partial z} + (\Theta - \Theta_E)\tau^{-1}$$

and utilizing (15).] The time required for the flow to traverse the distance H therefore, is $\sim\tau_D \equiv \tau/(R\Delta_H/\Delta_v)$. We refer to τ_D as the dynamic or overturning time scale; it plays a central role in the discussion of self-consistency which follows.

b. Discussion of approximations

Inspection of the set of equations (1) and the boundary conditions (1a) reveals that the solutions are determined by the values of five nondimensional parameters. The following choice of independent parameters proves to be convenient:

$$R, \quad \tau_D/\tau_c, \quad \tau_D/\tau_\nu, \quad \Omega\tau_D \quad \text{and} \quad \tau_D/\tau.$$

$\tau_\nu \equiv H^2/\nu$ is a diffusive relaxation time and $\tau_c \equiv H/C$ a time scale determined by the surface drag coefficient. We consider only the special case $R \ll 1$. In this limit, the following conditions seem to be required for self-consistency:

$$\left.\begin{array}{ll} \text{(i)} & \tau_D/\tau_c \gg 1 \\ \text{(ii)} & \tau_D/\tau_\nu \ll 1 \\ \text{(iii)} & R^{1/2}\Omega\tau_D \gg 1 \\ \text{(iv)} & \tau_D/\tau \gg 1 \end{array}\right\} . \tag{21}$$

Each of these inequalities is discussed in turn below.

(i) The inequality $\tau_D/\tau_c \gg 1$ is required in order that $u(0)$ be much less than $u(H) \approx \Omega a\theta^2$, as assumed in the derivation of (12). If the assumptions leading to the estimate (19) for the surface winds are adequate, then $u(0) \ll u(H)$ for all $\theta < \theta_H$ if

$$\frac{\Omega a H \Delta_H}{C\tau\Delta_v} R \ll \Omega a,$$

i.e., if

$$\tau_c \equiv \frac{H}{C} \ll \tau/(R\Delta_H/\Delta_v) \equiv \tau_D.$$

According to (19) the maximum strength of the surface wind is ≈ 0.2 in units of $\Omega a H\Delta_H R^2/(C\tau\Delta_v)$, while $u(H)$ rises to $5/3$ in units of ΩaR. $u(0)$ will therefore be considerably less than $u(H)$ throughout most of the cell even when $\tau_D/\tau_c \approx 1$.

(ii) The inequality $\tau_D/\tau_\nu \ll 1$ is identical to the condition $\nu \ll \nu_m$ under which the linear equation (10) produces a solution inconsistent with the angular momentum constraint, $u \leq u_M$. The same inequality is evidently required for the assumption $u \approx u_M$ to be self-consistent, as can be seen with the following qualitative argument.

Assuming that the poleward flow is concentrated near $z = H$ and integrating the momentum equation over this layer of concentrated flow, one finds

$$\frac{V}{a}\frac{\partial M}{\partial \theta} \approx \nu \frac{\partial M}{\partial z}\bigg|_{z=H-\delta}^{z=H},$$

where V is the mass flux in this layer and δ the depth of the layer. Using the stress-free upper boundary condition and estimating

$$\frac{\partial M}{\partial z}\bigg|_{z=H-\delta} \approx u(\theta,H)a/H,$$

the result for the angular momentum gradient is

$$-\frac{1}{a}\frac{\partial M}{\partial \theta} \approx \nu u(\theta,H)a/V(\theta)H.$$

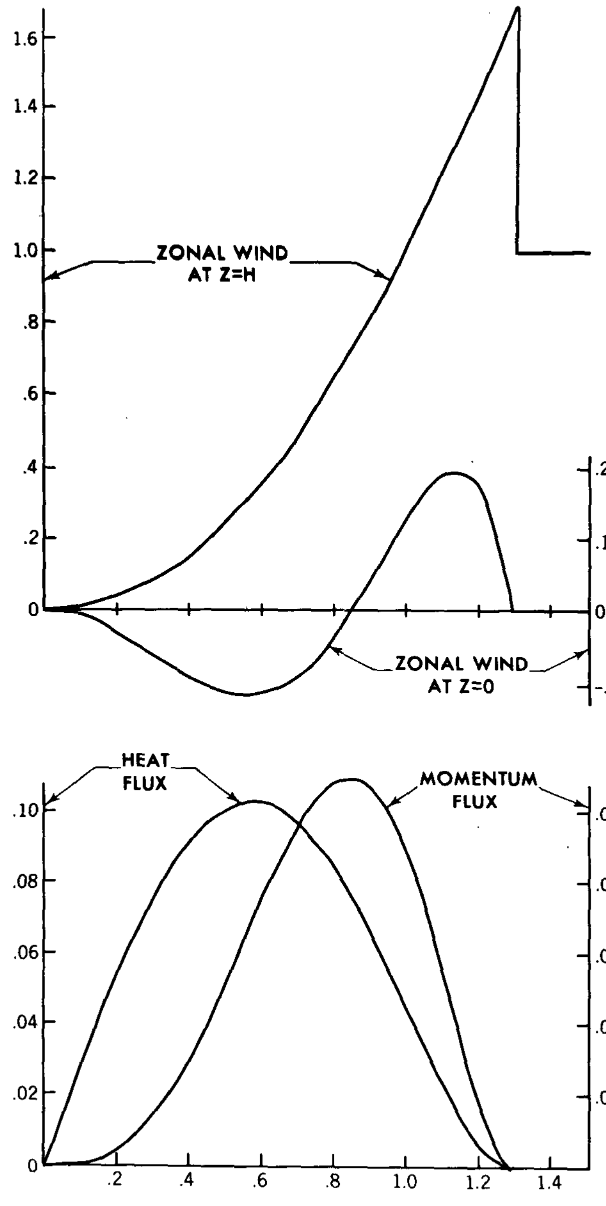

FIG. 3. Zonal wind, heat and momentum fluxes according to the theory for $R \ll 1$, with $u(z = H)$ normalized by ΩaR, $u(z = 0)$ normalized by $\Omega aH\Delta_H R^2/(C\tau\Delta_v)$, the heat flux $\Theta_0^{-1}\int_0^H v\Theta dz$ normalized by $Ha\Delta_H R^{3/2}/\tau$, and the momentum flux $\int_0^H vudz$ normalized by $\Omega a^2 H\Delta_H R^{5/2}/(\tau\Delta_v)$.

The scaling for V is discussed in Section 4a. For the sake of definiteness, we let V have the same meridional structure as the estimated heat flux:

$$V(\theta) \approx \frac{aH}{\tau}\frac{\Delta_H}{\Delta_v}R^{3/2}\left[\frac{\theta}{\theta_H} - 2\left(\frac{\theta}{\theta_H}\right)^3 + \left(\frac{\theta}{\theta_H}\right)^5\right].$$

Setting $u(\theta,H) \approx u_M$, one can check for consistency by computing the fractional error in zonal wind at latitude θ:

$$\frac{\Omega a\theta^2 - u}{\Omega a\theta^2} = -\frac{\int_0^\theta \frac{\partial M}{\partial \xi} d\xi}{\Omega a^2\theta^2} \approx \frac{\tau_D}{\tau_\nu}\frac{1}{1-(\theta/\theta_H)^2}. \quad (22)$$

Since $V \to 0$ at $\theta = \theta_H$ in the inviscid limit, one expects the viscosity, no matter how small, to play some role in the vicinity of θ_H. However, the loss of angular momentum due to viscous stresses should be negligible over the bulk of the cell if $\tau_D/\tau_\nu \ll 1$.

If one sets $\partial u/\partial z|_\delta \approx u_M/H$ in Eq. (20), one finds that the internal stresses are negligible compared with the surface drag if, once again, $\tau_D/\tau_\nu \ll 1$. Thus, one doesn't expect the Ferrel cell to appear above the surface westerlies until the poleward flow in the Hadley cell is nearly momentum conserving.

(iii) The third assumption used in deriving (12) is that the terms

$$\left[\nabla\cdot(\mathbf{v}v) - \nu\frac{\partial^2 v}{\partial z^2}\right]\Bigg|_0^H$$

are negligible compared with $fu(H)$. This will not be true in the immediate vicinity of the equator, but it is sufficient for our purposes if it holds over the bulk of the Hadley cell. Estimating the magnitude of these terms presents a problem, since the arguments of Section 4a provide estimates of the mass flux V but not of the actual strength of the meridional flow.

One can try to sidestep this problem by examining the linear model, Eq. (10). This equation predicts that for sufficiently small ν, the thickness of the boundary layers in the tropics is

$$\delta \approx (\nu/f_*)^{1/2} \approx (\nu/2\Omega\theta_*)^{1/2},$$

f_* being the Coriolis parameter at the latitude θ_*, and

$$\theta_* \equiv \left(\frac{\nu}{\nu_m}R^2\right)^{1/4}.$$

The maximum mass flux within this boundary layer is proportional to

$$V_{\max} \approx \frac{aH}{\tau}\frac{\Delta_H}{\Delta_v}\left(R^2\frac{\nu}{\nu_m}\right)^{3/4}.$$

The transition to the nonlinear regime occurs when $\nu \approx \nu_m$, and at this transition

$$\delta^2 \approx \nu_m/\Omega R^{1/2} = H^2/(\Omega\tau_D R^{1/2}) \quad (23)$$

and

$$|v_{\max}|^2 \approx |\delta^{-1}V_{\max}|^2 \approx \frac{\Omega a^2}{\tau_D}R^{3/2}.$$

Close to the equator, the magnitude of the meridional flow should be roughly

$$v \approx |v_{\max}|(\theta/\theta_*)$$

so that

$$v^2 \approx \frac{\Omega a^2}{\tau_D}R^{1/2}\theta^2 \quad \text{if} \quad \nu \approx \nu_m.$$

Near the equator the ratios

$$fu(H) : \nabla\cdot(\mathbf{v}v) : \nu\frac{\partial^2 v}{\partial z^2}$$

therefore, are of the order

$$\Omega^2 a\theta^2 : \frac{\Omega a}{\tau_D}R^{1/2} : \left(\frac{\Omega a^2}{\tau_D}R^{1/2}\right)^{1/2}\Omega R^{1/2}$$

or

$$\left(\frac{\theta}{\theta_H}\right)^2 : (\Omega\tau_D R^{1/2})^{-1} : (\Omega\tau_D R^{1/2})^{-1/2}.$$

If

$$\Omega\tau_D \mathrm{R}^{1/2} \gg 1 \quad (24)$$

and $\nu \approx \nu_m$, the terms

$$\nabla\cdot(\mathbf{v}v) + \nu\frac{\partial^2 v}{\partial z^2}$$

will be negligible compared with $fu(H)$ at most latitudes. If (24) is satisfied by a very wide margin, one can argue that there should be a wide range of ν smaller than ν_m for which the assumptions that the flow is balanced and that $u(H) \approx u_M$ throughout the bulk of the cell are simultaneously appropriate. We can say no more than this without more information about the circulation in the nonlinear regime.

Using (23), one sees that the condition $\Omega\tau_D R^{1/2} \gg 1$ insures that the meridional flow will have a boundary-layer character at the nonlinear transition.

(iv) A key assumption utilized in the preceding discussion is that neither the circulation nor diffusion change the mean static stability appreciably. Although this assumption is not directly required in deriving the inviscid limit of the heat flux and the Hadley cell width, it is required for our estimates of the surface stress and the mass flux, and, therefore, is central to the self-consistency analysis of this section.

The inequality $\tau_D \gg \tau$ must be satisfied if the effects of the circulation of the static stability are to be negligible. In the region of rising motion near the equator, in particular, if τ is much larger than τ_D the flow will force the static stability close to neutral, since Θ will be nearly conserved as the rising parcel traverses the distance H. If $\tau_D \gg \tau$, on the other hand, the radiative equilibrium static stability has time to assert itself and one expects changes in stability due to vertical mixing to be negligible.

The time required for a fluid particle to travel from $\theta \approx 0$ to $\theta \approx \theta_H$ within the boundary layers at $z = 0$ and $z = H$ is generally much smaller than τ_D. However, in the absence of substantial vertical mixing, meridional mixing in these layers can only change the potential temperature by an amount comparable to the temperature drop from the equator to θ_H, approximately $R\Delta_H$ in the inviscid limit. The resulting changes in stability will be negligible provided once again that $R\Delta_H/\Delta_v \equiv \tau/\tau_D \ll 1$.

$\tau \approx \tau_D$ thus seems to mark the transition from flows in which the effects of the circulation on the temperature field in the inviscid limit are minimal to flows in which the temperature field is distorted in possibly complex ways.

When $R \ll 1$, τ_D/τ is the Richardson's number for the exact inviscid solution described in Section 2 since $N^2 = g\Delta_v/H$ and $\partial u/\partial z = \Omega aR/H$ imply $N^2/(\partial u/\partial z)^2 = \Delta_v/R\Delta_H$. We have, therefore, not only restricted consideration to the case of gravitationally stable radiative equilibria, but also to the case of radiative equilibria for which the balanced zonal wind produces a large Richardson's number.

From the preceding arguments, we conclude that no thermal boundary layers with temperature drops of the order of Δ_v will be produced by the circulation if $\tau_D \gg \tau$. The vertical diffusion will, therefore, have an insignificant effect on the static stability if $H^2/\nu \gg \tau$, a condition already guaranteed by the inequalities (ii) and (iv).

Even if the inequalities (i)–(iv) are satisfied, there is still one remaining problem with the analysis of Section 4a. If $u \approx 0$ throughout the surface boundary layer, then the flow rising out of this layer at latitude θ will have the angular momentum $\Omega a^2(1 - \theta^2)$. The average angular momentum of the rising branch of the Hadley cell will therefore be $\Omega a^2(1 - \theta_R^2)$, where θ_R can be thought of as the effective half-width of this rising ranch. If the rising fluid is then channelled into a thin boundary layer near $z = H$, and if this boundary layer thins progressively as $\nu \to 0$ so that the momentum on the bounding streamline at $z = H$ is always well mixed with the rest of the momentum in this layer, then one expects the zonal wind at $z = H$ to be bounded by $\Omega a(\theta^2 - \theta_R^2)$ poleward of the rising motion. Predictions resulting from the assumption that $u \approx \Omega a\theta^2$ should therefore be considered as limiting cases which are likely to be closely approached only if the rising branch of the Hadley cell is sufficiently well localized at the equator. If u does not achieve the momentum conserving limit, then vertical shears and horizontal temperature gradients will be smaller, the Hadley cell will be wider and the heat flux larger than predicted in Section 4a, as one can easily demonstrate by modifying Fig. 1 appropriately.

c. Symmetric instabilities

Solutions to Eq. (1) will develop regions of negative potential vorticity as $\nu \to 0$. Air leaving the equator near $z = H$ and conserving its angular momentum eventually enters a region of radiative cooling. In such a region Θ decreases moving poleward along a line of constant M. Since one expects $\partial M/\partial z$ to be positive, it follows that the potential vorticity

$$q \equiv \frac{1}{a}\left(\frac{\partial M}{\partial z}\frac{\partial \Theta}{\partial \theta} - \frac{\partial M}{\partial \theta}\frac{\partial \Theta}{\partial z}\right) = \frac{1}{a}\frac{\partial M}{\partial z}\frac{\partial \Theta}{\partial \theta}\bigg|_{M=\text{constant}}$$

is negative in such a region, the condition for symmetric instability in an inviscid, adiabatic balanced flow.

As long as one restricts oneself to the problem of finding steady solutions to the set (1), the fact that the solution is unstable to symmetric disturbances is of no more concern than its undoubted instability to asymmetric disturbances. The problem of symmetric instabilities must be faced, however, if one tries to find these steady solutions numerically by marching forward in time. Also, if one addresses the question of determining the circulation when the large-scale asymmetric barotropic and baroclinic instabilities are artificially suppressed, one should face the problem of estimating the mixing due to zonally symmetric transience. Of particular concern is the question of whether or not the time-averaged circulation produced by the time-dependent version of (1) resembles the steady circulation described in Section 4a.

Rather than determining the zonal wind profile at the upper boundary of the Hadley cell by setting $\partial M/\partial \theta = 0$, one can, as an exercise, determine this profile by setting $q = 0$ and estimating

$$-\frac{g}{a\Theta_0}\frac{\partial \Theta}{\partial \theta} \approx \frac{f}{a}\frac{\partial M}{\partial z} \approx -\frac{fu(H)}{H}$$

just beneath the upper boundary layer. We again restrict the discussion to the case $R \ll 1$. In this limit

$$\frac{\partial M}{\partial \theta} \approx \frac{\partial M}{\partial z}\frac{\partial \Theta}{\partial \theta}\bigg/\frac{\partial \Theta}{\partial z}$$

implies that

$$f - \frac{1}{a}\frac{\partial u}{\partial \theta} \approx \frac{fu^2}{gH\Delta_v}.$$

Substituting $u \approx \Omega a\theta^2$ on the right-hand side, one finds

$$\frac{\partial u}{\partial \theta} \approx 2\Omega a\theta\left(1 - \frac{\Omega^2 a^2}{gH\Delta_v}\theta^4\right)$$

or

$$u \approx \Omega a\theta^2\left(1 - \frac{\Delta_H}{3R\Delta_v}\theta^4\right).$$

The fractional error one makes in setting $u \approx \Omega a\theta^2$, therefore, is $\approx R\Delta_H/\Delta_v$ at $\theta = \theta_H \approx R^{1/2}$. Whether or not mixing induced by symmetric instabilities results in a state with q everywhere non-negative, we suspect that when the circulation can be stabilized by making small changes in the zonal flow and temperature fields, then mixing of angular momentum and potential temperature due to these instabilities will also be small. We suspect, therefore, that symmetric instabilities will have little effect on the gross features of the zonal wind and temperature fields when $R\Delta_H/\Delta_v = \tau/\tau_D \ll 1$, an inequality we have already required for self-con-

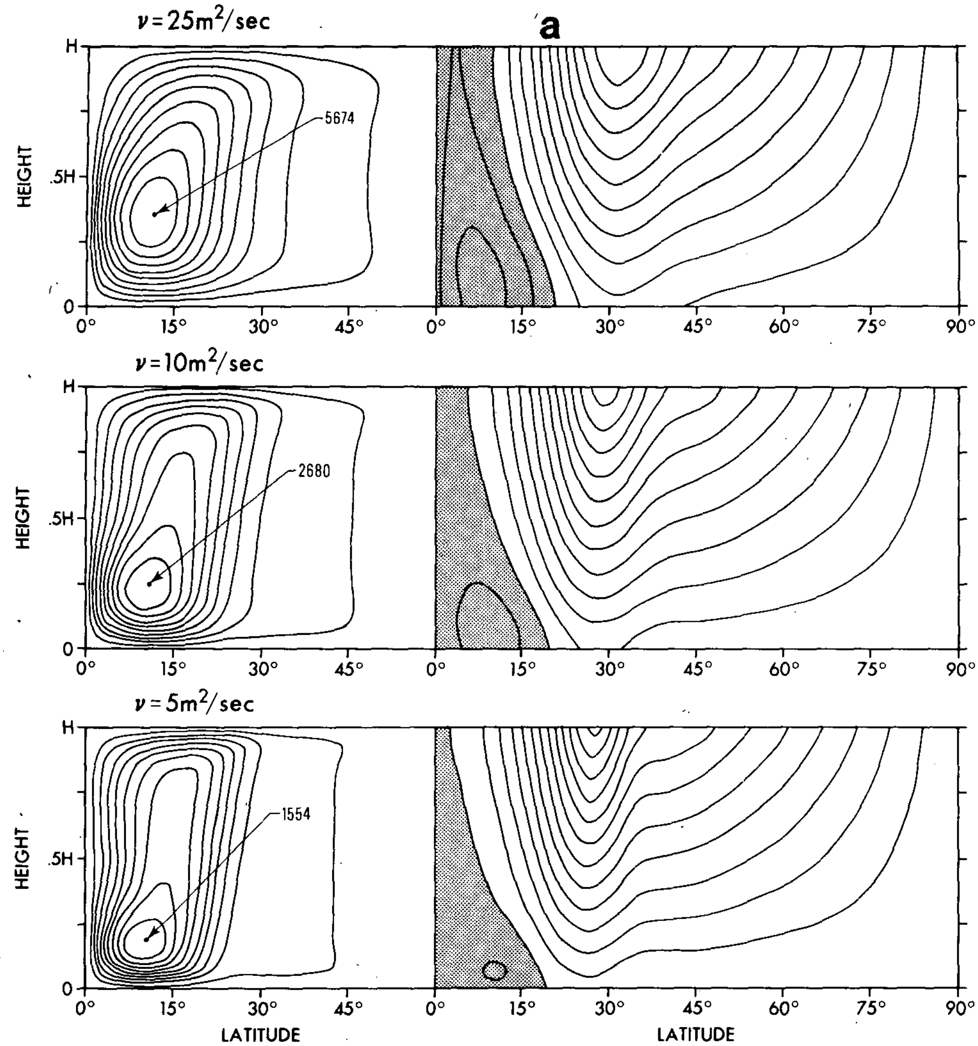

FIG. 4a. Calculated meridional streamfunctions and zonal wind fields in the standard case. In the left part of the figure, the streamfunction ψ is given for ν = 25, 10 and 5 $m^2 s^{-1}$, with a contour interval of 0.1 ψ_{max}. The value of ψ_{max} ($m^2 s^{-1}$) is marked by a pointer. The right part of each figure is the corresponding zonal wind field, with contour intervals of 5 m s^{-1}. The shaded area indicates the region of easterlies.

sistency in Section 4b. Furthermore, according to the criterion $q < 0$, the exact inviscid radiative equilibrium solution is itself symmetrically unstable if its Richardson's number $\Delta_v/R\Delta_H = \tau_D/\tau$ is less than unity. By assuming $\tau_D/\tau \gg 1$, we have also required this inviscid radiative equilibrium solution to be stable everywhere, thus preventing symmetric instability from developing outside of the Hadley circulation.

We have, admittedly, oversimplified this discussion of symmetric instabilities by using the inviscid criterion $q < 0$. As discussed by McIntyre (1970a), diffusion and/or radiative damping can destabilize a zonal flow with positive q. We have chosen the Prandtl number equal to unity to minimize this effect in the time-dependent calculations described below, but Newtonian radiative damping can still destabilize the flow, at least according to the simplest instability calculation analogous to McIntyre's, ignoring boundaries, inhomogeneities in the angular momentum and potential temperature gradients, and the meridional circulation. It is conceivable that some of the structure in the circulations described below can be thought of as resulting from equilibrated finite amplitude instabilities (see McIntyre, 1970b; Williams 1970) but this is an aspect of the problem we have not pursued.

5. Numerical solutions

We obtain approximate solutions to the set (1) by finite-differencing the time-dependent version of

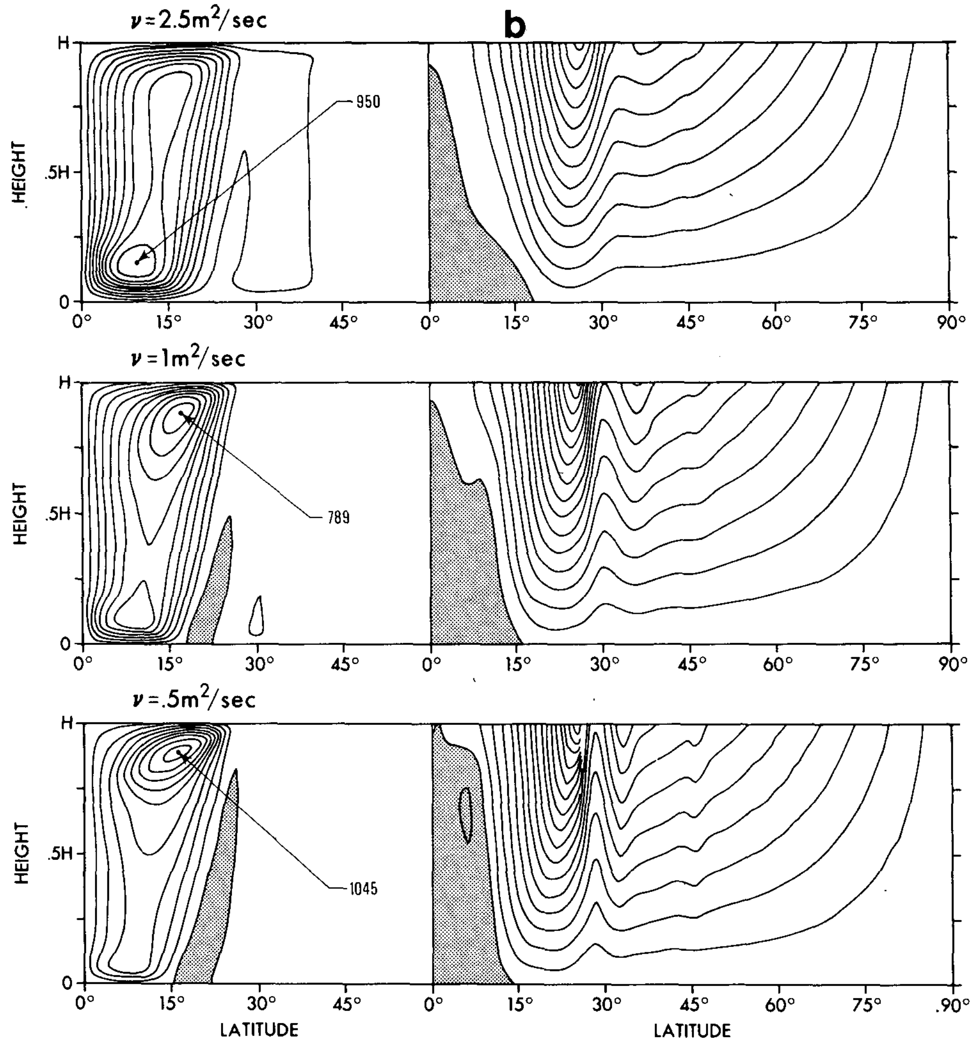

FIG. 4b. Calculated meridional streamfunctions and zonal wind fields as described in Fig. 4a. The shaded region in the ψ field corresponds to a Ferrel cell, $\psi < 0$.

these equations and integrating forward in time until a steady state is achieved. The numerical model utilized is standard in all respects. The second-order spatial finite-differencing on a staggered grid is a straightforward extension to multiple levels of the scheme utilized by Held and Suarez (1978) in a two-level model. The time integration is performed using Matsuno's (1966) explicit "simulated backward difference" method. The results described below are obtained using 50 grid points in the vertical and 90 points from equator to pole.

Considered as one of a number of possible iteration schemes for solving the boundary value problem, time marching has certain disadvantages. In particular, when our time-dependent model fails to achieve a steady state it is difficult to determine whether the failure is due to an instability of the differential equations or to numerical instability, since we invariably find that the transients which develop in such cases are not well resolved by our grid. When time marching does yield a steady solution, however, we immediately learn that this solution is stable to those perturbations resolved by the numerics. It is still possible that the steady solution is unstable to axisymmetric modes unresolved by the spatial finite-differencing, or to weakly unstable axisymmetric modes that do not grow due to the dissipative character of the Matsuno step.

We begin by describing the steady states obtained for a series of values of ν = 25, 10, 5, 2.5, 1 and 0.5 m^2/s^{-1}, with the other model parameters fixed at the following values:

$$\left.\begin{array}{ll} \Omega = 2\pi/(8.64 \times 10^4\ \mathrm{s}) & \Delta_H = 1/3 \\ a = 6.4 \times 10^6\ \mathrm{m} & \Delta_v = 1/8 \\ g = 9.8\ \mathrm{m^2\ s^{-1}} & C = 0.005\ \mathrm{m\ s^{-1}} \\ H = 8.0 \times 10^3\ \mathrm{m} & \tau = 20\ \mathrm{days} \end{array}\right\} . \quad (25)$$

At a lower value of ν (0.25 m^2 s^{-1}) a steady state is

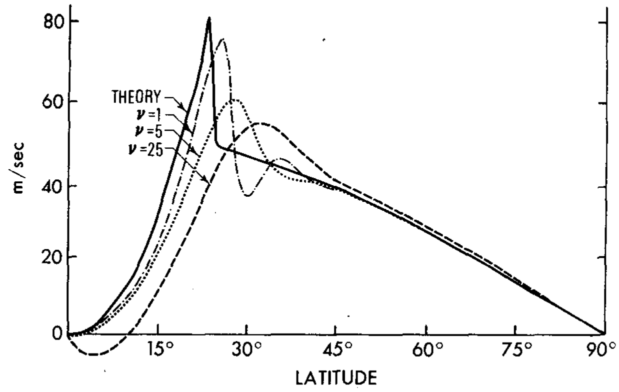

FIG. 5. Zonal winds at $z = H$ in the standard case for three values of ν, compared with the theoretical prediction for $\nu \to 0$.

not obtained. The corresponding nondimensional parameters are

$$R = 0.121, \qquad \tau_D/\tau = 3.12$$

$$\tau_D/\tau_c = 3.37, \ \ \Omega\tau_D R^{1/2} = 136$$

and $\tau_D/\tau_\nu = \nu/\nu_m = 2.1$, 0.84, 0.42, 0.21, 0.084 and 0.042. These parameters have been chosen to be reasonably earthlike but with the inequalities (21) in mind. The values given in (25) shall be referred to as the "standard case." The steady meridional circulations and zonal wind distributions are displayed in Figs. 4a and 4b. In order to emphasize changes in the structure of the tropical circulation, the contouring interval for the streamfunction is chosen to be one-tenth of the maximum streamfunction value in each case. There is always a weak direct circulation in the extratropics, with the meridional fluxes confined to Ekman layers, but it is too weak to be visible in the contoured field for small ν.

In the more viscous cases the Hadley cell has the general appearance expected from linear viscous models; the direct cell gradually decays poleward, with meridional mass transports occurring in broad boundary layers. As the viscosity is reduced the cell begins to develop a well-defined meridional extent as well as significant tilt in the subtropics. This tilt presages the appearance of the Ferrel cell above the region of the surface westerlies. The Ferrel cell appears when $\nu = 1$ m^2 s^{-1}, at which point the drag associated with the westerlies finally overcomes the internal stresses. At these low values of ν the circulation has the same qualitative appearance as that computed by Schneider (1977) and depicted schematically by Eliassen (1952). The steady-state flow continues to evolve as ν is decreased further, however. In the least viscous steady state obtained, $\nu = 0.5$ m^2 s^{-1}, most of the flow recirculates in the upper third of the domain.

The transition from a smooth zonal wind structure to a very sharp shear zone on the poleward boundary of the Hadley cell is also evident in Fig. 4. For $\nu = 0.5$ m^2 s^{-1} the zonal wind drops in strength by more than 60 m s^{-1} within 200 km (two grid points) at 25°. Our meridional resolution is evidently insufficient for examining the detailed structure of this subtropical boundary layer. At these small values of ν a secondary maximum in zonal wind appears poleward of this shear zone, due to weak divergence above the rising motion in the Ferrel cell; the transport by the equatorward flow south of this rising motion depresses the zonal winds, while the poleward flow north of this point enhances the winds. Some grid point noise is also evident in the zonal winds at these smaller values of ν.

Fig. 5 compares the zonal winds at $z = H$ in three experiments ($\nu = 25$, 5 and 1 m^2 s^{-1}) with the wind profile predicted as in Section 4a, but without making the approximation $R \ll 1$. In the least viscous case shown (as well as in the case $\nu = 0.5$ m^2 s^{-1} not shown) the maximum wind is located 2° latitude poleward of the predicted position. A major deficiency of the theory is evidently its oversimplified description of the transition at this latitude from the momentum-conserving wind to the radiative-equilibrium wind. The oscillations in u poleward of the subtropical boundary of the Hadley cell do not disappear as $\nu \to 0$, but rather become more pronounced as the Ferrel cell strengthens.

The latitudinal momentum profiles in the poleward flow within the Hadley cell are displayed in more detail in Fig. 6, a plot of $\Omega a \sin^2\theta - u \cos\theta$ at $z = H$ for each case. The zonal wind approaches its momentum-conserving limit more or less uniformly in this region, until $\nu \approx 1$ m^2 s^{-1}. For $\nu = 1.0$ and 0.5 m^2 s^{-1} the flow is very nearly momentum conserving poleward of 15°, but there is still significant mixing equatorward of 10° between lower momentum air and the air on the bounding streamline at $z = H$. The substantial width of the rising branch of the Hadley cell evident in Fig. 4 seems to be at least

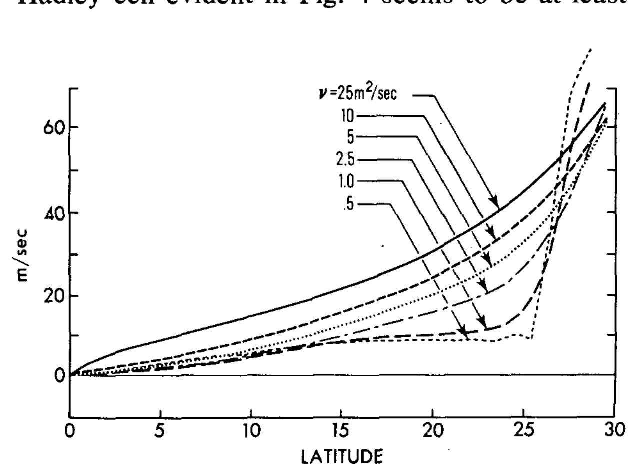

FIG. 6. $\Omega a \sin^2(\theta) - u \cos(\theta)$ at $z = H$ for various values of ν in the standard case.

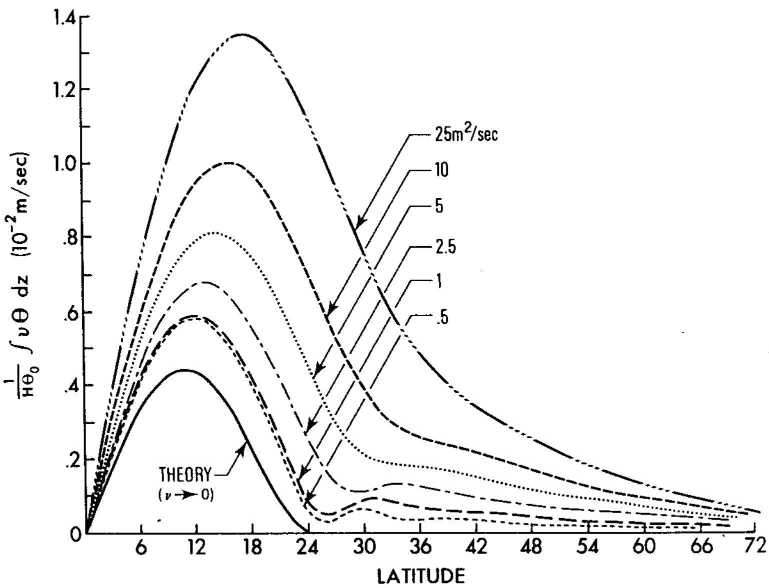

FIG. 7. Calculated meridional heat fluxes in the standard case for various values of ν and the theoretical prediction for $\nu \rightarrow 0$.

partly responsible for the failure of the winds to approach even closer to their momentum conserving limit.

Fig. 7 depicts the vertically averaged horizontal flux of potential temperature in each of these circulations, as well as the predicted inviscid limit. The approach of this flux to its predicted limit is broadly similar to that of the zonal winds at $z = H$, the approach being more or less uniform until $\nu = 1$ m^2 s^{-1}. In fact, one can show by direct calculation that of the three assumptions required in Section 4a to obtain the theoretical heat flux curve—$u(H) \approx u_M$, $u(0) \approx 0$, and a balanced zonal flow—it is the violation of the first of these which results in almost all of the disagreement in Fig. 7. The effects of the Ferrel cell are again evident near the subtropical boundary of the Hadley cell in the sharp minimum and secondary weak maximum in the heat flux. Further poleward, the flux quickly asymptotes to that produced by simple Ekman boundary layers at $z = 0$ and $z = H$, $\nu u_E \Delta_v / f H^2$.

Surface winds in three of these circulations are plotted in Fig. 8, once again accompanied by the predicted inviscid limit. The prediction in this case depends either on the assumption of similarity in the vertical profiles of M and Θ, or on the assumption that the meridional circulation is confined to thin boundary layers at $z = 0$ and H. Since M is nearly conserved following the flow while Θ is not (because of radiative damping), one does not expect the similarity assumption to the particularly good. There is, in addition, considerable meridional flow in the interior of the fluid at the smaller values of ν, particularly above the region of surface westerlies. The predictions for the strength of the winds and for the location of the boundary between surface easterlies and westerlies still seem to retain some qualitative validity, however.

Contours of M and Θ equatorward of 30° in the least viscous steady state obtained ($\nu = 0.5$ m^2 s^{-1}) are displayed in Fig. 9. Within the shaded region the orientation of the contours is such that the Ertel potential vorticity q is negative. q is near zero in the striped region near $z = H$, but a small amount of grid-point noise in M causes q to change sign from point to point. (The stripes are meant only to be suggestive of these changes in sign.) This region of near zero q corresponds roughly to the region of strong recirculating flow. q is definitely negative in the region centered at ~12° and $z = H/2$.

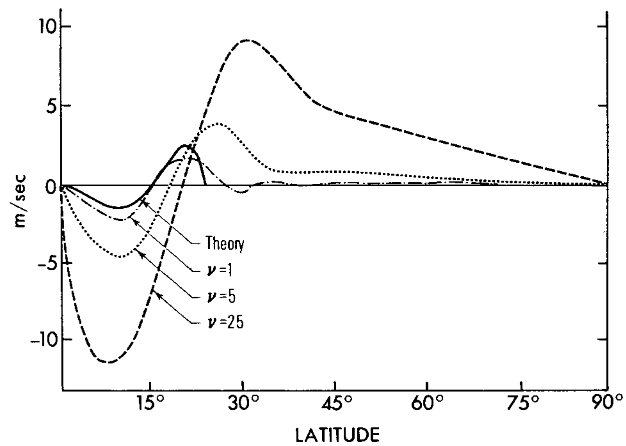

FIG. 8. Calculated surface wind in the standard case for various values of ν and the theoretical prediction for $\nu \rightarrow 0$.

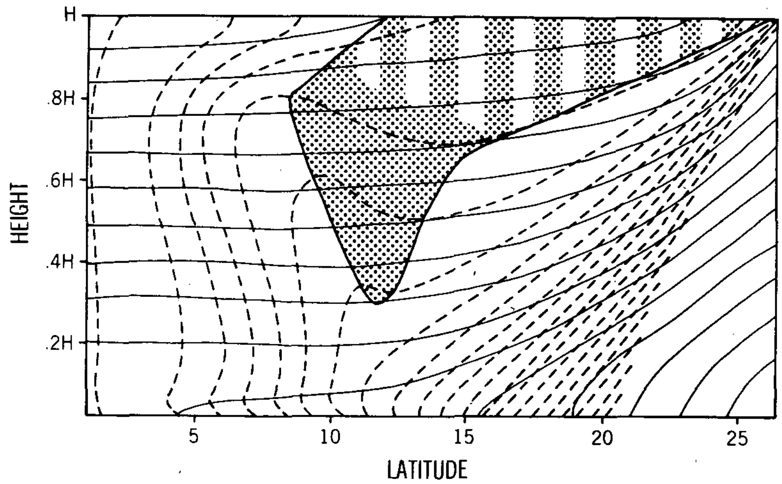

FIG. 9. Contours of constant M (broken line) and constant Θ/Θ_0 (solid line) for $\nu = 0.5$ m² s⁻¹ in the standard case. The potential vorticity is approximately zero in the striped region and negative in the shaded region. The contour interval for M/a is 2.5 m s⁻¹, and that for Θ/Θ_0 is 0.01.

After viewing Fig. 9, it is not surprising that steady circulations for still smaller values of ν could not be found by time-marching. Fig. 10 is a plot of the meridional streamfunction obtained after 1000 days of integration in a calculation with $\nu = 0.25$ m^2 s^{-1}. The flow is still evolving at this time, and has developed sufficient grid-point noise in the zonal winds to force abandonment of the integration. Presumably a steady solution of these equations does exist for this value of ν. We suspect that vertical structure of the sort depicted in Fig. 10 is present in this steady state.

A variety of other calculations have been performed varying the model parameters. Figs. 11 and 12 summarize a number of these calculations. The vertically averaged horizontal fluxes of potential temperature are displayed in Fig. 11. Fig. 12 compares the position of the maximum zonal wind at $z = H$ in each circulation with the position predicted by (17). The cases presented are (a) larger static stability $\Delta_v = 2\Delta_{v0}$; (b) larger rotation rate $\Omega = \sqrt{2}\Omega_0$; (c) larger vertical scale, $H = 2H_0$; with $\Delta_v = 2\Delta_{v0}$ so that $\partial\Theta/\partial z$ in radiative equilibrium remains unaltered; and (d) smaller rotation rate, $\Omega = \Omega_0/\sqrt{2}$. The subscript zero refers to the parameter values listed in (25). For each case we compute a series of steady circulations, decreasing ν until a steady flow cannot be found with this numerical model.

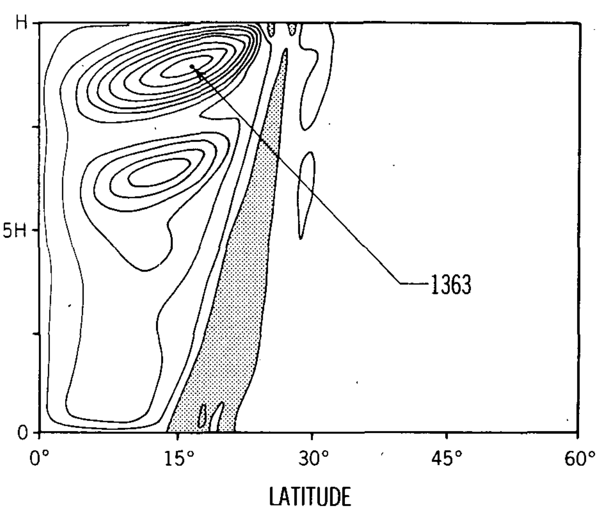

FIG. 10. Meridional streamfunction for $\nu = 0.25$ m² s⁻¹ after 1000 days of time integration in the standard case. The contouring interval is 0.1 times the maximum streamfunction value.

The value of R increases as H increases and as Ω decreases, but is unaffected by the value of Δ_v. The width of the Hadley cell and the latitude of the subtropical jet should, therefore, decrease in case (b), increase in (c) and (d), and remain unchanged in (a). This is precisely the behavior observed in these solutions. Fig. 12 shows that for the least viscous circulation in each series the jet is typically only ~2° poleward of its predicted position.

Compared with the standard case, the value of $\nu_m \equiv (H^2/\tau)(R\Delta_H/\Delta_v)$ is smaller by a factor of 2 in cases (a) and (b), larger by a factor of 4 in (c), and larger by a factor of 2 in (d). One therefore expects smaller values of ν to be required in (a) and (b) than in the standard case in order to approach the inviscid limit, and larger values to be sufficient in (c) and (d). This is the qualitative behavior observed in the figures. One complication is that in case (d) the least viscous steady solution obtained is not quite inviscid enough for the flux to be insensitive to ν. This may be related to the fact that (d) is the only case for which τ_D/τ, the radiative equilibrium Richardson's number, is smaller than in the standard case. The smaller τ_D/τ, the further the nearly inviscid circulation from a state with non-

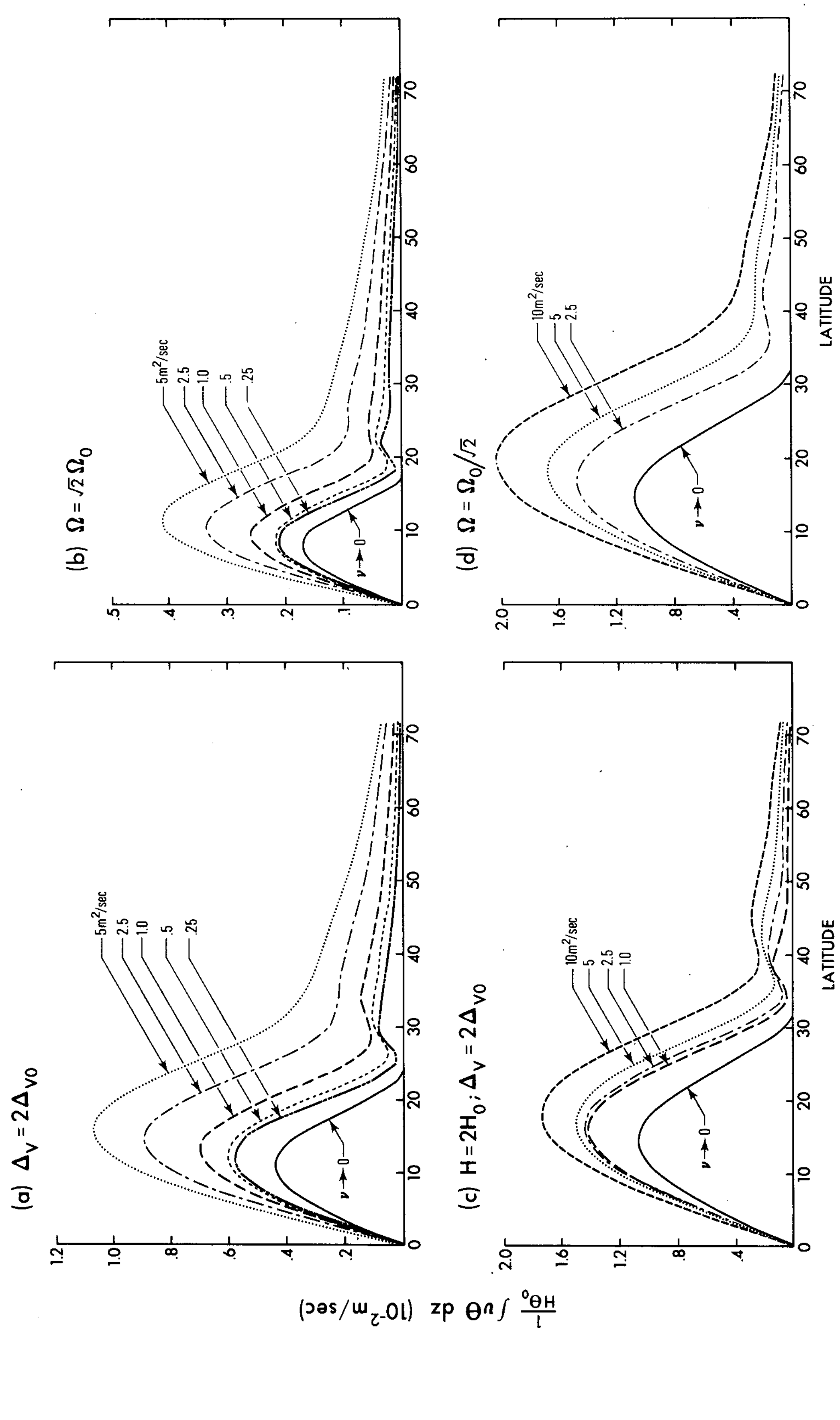

FIG. 11. Meridional heat fluxes for various parametric values: (a) $\Delta_v = 2\Delta_{v0}$; (b) $\Omega = \sqrt{2}\Omega_0$; (c) $H = 2H_0$, $\Delta_v = 2\Delta_{v0}$; and (d) $\Omega = \Omega_0/\sqrt{2}$.

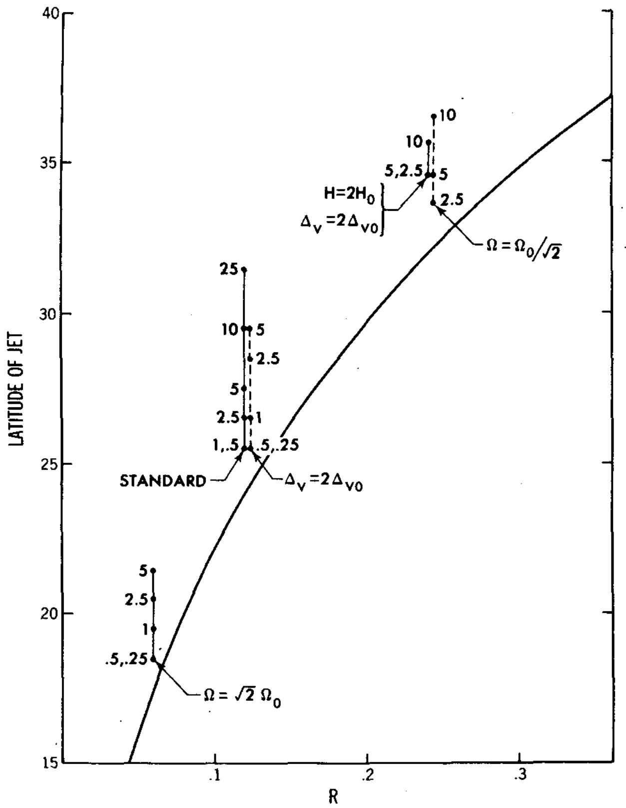

FIG. 12. Position of the maximum zonal wind at $z = H$ at different values of Δ_v, H and Ω, and the predicted position according to (17).

negative potential vorticity, according to Section 4c. ν/ν_m for the least viscous solution is roughly a factor of two larger in this case than for the least viscous solutions in the other cases, and in this sense, at least, instability does set on further from the inviscid limit in (d). In cases (a), (b) and (c), closer approach to the predicted limit seems to be prevented once again by the substantial width of the rising branch of the Hadley cell.

Calculations have also been performed varying the Prandtl number. Attempts at finding a steady circulation with the diffusion of heat, ν^H, set equal to zero proved unsuccessful. A number of steady circulations were obtained with $\nu^H = 0.25\ \nu^M$, however, and the qualitative structure of the circulations remains unchanged from that with unit Prandtl number. For example, in the case with $\nu^M = 1\ \mathrm{m^2\ s^{-1}}$ and with the other parameters as in (25), the maximum heat flux increases $\sim$3% and the jet decreases in strength by 4.5 m s^{-1} without changing its latitude. The static stability increases slightly when the diffusion of heat is reduced. As a result, a weaker circulation is needed to transport the same amount of heat, and therefore, viscous stresses have more time to act on a fluid parcel before it traverses the cell. The heat flux must increase slightly to be consistent with the resulting decrease in the vertical shears. We feel confident that the steady solutions with $\nu^H = 0$ are also qualitatively similar to those described here.

6. Summary and discussion

The Hadley cell in a dry, stably stratified axisymmetric flow in a Boussinesq fluid is strongly constrained by conservation of angular momentum and potential temperature. The assumptions that the zonal wind is balanced, equal to $\Omega a\ \sin^2\theta/\cos\theta$ near the top of the cell, and equal to zero near the ground determine the meridional profile of the vertically averaged potential temperature within the cell. If the flow is forced by linear, radiative damping back to a specified radiative equilibrium temperature, then conservation of potential temperature and continuity of potential temperature at the polar boundary of the cell lead to an expression for the latitude of the polar boundary and for the vertically averaged heat transport by the cell (as illustrated in Fig. 1). For the simple radiative equilibrium temperature profile given by Eq. (2) characterized by the parameters Δ_H, the fractional change in potential temperature from equator to pole, and Δ_v, the fractional change from the top to the bottom of the fluid, one finds that the latitude of the poleward boundary of the cell, θ_H, is $(\frac{5}{3}R)^{1/2}$ when $R \equiv gH\Delta_H/\Omega^2a^2 \ll 1$ and that the vertically averaged heat flux is proportional to $(a\Delta_H/\tau)R^{3/2}$. The additional assumptions that the static stability is unaffected by the circulation and that the vertical profiles of u and Θ are similar result in a surface stress proportional to $\Omega aH\Delta_H R^2/\tau\Delta_v$, with surface easterlies (westerlies) equatorward (poleward) of $(3/7)^{1/2}\theta_H$.

Qualitative arguments limiting the domain in the model's parameter space within which the resulting circulation is self-consistent have been presented in Section 4b. It is not clear that these qualitative arguments are entirely adequate. In particular, it is not clear what form the circulation takes for values of the viscosity smaller than those for which we were able to obtain steady numerical solutions. As seen in Fig. 11, the approach of the heat flux to its predicted inviscid limit ceases at small values of ν. We have interpreted this behavior in terms of the substantial width of the rising branch of the Hadley cell. It may be the case, however, that for still smaller values of ν the heat fluxes being to depart further from the predicted limit. In any case, we conclude from the numerical results that the simple arguments of Section 4a are of qualitative, and in some cases, at least, of quantitative, value in interpreting the nearly inviscid solutions to (1).

But what does one learn from these arguments or from this set of equations about axisymmetric circulations in more realistic model atmospheres?

It is evident, in particular, that latent heat release is crucially important for the structure of the Hadley cell and that the distribution of latent heating is controlled in part by the large-scale flow. No such effect is included in (1).

To address this question, we retain the Boussinesq, Newtonian cooling framework for the moment, and consider a "moist" model in which the circulation is allowed to alter the heating distribution Θ_E in some complex manner, without altering the global mean Θ_E. More precisely, in a "moist" model one can divide the heating, $\Theta_E \equiv \Theta_E{}^* + L$, into the vertically averaged convergence of the latent heat flux L and the remainder, $\Theta_E{}^*$, the global integral of L being zero. Even though L may be dependent on the circulation and may be of such a form that Θ_E has a very sharp maximum at the equator, these arguments still apply if the circulation still has the basic form postulated in Section 4a. The meridional temperature profile within the Hadley cell will once again be given by (12). If the circulation is confined to the region $\theta < \theta_H$, then $\bar{\Theta}_E(\theta_H) = \bar{\Theta}_E^*(\theta_H)$ and

$$\int_0^{\theta_H} \bar{\Theta}_E \cos(\theta) d\theta = \int_0^{\theta_H} \bar{\Theta}_E^* \cos(\theta) d\theta.$$

The form and magnitude of L are irrelevant to the argument; the value of θ_H, determined as before by continuity of Θ at θ_H and conservation of Θ, remains unchanged. The only effect of the circulation dependent heating is on the strength of the circulation itself. If Θ_E increases in the region of low-level convergence, crudely accounting for the effects of latent heating, then the flow will certainly be stronger than its "dry" (fixed Θ_E) analogue, and the region of rising motion will undoubtedly be more localized, but there should be little effect on the width of the Hadley cell or on the strength of the jet if the flow is sufficiently inviscid. The most significant effect might just be a closer approach to the predicted inviscid limit due to the more localized rising motion. If substantial internal viscous stresses are present, however, a more intense circulation will leave the stresses less time to act, causing the zonal winds to increase and the cell to contract.

This is not to say that moist processes would have no other effects on the Hadley cell in a more realistic model. In particular, moist convection can alter the height of the momentum conserving flow by altering the height of the tropical tropopause, thereby altering R and the width of the cell. Also, momentum mixing due to moist convection can lead to a fairly viscous circulation with a structure determined, at least in part, by this "cumulus friction". However, to the extent that the circulation is successful in localizing convective activity in a thin convergence zone, the stresses associated with the convection will also be limited to this zone, leaving a fairly inviscid interior in the rest of the Hadley cell. A nonlinear, nearly inviscid analysis would then be applicable to this moist model atmosphere.

The problem of the earth's Hadley cell is also complicated by the presence of large heat fluxes across the lower boundary into the ocean. To take this effect into account, one need only incorporate this flux F into the function $\bar{\Theta}_E$, i.e.,

$$\bar{\Theta}_E \tau^{-1} \rightarrow \bar{\Theta}_E \tau^{-1} - F.$$

If F happens to be non-negligible only in the region $\theta < \theta_H$, then one can easily show that an increase in the oceanic heat flux σ, [$F \equiv (a \cos\theta)^{-1} \partial(\sigma \cos\theta)/\partial\theta$ in a steady state)] will be compensated for exactly by a decrease in the atmospheric flux and will have no effect on the width of the cell, the strength of the jet, or on the atmospheric mean temperatures—as long as the flow is sufficiently inviscid and as long as the various assumptions utilized in the analysis remain valid. However, if the oceanic flux forces the rising branch of the Hadley cell off the equator, then the total atmospheric plus oceanic flux can change, a point we hope to return to in a forthcoming paper.

More serious problems arise when one tries to relax the Boussinesq and Newtonian cooling approximations. One can still argue that the flow in the inviscid limit is constrained to some extent if the vertical structure of the atmosphere is itself strongly constrained by processes other than the large-scale flow. To see this, it is convenient to examine the vertically integrated atmospheric energy balance. In a steady state

$$0 = S(\theta) - F(\theta) - L(\theta) - H(\theta), \qquad (26)$$

where $S(\theta)$ is the net incoming solar flux and $F(\theta)$ the heat flux into the ocean, both of which are assumed known. $L(\theta)$ is the outgoing longwave flux at the top of the atmosphere and $H(\theta)$ the divergence of the atmospheric latent plus sensible heat transport. If a simple Hadley cell exists with nearly momentum-conserving flow aloft and relatively small surface winds, then geostrophic balance in a compressible atmosphere still provides information about the meridional structure of some weighted vertical average of the temperature. In the simplest case, in which the poleward flow is at constant pressure p_T and the surface pressure p_B is constant, one has

$$f u_M \approx -R_0 \frac{\partial}{\partial y} \{T\},$$

where

$$\{T\} \equiv \int_{p_B}^{p_T} \frac{T}{p} dp,$$

R_0 being the gas constant. One therefore obtains a formula analogous to (12) for $\{T\}$.

The outgoing longwave flux can be thought of as a second weighted average of atmospheric and surface temperatures, the weighting being a function of the atmospheric composition. If the vertical structure of the atmosphere is strongly controlled by small-scale mixing in a manner not affected by the large-scale flow, then these two weighted averages might be sufficiently closely linked in practice that L could be thought of as a fixed function of $\{T\}$. If this is a legitimate approximation, then the assumptions that $H(\theta_H) = 0$ and the constraint

$$0 = \int_0^{\theta_H} H(\theta)\cos(\theta)d\theta$$

$$= \int_0^{\theta_H} [S(\theta) - F(\theta) - L(\{T(\theta)\})]\cos\theta d\theta$$

result in precisely the same geometric construction for the width of the Hadley cell as illustrated in Fig. 1. The meridional profile of the total heat flux by the atmosphere is thereby determined. The details of the moist convective heating, besides controlling the height of the poleward flow, control the partitioning of this transport between latent and sensible fluxes.

The factor which is most likely to upset this line of argument is cloudiness. The cloud distribution is controlled to a great extent by the circulation and does certainly affect the outgoing longwave flux and the shortwave heating. To the extent that these two effects do not compensate, one must develop a theory for the circulation-dependent cloudiness and couple it to the sort of argument presented above in order to determine the gross structure of the cell.

Based on these considerations, we would argue that the applicability of the analysis of Section 4 is not confined to the particular system of equations treated in this paper. As long as the atmosphere is nearly inviscid, these arguments can be modified appropriately to show that the Hadley cell is strongly constrained by geostrophy and conservation of heat and momentum. The crucial requirement is that the internal momentum mixing be small. Should viscous stresses in the interior be sufficiently large that $\partial u/\partial y \ll f$, the momentum balance in the poleward branch of the Hadley cell would be (i) $fv \approx$ divergence of stresses rather than (ii) $(f - \partial u/\partial y) \approx 0$. If (i) holds, then the meridional circulation is determined by the stresses, and the thermodynamic equation determines the temperature field consistent with this circulation. If (ii) holds, then this momentum equation effectively determines the mean atmospheric temperature profile, as we have seen, and the thermodynamics determines the circulation consistent with these temperatures.

The response to a perturbation in diabatic heating [or oceanic heat flux in (26)] must be dramatically different in these two limiting cases. In the linear viscous flow the circulation remains unchanged to the extent that the stresses remain unchanged, and the atmospheric temperatures must adjust until the radiative deficit balances the heating perturbation. In the nearly inviscid flow these arguments suggest that temperatures remain more or less unchanged, leaving the heating anomaly to be balanced by adiabatic cooling.

In low latitudes, the earth's atmosphere is intermediate between these two extremes. Stress of various kinds are far from negligible, but neither is $\partial u/\partial y$ negligible compared with f. An understanding of nearly inviscid as well as viscous axisymmetric flows should be of value in the study of the Hadley circulation.

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REFERENCES

Charney, J., 1973: Planetary fluid dynamics. *Dynamic Meteorology*, P. Morel, Ed., D. Reidel, 97–351.

Dickinson, R. E., 1971: Analytic model for zonal winds in the tropics. I. Details of the model and simulation of gross features of the zonal mean troposphere. *Mon. Wea. Rev.*, **99**, 501–510.

Eliassen, A., 1951: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophys. Norv.*, **5**, 19–60.

Held, I., and M. Suarez, 1978: A two-level primitive equation atmospheric model designed for climatic sensitivity experiments. *J. Atmos. Sci.*, **35**, 205–229.

Hide, R., 1969: Dynamics of the atmospheres of the major planets with an appendix on the viscous boundary layer at the rigid boundary surface of an electrically conducting rotating fluid in the presence of a magnetic field. *J. Atmos. Sci.*, **26**, 841–853.

Leovy, C., 1964: Simple models of thermally driven mesospheric circulation. *J. Atmos. Sci.*, **21**, 327–341.

Matsuno, T., 1966: Numerical integrations of primitive equations by use of a simulated backward difference method. *J. Meteor. Soc. Japan*, **44**, 76–84.

McIntyre, M. E., 1970a: Diffusive destabilization of the baroclinic circular vortex. *Geophys. Fluid Dyn.*, **1**, 19–57.

——, 1970b: Role of diffusive overturning in nonlinear axisymmetric convection in a differentially heating rotating annulus. *Geophys. Fluid. Dyn.*, **1**, 59–89.

Nakamura, H., 1978: Dynamical effects of mountains on the general circulation of the atmosphere. III. Effects on the general circulation of the baroclinic atmosphere. *J. Meteor. Soc. Japan*, **56**, 353–366.

Pedlosky, J., 1969: Axially symmetric motion of stratified rotating fluid in a spherical annulus of narrow gap. *J. Fluid Mech.*, **36**, 401–415.

Schneider, E. K., 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part II: Nonlinear calculations. *J. Atmos. Sci.*, **34**, 280–297.

——, and R. S. Lindzen, 1976: The influence of stable stratification on the thermally driven tropical boundary layer. *J. Atmos. Sci.*, **33**, 1301–1307.

——, and ——, 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part I: Linearized calculations. *J. Atmos. Sci.*, **34**, 263–279.

Schoeberl, M. R., and D. F. Strobel, 1978: The zonally averaged circulation of the middle atmosphere. *J. Atmos. Sci.*, **35**, 577–591.

Williams, G., 1970: Axisymmetric annulus convection at unit Prandtl number. *Geophys. Fluid Dyn.*, **1**, 357–369.