A Quasi-equilibrium Turbulent Energy Model for Geophysical Flows

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ABSTRACT

The Mellor–Yamada hierarchy of turbulent closure models is reexamined to show that the elimination of a slight inconsistency in their analysis leads to a quasi-equilibrium model that is somewhat simpler than their level 2½ model. Also the need to impose realizability conditions restricting the dependence of exchange coefficients on shearing rates is eliminated. The model is therefore more robust while the principal advantage of the level 2½ model, namely the solution of a prognostic equation for turbulent kinetic energy is retained. Its performance is shown to be not much different from that of level 2½.

1. Introduction

A hierarchy of turbulent closure models described by Mellor and Yamada (1974) has been in use in geophysical flows. It is based on an order-of-magnitude analysis of small deviations of Reynolds stresses and heat flux from the state of local isotropy. In the limiting case of near isotropy, one obtains an algebraic set of equations for all turbulence quantities, including turbulent kinetic energy and temperature variance. This fully algebraic model describes a situation in which mechanical and buoyancy production are exactly balanced by dissipation in turbulent energy and temperature variance equations. Production terms balance pressure-redistribution terms in the Reynolds stress and heat flux equations. The advective and diffusive terms are neglected in all equations, thus making the state of near isotropy identical with the state of local equilibrium. Donaldson (1973) called such a model "super-equilibrium" model.

This model is classified as level 2 closure in the Mellor and Yamada (1974, 1977) hierarchy and is attractive mainly because of its simplicity and robustness. However, it is deficient in situations where the neglected advection and diffusion terms are not small, as for example, in convective entrainment at a density interface in stably stratified environments. An alternative is the 2½ level closure model, in which turbulent energy is calculated prognostically from the transport equation, but Reynolds stress equations are solved assuming local equilibrium. This model has been widely used in different geophysical applications, including small- and meso-scale oceanography, micro- and mesoscale meteorology, and global circulation models.

The application of the Mellor and Yamada 2½ level model could, however, cause some problems for strongly stable or unstable stratification. For example, for strong unstable stratification, the model can, in some circumstances, lead to $\overline{w^2} > q^2$. To deal with these problems, Mellor and Yamada (1982), Hassid and Galperin (1983) and Helfand (1985) suggested inclusion of various limitations on the dependence of the exchange coefficients on parameters involving velocity and density gradients and on the turbulent energy and length scale. These methods succeed in preventing singularities in the computed results, but because of their arbitrary nature, it is desirable to avoid imposing these realizability conditions, when feasible.

The model proposed here is based on the same systematic expansion of Reynolds stress and heat flux equations as in Mellor and Yamada (1974). However, the removal of a slight inconsistency in their scaling arguments results in a simpler but more robust quasi-equilibrium model. The resulting turbulent exchange coefficients are simpler than the corresponding ones for the level 2½ model and at the same time, the need to impose realizability conditions restricting their dependence on shearing rates is eliminated. The turbulent energy is however still calculated prognostically using transport equations, thus preserving the principal advantage of the level 2½ model.

2. Derivation of the quasi-equilibrium model

We shall follow closely the paper of Mellor and Yamada (1974, hereafter referred to as MY) in which the reader will find more details.

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A sequence of closure assumptions made in MY leads to a system of equations for the Reynolds stress tensor, $\overline{u_iu_j}$, forming the basis of the level 4 model. An equation can then be formed which is the difference between the Reynolds stress equation and the product of $\delta_{ij}/3$ and the trace of the same equation [see MY Eqs. (9) and (8)]. Defining $q^2 = \overline{u_i^2}$ the equation is

$$\frac{D}{Dt}\left(\overline{u_iu_j} - \frac{\delta_{ij}}{3}q^2\right) - \frac{\partial}{\partial x_k}\left\{q\lambda_1\left[\frac{\partial\overline{u_iu_j}}{\partial x_k} + \frac{\partial\overline{u_iu_k}}{\partial x_j} + \frac{\partial\overline{u_ju_k}}{\partial x_i} - \frac{\delta_{ij}}{3}\left(\frac{\partial q^2}{\partial x_k} + 2\frac{\partial\overline{u_lu_k}}{\partial x_l}\right)\right]\right\} = -\overline{u_ku_i}\frac{\partial U_j}{\partial x_k} - \overline{u_ku_j}\frac{\partial U_i}{\partial x_k} + \frac{2}{3}\delta_{ij}\overline{u_ku_l}\frac{\partial U_l}{\partial x_k} - \beta\left(g_j\overline{u_i\theta} + g_i\overline{u_j\theta} - \frac{2}{3}\delta_{ij}g_l\overline{u_l\theta}\right) - \frac{q}{3l_1}\left(\overline{u_iu_j} - \frac{\delta_{ij}}{3}q^2\right) + C_1q^2\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad (1)$$

$$\frac{Dq^2}{Dt} - \frac{\partial}{\partial x_k}\left[q\lambda_1\left(\frac{\partial q^2}{\partial x_k} + 2\frac{\partial\overline{u_iu_k}}{\partial x_i}\right)\right] = -2\,\overline{u_ku_i}\frac{\partial U_i}{\partial x_k} - 2\beta g_i\overline{u_i\theta} - 2\frac{q^3}{\Lambda_1}, \quad (2)$$

where

$$(l_1, l_2, \Lambda_1, \Lambda_2, \lambda_1) = (A_1, A_2, B_1, B_2, 0.12)l. \quad (3)$$

Here l is the master length scale of turbulence, $g_j = (0, 0, -g)$, the acceleration due to gravity, $f_j = (0, f_y, f)$, the planetary vorticity vector, and $\beta = -(\partial\rho/\partial T)_p/\rho$.

Following MY, let us introduce nondimensional tensors a_{ij} and b_j characterizing the departure from local isotropy:

$$\overline{u_iu_j} \equiv \left(\frac{\delta_{ij}}{3} + a_{ij}\right)q^2, \quad a_{ii} = 0, \quad (4)$$

$$\overline{u_j\theta} \equiv b_jq\phi, \quad (5)$$

where

$$\phi^2 \equiv \overline{\theta^2}$$

$$a \equiv \|a_{ij}\|, \quad (6a)$$

$$b \equiv \|b_j\|, \quad (6b)$$

$$O(a) = O(b). \quad (6c)$$

$\|\cdot\|$ denotes the norm of the matrix.

Recall the scaling assumptions of MY, valid for all the models in their hierarchy. Without going into the details they are (using Λ instead of λ, for convenience in scaling notation)

$$U_x^2 \equiv O(\|\partial U_i/\partial x_j\|^2), \quad (7a)$$

$$\Theta_x^2 \equiv O(\|\partial\Theta/\partial x_i\|^2), \quad (7b)$$

$$l = O(l_1) = O(l_2), \quad (7c)$$

$$a^2 = l/\Lambda, \quad U_x = a^{-1}q/\Lambda, \quad (7d)$$

$$b^2 = l/\Lambda, \quad \theta_x = b^{-1}\phi/\Lambda, \quad (7e)$$

$$g\beta\phi = b^{-1}q^2/\Lambda. \quad (7f)$$

Let us rewrite Eqs. (1) and (2) in analogy with Eqs. (12), (13) of MY using scaling hypotheses (7a–f) and the additional assumption [Eq. (19) in MY]

$$Uq^2/L = aq^3/\Lambda, \quad (8)$$

which, when rewritten in the form $L/U = a^{-1}\Lambda/q$, assumes that eddy-turnover time $t_e = \Lambda/q$ is smaller than the advective time scale $t_a = L/U$ by the same factor, a, which characterizes the departure from the local isotropy. (Note that this assumption relates expansion by the deviation from local isotropy with the departure from local equilibrium. This connection is behind the above mentioned analogy between the level 2 model of MY and the superequilibrium model of Donaldson, 1973).

We shall now evaluate twice the turbulent energy equation (the trace of the Reynolds stress equation):

$$\begin{array}{cccccc} \dfrac{Dq^2}{Dt} & - & \dfrac{\partial}{\partial x_k}\left\{\dfrac{5}{3}q\lambda_1\dfrac{\partial q^2}{\partial x_k}[1 + O(a)]\right\} & & & \\ \dfrac{Uq^2}{L} & & \dfrac{Uq^2}{L} & & & \\ \dfrac{aq^3}{\Lambda} & & \dfrac{aq^3}{\Lambda} & & & \\ & = & -2a_{ki}q^2\dfrac{\partial U_i}{\partial x_k} & -2b_kg_k\beta\phi & -2\dfrac{q^3}{\Lambda}, & (9) \\ & & aq^2U_x & \beta bgq\phi & \dfrac{q^3}{\Lambda} & \\ & & \dfrac{q^3}{\Lambda} & \dfrac{q^3}{\Lambda} & \dfrac{q^3}{\Lambda} & \end{array}$$

where, to obtain an estimate of the order of each term, use is made of Eqs. (7) and (8). Repeating the process with Eq. (1) we have

$$\begin{array}{ccc} \dfrac{D}{Dt}(a_{ij}q^2) & - & \dfrac{\partial}{\partial x_k}\left\{\dfrac{1}{3}q\lambda_1\left(\delta_{ik}\dfrac{\partial q^2}{\partial x_j} + \delta_{jk}\dfrac{\partial q^2}{\partial x_i} - \dfrac{2}{3}\delta_{ij}\dfrac{\partial q^2}{\partial x_k}\right)[1 + O(a)]\right\} \\ \dfrac{aUq^2}{L} & & \dfrac{Uq^2}{L}[1 + O(a)] \\ \dfrac{a^2q^3}{\Lambda} & & \dfrac{aq^3}{\Lambda}[1 + O(a)] \end{array}$$

$$= -\underset{\substack{q^2U_x[1+O(a)]\\ \frac{a^{-1}q^3}{\Lambda}[1+O(a)]}}{q^2\left(\tfrac{1}{3}\delta_{ki}+a_{ki}\right)\frac{\partial U_j}{\partial x_k}} - \underset{\substack{q^2U_x[1+O(a)]\\ \frac{a^{-1}q^3}{\Lambda}[1+O(a)]}}{q^2\left(\tfrac{1}{3}\delta_{kj}+a_{kj}\right)\frac{\partial U_i}{\partial x_k}} - \underset{\substack{aq^2U_x\\ \frac{q^3}{\Lambda}}}{\tfrac{2}{3}q^2\delta_{ij}a_{kl}\frac{\partial U_l}{\partial x_k}} - \underset{\substack{q^2U_x\\ \frac{a^{-1}q^3}{\Lambda}}}{C_1q^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)}$$

$$-\underset{\substack{b\beta g\phi q\\ \frac{q^3}{\Lambda}}}{\beta q\phi\left(g_jb_i+g_ib_j-\tfrac{2}{3}\delta_{ij}g_lb_l\right)} - \underset{\substack{\frac{aq^3}{l}\\ \frac{a^{-1}q^3}{\Lambda}}}{\frac{q^3}{3l_1}a_{ij}}. \quad (10)$$

Neglecting terms of $O(a^2)$ in Eq. (10) will result in the level 3 model of MY. Instead, let us proceed one step further wherein the right hand side of Eq. (9), after multiplication by $\delta_{ij}/3$, is used to replace similar terms in Eq. (10). After re-ordering the terms, the resulting equation is

$$\underset{\frac{a^2q^3}{\Lambda}}{\frac{D}{Dt}(a_{ij}q^2)} - \frac{\partial}{\partial x_k}\left\{\underset{\frac{aq^3}{\Lambda}[1+O(a)]}{-q\lambda_1\left(\delta_{ik}\frac{\partial q^2}{\partial x_j}+\delta_{jk}\frac{\partial q^2}{\partial x_i}-\tfrac{2}{3}\delta_{ij}\frac{\partial q^2}{\partial x_k}\right)[1+O(a)]}\right\}$$

$$= -\underset{\frac{a^{-1}q^3}{\Lambda}[1+O(a)]}{q^2\left(\tfrac{1}{3}\delta_{ki}+a_{ki}\right)\frac{\partial U}{\partial x_k}} - \underset{\frac{a^{-1}q^3}{\Lambda}[1+O(a)]}{q^2\left(\tfrac{1}{3}\delta_{kj}+a_{kj}\right)\frac{\partial U_j}{\partial x_k}} + \underset{\frac{a^{-1}q^3}{\Lambda}}{C_1q^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)} - \underset{\frac{q^3}{\Lambda}}{\beta q\phi(g_jb_i+g_ib_j)}$$

$$-\tfrac{1}{3}\delta_{ij}\left\{\underset{\frac{aq^3}{\Lambda}}{\frac{Dq^2}{Dt}} - \frac{\partial}{\partial x_k}\left[\underset{\frac{aq^3}{\Lambda}[1+O(a)]}{\tfrac{5}{3}q\lambda_1\frac{\partial q^2}{\partial x_k}(1+O(a))}\right] + \underset{\frac{q^3}{\Lambda}}{\frac{2q^3}{\Lambda}}\right\} - \underset{\frac{a^{-1}q^3}{\Lambda}}{\frac{q^3}{3l_1}a_{ij}}. \quad (11)$$

Lumping all terms of $O(a^2)$ together in both Eqs. (9) and (11) we obtain

$$\frac{Dq^2}{Dt} - \frac{\partial}{\partial x_k}\left(\frac{5}{3}q\lambda_1\frac{\partial q^2}{\partial x_k}\right) = -2\overline{u_ku_l}\frac{\partial U_k}{\partial x_l} - 2\beta g_k\overline{u_k\theta} - 2\frac{q^3}{\Lambda_1} + O(a^2), \quad (12)$$

$$\overline{u_iu_j} = \tfrac{1}{3}\delta_{ij}q^2 - \frac{3l_1}{q}\left[(\overline{u_ku_i} - C_1q^2\delta_{ki})\frac{\partial U_j}{\partial x_k} + (\overline{u_ku_j} - C_1q^2\delta_{kj})\frac{\partial U_i}{\partial x_k}\right] - \frac{3l_1}{q}\beta(g_j\overline{u_i\theta} + g_i\overline{u_j\theta}) - \frac{l_1}{q}\delta_{ij}\frac{2q^3}{\Lambda_1} + O(a^2). \quad (13)$$

Equation (13) replaces Eq. (21) in MY for the level 3 model and it does not contain, explicitly or implicitly, any term of the order $O(a^2)$ or higher, in contrast to MY 2½ and 3 level models. Following MY, let us rewrite Equations (12) and (13) in the usual format convenient for geophysical boundary layers, omitting $O(a^2)$ terms:

$$\frac{Dq^2}{Dt} - \frac{\partial}{\partial z}\left(\frac{5}{3}0.12ql\frac{\partial q^2}{\partial z}\right) = -2\overline{uw}\frac{\partial U}{\partial z} - 2\overline{vw}\frac{\partial V}{\partial z} + 2\beta g\overline{w\theta} - 2\frac{q^3}{\Lambda_1}, \quad (14)$$

$$\begin{pmatrix}\overline{u^2}\\ \overline{v^2}\\ \overline{w^2}\end{pmatrix} = \frac{q^2}{3}\left(1 - \frac{6A_1}{B_1}\right)\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{3l_1}{q}\begin{pmatrix}2\frac{\partial U}{\partial z} & 0 & 0\\ 0 & 2\frac{\partial V}{\partial z} & 0\\ 0 & 0 & -2\beta g\end{pmatrix}\begin{pmatrix}\overline{uw}\\ \overline{vw}\\ \overline{w\theta}\end{pmatrix}, \quad (15)$$

$$\begin{pmatrix}\overline{uv}\\ \overline{uw}\\ \overline{vw}\end{pmatrix} = -\frac{3l_1}{q}\begin{pmatrix}\overline{uw}\frac{\partial V}{\partial z} + \overline{vw}\frac{\partial U}{\partial z}\\ (\overline{w^2} - C_1q^2)\frac{\partial U}{\partial z} - \beta g\overline{u\theta}\\ (\overline{w^2} - C_1q^2)\frac{\partial V}{\partial z} - \beta g\overline{v\theta}\end{pmatrix}. \quad (16)$$

One should keep in mind that contraction of the diagonal components of the Reynolds stress tensor, Eq. (15), implies

$$\overline{u^2} + \overline{v^2} + \overline{w^2} = q^2\left(1 - \frac{6A_1}{B_1}\right) - \frac{6l_1}{q}\left(\overline{uw}\frac{\partial U}{\partial z} + \overline{vw}\frac{\partial V}{\partial z} - \beta g\overline{w\theta}\right) = q^2 + O(a^2).$$

This result is consistent with the order-of-magnitude analysis performed above in which all $O(a^2)$ terms were neglected. In the level 2½ model $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$ sum up to q^2 exactly. However, the difference is not significant since only q^2 is calculated prognostically; the components of q^2 can be then found diagnostically and they have no feed back on the turbulent kinetic energy equation, Eq. (14). Besides, after some algebraic manipulations correct to order $O(a^2)$, Eq. (15) can be rewritten so that $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ add up to q^2 exactly.

Similar scaling analysis can be performed on the temperature variance equations (5) of MY, where we shall distinguish between l_1, Λ_1 and l_2, Λ_2:

$$\begin{array}{cccc} \dfrac{D\overline{\theta^2}}{Dt} - & \dfrac{\partial}{\partial x_k}\left(q\lambda_2\dfrac{\partial\overline{\theta^2}}{\partial x_k}\right) & = -2q\phi b_k\dfrac{\partial\Theta}{\partial x_k} & - 2\dfrac{q\theta^2}{\Lambda_2} \\ \dfrac{U\phi^2}{L} & \dfrac{U\phi^2}{L} & 2q\phi b\Theta_x & \dfrac{2q\phi^2}{\Lambda_2} \\ \dfrac{U\phi^2}{L} & \dfrac{U\phi^2}{L} & \dfrac{2q\phi^2}{\Lambda_2} & \dfrac{2q\phi^2}{\Lambda_2} \end{array} \tag{17}$$

Following MY we assume $\lambda_1 = \lambda_2$. The ratio of the terms in the left hand and right hand sides of Eq. (17) is

$$\left(\frac{U\phi^2}{L}\right)\Big/\left(\frac{2q\phi^2}{\Lambda_2}\right) = \frac{a}{2}\frac{\Lambda_2}{\Lambda_1}, \tag{18}$$

where Eq. (8) was used and an assumption was made that the advective length scale, L, is the same for momentum and heat transfer. Using Eqs. (3) and (21) one can evaluate the numerical value of the ratio, $(a/2)(\Lambda_2/\Lambda_1)$, appearing in Eq. (18), to be 0.072. This value is very close to b^2 (=0.073). Therefore, the advective and diffusive terms in Eq. (17) may be dropped as small $O(b^2)$ terms, which is consistent with our approximation. It means that the production and dissipation of the temperature variance, $\overline{\theta^2}$, are in local equilibrium to $O(b^2)$. One should keep in mind, however, that this kind of argument is not a result of rigorous application of asymptotic analysis. It provides only *a posteriori* justification for neglecting all advective and diffusive terms in Eq. (17), an approximation known to perform well in geophysical boundary layers. Now, turbulence heat flux and temperature variance equations can be written as follows (Mellow and Yamada, 1982):

$$\begin{pmatrix} \overline{u\theta} \\ \overline{v\theta} \\ \overline{w\theta} \end{pmatrix} = -\frac{3l_2}{q}\begin{pmatrix} \overline{uw}\dfrac{\partial\Theta}{\partial z} + \overline{w\theta}\dfrac{\partial U}{\partial z} \\ \overline{vw}\dfrac{\partial\Theta}{\partial z} + \overline{w\theta}\dfrac{\partial V}{\partial z} \\ \overline{w^2}\dfrac{\partial\Theta}{\partial z} - \beta g\overline{\theta^2} \end{pmatrix}, \tag{19}$$

$$\overline{\theta^2} = -\frac{\Lambda_2}{q}\overline{w\theta}\frac{\partial\Theta}{\partial z}. \tag{20}$$

The constants of the model as specified in Mellor and Yamada (1982) are

$$(A_1, B_1, C_1, A_2, B_2) = (0.92, 16.6, 0.08, 0.74, 10.1). \tag{21}$$

The master length scale, l, can be found from an algebraic (MY, Hassid and Galperin, 1983) or a differential (Mellor and Yamada, 1982) equation. We have found it necessary to limit l in stably stratified flows according to

$$l \leqslant \frac{0.53q}{N}, \tag{22}$$

where

$$N = \left(\beta g\frac{\partial\Theta}{\partial z}\right)^{1/2}$$

is Brunt–Väisälä frequency. This restriction has been used, for example, by Andre et al. (1978) and Hassid and Galperin (1983) to reflect the limiting effect of stable stratification on the size of turbulent eddies.

The set of equations (14)–(22) along with the mean momentum, continuity and heat balance equations forms the basis of the present model, which could be referred to as 2¼ level model, since it is intermediate between the 2 and 2½ level models of MY hierarchy in complexity. However, it is, perhaps, more precise to classify it as level 3 because it is derived by a consistent elimination of $O(a^2)$ and $O(b^2)$ terms in the level 4 model. (In this sense, both MY 3 and 2½ level models actually fall between levels 3 and 4.) The next lower order yields level 2 equations just as in MY.

3. Vertical turbulent exchange coefficients

From Equations (15), (16), (19) and (20) expressions for these coefficients can be obtained:

$$K_M = qlS_M, \quad K_H = qlS_H, \tag{23}$$

where

$$S_M = A_1\frac{1 - 3C_1 - (6A_1/B_1) - 3A_2G_H[(B_2 - 3A_2)[1 - (6A_1/B_1)] - 3C_1(B_2 + 6A_1)]}{[1 - 3A_2G_H(6A_1 + B_2)](1 - 9A_1A_2G_H)}, \tag{24}$$

$$S_H = A_2 \frac{1 - (6A_1/B_1)}{1 - 3A_2G_H(6A_1 + B_2)}, \quad (25)$$

$$G_H = -\left(\frac{l}{q}\right)^2 \beta g \frac{\partial \Theta}{\partial z}. \quad (26)$$

Equations (24) and (25) are identical to Mellor and Yamada's (1982) equations (38) and (39) after substitution $(P_s + P_b)/\epsilon = 1$.

An advantage of the present model is that the turbulent exchange coefficients and the vertical component of turbulent kinetic energy depend only on G_H. Absence of dependence of S_M and S_H on velocity gradient removes the necessity to impose a realizability condition related to it, as described, for instance, in Hassid and Galperin (1983).

The upper bound of G_H corresponding to the case of unstable stratification should be set by the requirement that G_M remains non-negative in the level 2 model, where

$$S_M G_M + S_H G_H = \frac{1}{B_1}, \quad (27)$$

$$G_M = \left(\frac{l}{q}\right)^2 \left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]. \quad (28)$$

This requirement can be written as

$$G_H \leqslant [A_2(12A_1 + B_1 + 3B_2)]^{-1} = 0.0233. \quad (29)$$

For the limiting value of G_H one will find that $\overline{w^2}/q^2 = 0.555$, thus making the two other components equal to $\overline{u^2}/q^2 = \overline{v^2}/q^2 = 0.22$. These values are in rather good agreement with the data of Willis and Deardorff (1974) for convective turbulence.

In the case of stable stratification, using the definition of G_H, Eq. (26), and Inequality (22), one finds:

$$G_H \geqslant -(0.53)^2 = -0.28. \quad (30)$$

It can be shown, using Eqs. (15) and (25), that as G_H reaches its minimum given by (30), the ratio, $\overline{w^2}/q^2$, is bounded according to

$$\frac{\overline{w^2}}{q^2} \geqslant \frac{1}{3}\left(1 - \frac{6A_1}{B_1}\right) \frac{1 - 3A_2B_2G_H}{1 - 3A_2G_H(6A_1 + B_2)} = 0.15. \quad (31)$$

Inequality (31) shows that as small scale turbulence is suppressed by buoyancy forces and the vertical component of fluctuations and the total turbulence kinetic energy both go to zero, their ratio, $\overline{w^2}/q^2$, remains bounded. The model does not account for vertical fluctuations associated with internal wave motions that persist long after turbulent mixing is extinguished; few models do. Inequality (31) also suggests that even under strong stable stratification, the level of anisotropy does not exceed the critical value established by Mellor & Yamada (1982) as

$$\overline{u^2_{\min}}/q^2 \geqslant 0.12,$$

$\overline{u^2_{\min}}$ being the minimum among $\overline{u_i^2}$, $i = 1, 2, 3$, at which some of the basic assumptions of the model may become invalid. Figures 1 and 2 show S_M, S_H and $\overline{w^2}/q^2$ as functions of G_H.

4. Comparison of the quasi-equilibrium and the 2½ level models

Martin (1985) has compared the performance of both MY 2 and 2½ level models against mixed layer observational data in the Pacific Ocean. These data were collected at OWS November and Papa in the eastern North Pacific. The comparisons were made for a period of an entire year during 1961 when data coverage was excellent and relatively error and gap free. Unlike Mellor and Yamada (1982), Martin (1985) used an algebraic length scale equation in both the 2 and 2½ level models.

Martin (1985) found that both level 2 and 2½ models yield similar results over the entire year, despite big seasonal changes in external forcing. The differences between the two models were not significant. Since, as was stated above, the present quasi-equilibrium model is between levels 2 and 2½ of MY in the hierarchy, it is to be expected that its performance will be similar to that of the 2½ level model. Therefore, no comparisons of this model with ocean station Papa and November data will be attempted. Instead, we shall concentrate on comparisons of the relative performance of the level 2½ and quasi-equilibrium models when both are run with identical initial and boundary conditions to simulate available laboratory data.

To assess this relative performance, the experiments of Kato and Phillips (1969) (KP), Kantha et al. (1977) (KPA) and Deardorff and Willis (1985) (DW) have been chosen. We need to emphasize though that the

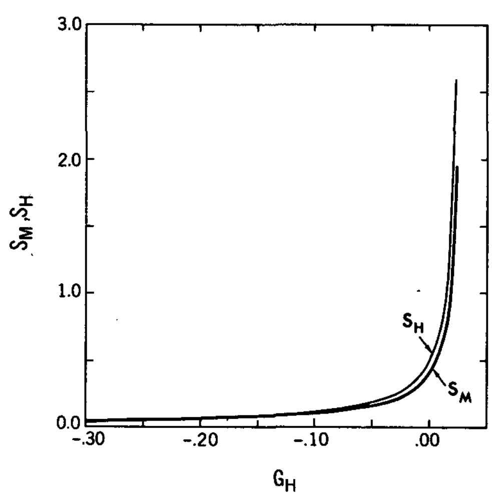

FIG. 1. Dependence of stability parameters S_M and S_H on G_H, as given by Eqs. (24) and (25).

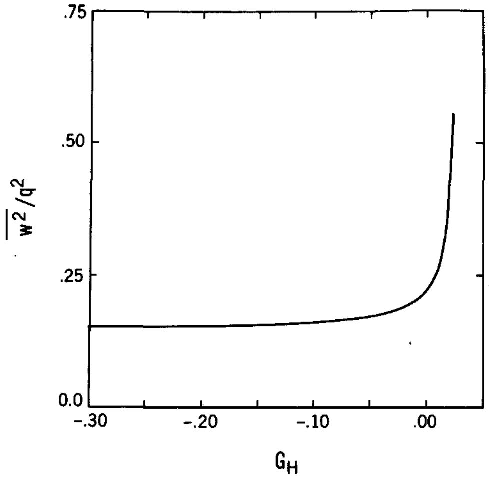

FIG. 2. Vertical component of the turbulence kinetic energy, $\overline{w^2}$, as a function of G_H as given by Eq. (15).

primary goal of this section is to compare the performance of the models against each other while the experimental data are used rather as benchmarks to place the results in proper context. In all the calculations, the same algebraic length scale as in Hassid and Galperin (1983) was used. However, the limitation on the length scale in stable stratification was slightly modified, so that strong density changes like the one occurring at the base of the mixed layer in the KPA experiment could also be treated properly. Thus, the length scale was taken as the smallest value that satisfies the following two inequalities:

$$K^2q^2 \geqslant \beta g \int_z^{z+l/\kappa} [T(z') - T(z)]dz', \qquad (32a)$$

$$K^2q^2 \geqslant \beta g \int_{z-l/\kappa}^{z} [T(z) - T(z')]dz', \qquad (32b)$$

which impose two restrictions on l. The physical basis for this is the limiting effect of gravitational forces on the size of the eddies (Bougeault and Andre, 1986). The constant K is chosen equal to $\frac{3}{8}\kappa$, so that for constant temperature gradient, the limitation on the length scale is the same as given by Inequality (22). The numerical value of A_2 was set to 0.587 providing a turbulent Prandtl number of 1.0, as suggested by Hassid and Galperin (1983). However, the results are not sensitive to small changes in A_2 or Pr_t.

In Fig. 3, the predicted mixed layer development is shown for the two models, along with the experimental results of DW, in which convectively-induced turbulence penetrates a region of constant stable temperature gradient of 20°C m^{-1}. The theoretical predictions for three different definitions of the boundary layer height are shown together with the experimental points. In spite of the different limitation on the value of G_H, both 2½ and the present models give results that are almost indistinguishable from each other and in good agreement with the experimental data.

In Figs. 4 and 5, the predictions of the two models are shown for penetration of mechanically-induced

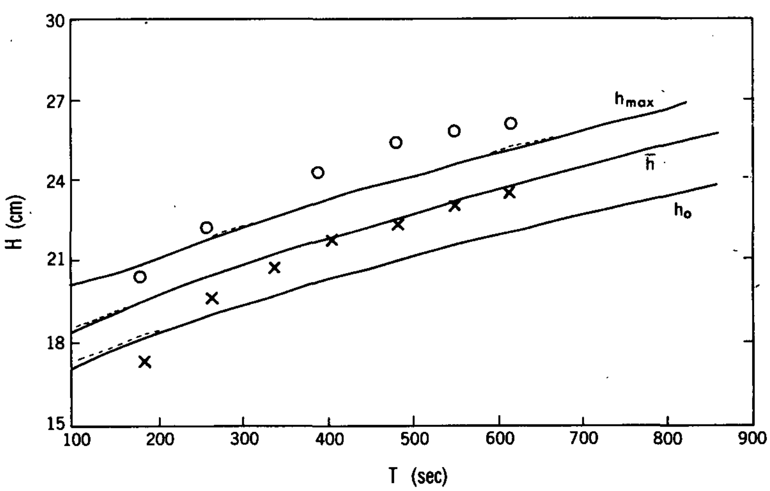

FIG. 3. Mixed layer growth in the penetrative convection experiment. Data of Deardorff and Willis (1985): ×—height at which the temperature profile first crosses the ambient profile; ○—height at which the temperature profile joins the ambient temperature profile. Numerical predictions: (dashed) quasi-equilibrium model and (solid) 2½ model. The three theoretical mixed-layer heights are h_0—the height at which the heat flux is equal to zero; $\bar{h}$—the height at which the negative heat flux reaches its maximum; h_{max}—the height of maximum penetration.

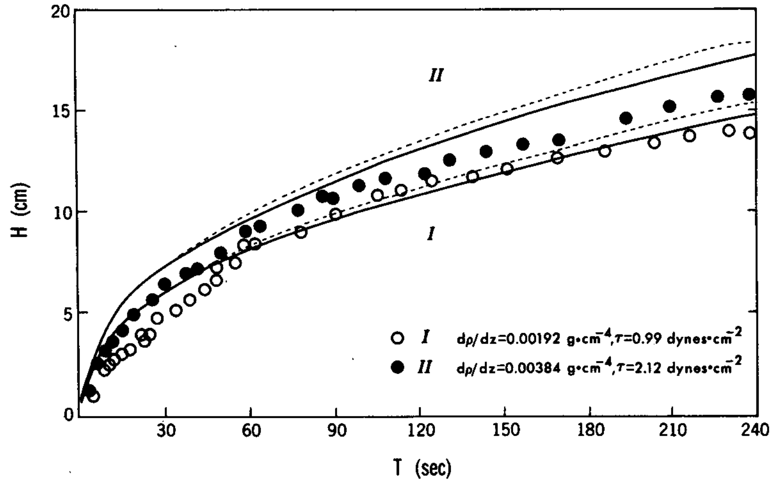

FIG. 4. Mixed layer growth in the experiment of Kato and Phillips (1969). Data: O, ●. Numerical predictions: (solid) quasi-equilibrium model and (dashed) 2½ model.

turbulence in stably stratified media, along with the data of KP for constant ambient density gradient and KPA for a sharp discontinuity in the initial ambient density profile. Again, the two models are shown to produce results close to each other. There are differences of 3%–4% for KP and of 10% for KPA. The 2½ level model gives a larger value for the mixed layer height than the present quasi-equilibrium model. Both models rather overpredict the experimental results, especially for strong stratifications. It is possible that the effect of the tank curvature, which was not taken into account in the present calculations but which appears to affect entrainment experiments in an annulus (Mellor and Strub, 1980, Deardorff and Yoon, 1984) might be appreciable in these experiments. More recent experiments on mixing in stratified fluids by Narimousa et al. (1986) in a race-track shaped flume are however in good agreement with KPA results, so that the extent of contamination by curvature is still rather uncertain. In any case, as was stated above, the relative performance of the two models is of primary interest in the present section and the results strongly suggest that there is little difference between the two.

5. Discussion and conclusions

The scaling analysis performed in this paper shows that the quasi-equilibrium turbulent energy model presented here removes a small inconsistency in the MY hierarchy. This model is simpler than the 2½

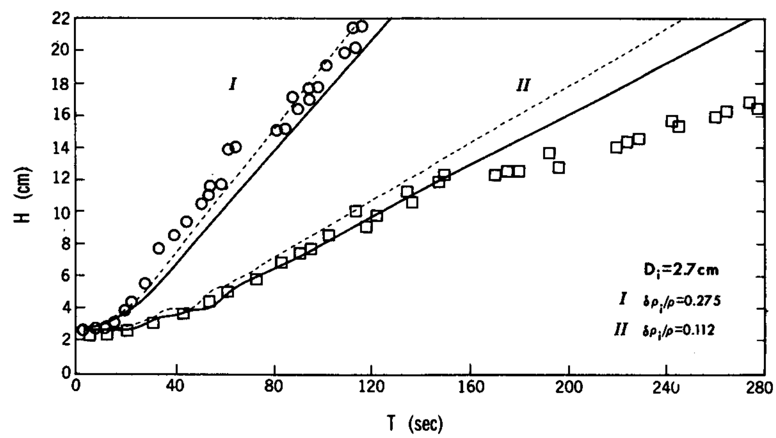

FIG. 5. Mixed layer growth in the experiment of Kantha et al. (1977). Data: O, □. D_i is the initial depth of the interface. Numerical predictions: (solid) quasi-equilibrium model, (dashed) 2½ model.

model of MY but retains its principal advantage because of the prognostic calculation of turbulent energy. Among the advantages of the present model are simpler expressions for the turbulent exchange coefficients and the fact that the need for restrictions on the values of shear strain rates appearing in these coefficients in the 2½ model is eliminated, thus making the model more robust. Also, both stability parameters, S_M and S_H, depend only on G_H, which makes the analytical analyses of the turbulent exchange processes under different circumstances more feasible. For example, when the rotational terms are retained in the equations for turbulent correlations (rather than neglected as is the conventional practice), it becomes hard to deduce and impose realizability conditions on exchange coefficients which are no longer simple scalar quantities. Under those circumstances, the quasi-equilibrium model with fewer realizability conditions possesses a distinct advantage. Also, the numerical performance of the model is expected to be very similar to that of the 2½ level model in most cases, and therefore it is believed that the proposed model could be as useful as the 2½ level model.

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