A VERSATILE THREE-DIMENSIONAL RAY TRACING COMPUTER PROGRAM FOR RADIO WAVES IN THE IONOSPHERE

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This report describes an accurate, versatile FORTRAN computer program for tracing rays through an anisotropic medium whose index of refraction varies continuously in three dimensions. Although developed to calculate the propagation of radio waves in the ionosphere, the program can be easily modified to do other types of ray tracing because of its organization into subroutines.

The program can represent the refractive index by either the Appleton-Hartree or the Sen-Wyller formula, and has several ionospheric models for electron density, perturbations to the electron density (irregularities), the earth's magnetic field, and electron collision frequency.

For each path, the program can calculate group path length, phase path length, absorption, Doppler shift due to a time-varying ionosphere, and geometrical path length. In addition to printing these parameters and the direction of the wave normal at various points along the ray path, the program can plot the projection of the ray path on any vertical plane or on the ground and punch the main characteristics of each ray path on cards.

The documentation includes equations, flow charts, program listings with comments, definitions of program variables, deck set-ups, descriptions of input and output, and a sample case.

Key words: Ray tracing, computer program, radio waves, ionosphere, three-dimensional, Appleton-Hartree formula, Sen-Wyller formula.

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1. INTRODUCTION

This report describes a three-dimensional ray tracing program written in FORTRAN language for the CDC-3800 computer. Copies of the program deck are available from the Institute for Telecommunication Sciences.

Earlier versions of this program have been in use now for over nine years, both by us and by people scattered all over the world. During that time we have improved and modified the program to the extent that we now need to document these changes so that the present program will be easier to use. We have included the input parameter forms that we use to request ray path calculations because they give nearly all the necessary input data and describe the electron density, collision frequency, and magnetic field models.

2. GENERAL DESCRIPTION

This computer program traces the path of radio wave through a user-specified model of the ionosphere when given the transmitter location (longitude, latitude, and height above the ground), the frequency of the wave, the direction of transmission (both elevation and azimuth), the receiver height, and the maximum number of hops wanted.

3. RAY TRACING EQUATIONS

The program calculates ray paths by numerically integrating Hamilton's equations. Lighthill (1965) gives Hamilton's equations in four dimensions (three spatial and one time) for Cartesian coordinates. Haselgrove (1954) gives Hamilton's equations in three dimensions for spherical polar coordinates. Combining the two gives Hamilton's equations in four dimensions in which the three spatial coordinates are spherical polar (see Table 1 for a definition of the symbols):

$$\frac{dr}{d\tau} = \frac{\partial H}{\partial k_r}, \qquad (1)$$

Table 1. List of the More Important Symbols

A	In section 3, absorption in decibels.
B_o	Magnetic induction of earth's magnetic field.
c	Speed of electromagnetic waves in free space.
C	Cosine of the angle of incidence on the ionosphere.
e	Charge of the electron (a negative number).
$F(w)$	$F(w) = wC_{3/2}(w) + i\frac{5}{2}C_{5/2}(w) = \frac{1}{3/2!}\int_o^\infty \frac{t^{3/2}\exp(-t)dt}{w - it}$. (Davies, 1965, p. 86)
f	Wave frequency.
Δf	Frequency shift of a wave due to a time varying ionosphere (sometimes called Doppler shift).
f_H	Gyro frequency for electrons, $\lvert e\rvert B_o/2\pi m$.
f_N	Plasma frequency, $(Ne^2/4\pi^2\varepsilon_o m)^{\frac{1}{2}}$.
$G(w)$	$wF(w)$.
H	Hamiltonian.
k_r, k_θ, k_φ	Components of the propagation vector in the r, θ, φ directions--a vector perpendicular to the wave front having a magnitude $2\pi/\lambda = \omega/v$.
m	Mass of electron.
N	Number of electrons per unit volume.
n	Phase refractive index (in general complex).
n'	Group refractive index (in general complex).
P	Phase path length, phase of wave divided by free space wave number $2\pi/\lambda_o$.
P'	Group path length ct.
r, θ, φ	Coordinates of a point in spherical polar coordinates.
s	Geometric ray path length.
S	Sine of the angle of incidence on the ionosphere.

Table 1. (Continued)

t	Time, travel time of a wave packet.
U	$1 - iZ$ in the Appleton-Hartree formula or $Z/F(1/Z) = 1/G(1/Z)$ in the Sen-Wyller formula.
V_r, V_θ, V_φ	Components of the wave normal direction in the r, θ, and φ directions, normalized so that $V_r^2 + V_\theta^2 + V_\varphi^2 =$ Real $\{n^2\}$.
v	Phase velocity.
X	$\omega_N^2/\omega^2 = f_N^2/f^2 = Ne^2/(\epsilon_o m\omega^2)$.
Y	$\omega_H/\omega = f_H/f$.
Y_L	$Y \cos \psi$.
Y_T	$Y \sin \psi$.
Z	υ/ω or υ_m/ω.
ϵ_o	Electric permittivity of free space.
θ	Colatitude in spherical polar coordinates.
λ	Wavelength.
λ_o	Wavelength in free space.
υ	Electron collision frequency.
υ_m	Mean electron collision frequency.
ρ	Characteristic wave polarization (definition in Table 6).
ρ_L	Longitudinal polarization (definition in Table 6).
τ	Independent variable in Hamilton's equations.
φ	Longitude in spherical polar coordinates.
ψ	Angle between wave normal and $-B_o$.
ω	$2\pi f$, angular wave frequency.
$\Delta\omega$	$2\pi\Delta f$, angular frequency shift.
ω_H	$2\pi f_H = \lvert e \rvert B_o/m$, angular gyrofrequency.
ω_N	$2\pi f_N = (Ne^2/\epsilon_o m)^{\frac{1}{2}}$, angular plasma frequency.

$$\frac{d\theta}{d\tau} = \frac{1}{r} \frac{\partial H}{\partial k_\theta} \quad , \tag{2}$$

$$\frac{d\varphi}{d\tau} = \frac{1}{r \sin\theta} \frac{\partial H}{\partial k_\varphi} \quad , \tag{3}$$

$$\frac{dt}{d\tau} = - \frac{\partial H}{\partial \omega} \quad , \tag{4}$$

$$\frac{dk_r}{d\tau} = - \frac{\partial H}{\partial r} + k_\theta \frac{d\theta}{d\tau} + k_\varphi \sin\theta \frac{d\varphi}{d\tau} \quad , \tag{5}$$

$$\frac{dk_\theta}{d\tau} = \frac{1}{r} \left(- \frac{\partial H}{\partial \theta} - k_\theta \frac{dr}{d\tau} + k_\varphi r \cos\theta \frac{d\varphi}{d\tau} \right) \quad , \tag{6}$$

$$\frac{dk_\varphi}{d\tau} = \frac{1}{r \sin\theta} \left(- \frac{\partial H}{\partial \varphi} - k_\varphi \sin\theta \frac{dr}{d\tau} - k_\varphi r \cos\theta \frac{d\theta}{d\tau} \right) \quad , \tag{7}$$

$$\frac{d\omega}{d\tau} = \frac{\partial H}{\partial t} \quad , \tag{8}$$

The variables r, θ, φ are the spherical polar coordinates of a point on the ray path; k_r, k_θ, and k_φ are the components of the propagation vector (wave normal direction normalized so that in free space

$$k_r^2 + k_\theta^2 + k_\varphi^2 = \frac{\omega^2}{c^2} \quad , \tag{9}$$

where $\omega = 2\pi f$ is the angular frequency of the wave and c is the speed of propagation of electromagnetic waves in free space); t is time, in (4) it is the propagation time of a wave packet, in (8) it expresses the variation with time of a time varying medium; τ is a parameter whose value depends on the choice of the Hamiltonian H.

For actual calculation, the ray tracing program uses group path $P' = ct$ as the independent variable because the derivatives with respect to P' are independent of the choice of Hamiltonian, allowing the program to switch Hamiltonians in the middle of a path. This choice automatically causes the program to take smaller steps in real path length near reflection where the calculations are more critical. The resulting equations obtained by dividing (1) through (8) by c times (4) are:

$$\frac{dr}{dP'} = -\frac{1}{c}\frac{\partial H/\partial k_r}{\partial H/\partial \omega} \quad , \tag{9}$$

$$\frac{d\theta}{dP'} = -\frac{1}{rc}\frac{\partial H/\partial k_\theta}{\partial H/\partial \omega} \quad , \tag{10}$$

$$\frac{d\varphi}{dP'} = -\frac{1}{rc\,\sin\theta}\frac{\partial H/\partial k_\varphi}{\partial H/\partial \omega} \quad , \tag{11}$$

$$\frac{dk_r}{dP'} = \frac{1}{c}\frac{\partial H/\partial r}{\partial H/\partial \omega} + k_\theta \frac{d\theta}{dP'} + k_\varphi \sin\theta \frac{d\varphi}{dP'} \quad , \tag{12}$$

$$\frac{dk_\theta}{dP'} = \frac{1}{r}\left(\frac{1}{c}\frac{\partial H/\partial \theta}{\partial H/\partial \omega} - k_\theta \frac{dr}{dP'} + k_\varphi\, r\cos\theta \frac{d\varphi}{dP'}\right) \quad , \tag{13}$$

$$\frac{dk_\varphi}{dP'} = \frac{1}{r\sin\theta}\left(\frac{1}{c}\frac{\partial H/\partial \varphi}{\partial H/\partial \omega} - k_\varphi \sin\theta \frac{dr}{dP'} - k_\varphi\, r\cos\theta \frac{d\theta}{dP'}\right) \quad , \tag{14}$$

$$\frac{d(\Delta f)}{dP'} = \frac{1}{2\pi}\frac{d\Delta\omega}{dP'} = \frac{1}{2\pi}\frac{d\omega}{dP'} \tag{15}$$

$$= -\frac{1}{2\pi}\frac{\partial H/\partial t}{\partial H/\partial \omega} \quad .$$

Equation (15) for the frequency shift of a wave propagating through a time varying medium follows directly from Hamilton's equations (4) and (8). An alternative derivation is given by Bennett (1967). For large frequency shifts, the frequency shift should be accumulated along the ray path and the shifted frequency used in calculations at each point on the ray path. Equations (1) through (8) imply that all eight dependent variables vary along the path, and that at each point on the path the instantaneous value of all parameters (including frequency) is used in further evaluations of the equations. However, the time variation of the ionosphere due to natural causes (such as solar flares) is so slow that the resulting frequency shifts are small enough (less than one part in 10^5) to have negligible effect on the propagation. For this reason, the program calculates frequency shift to compare with frequency shift measurements, but does not adjust the carrier frequency of the wave used in the propagation calculations.

The first six differential equations (9) through (14) are always integrated. The user can choose whether to have the program integrate (15) to calculate the frequency shift.

There are three other quantities that can be calculated by integration along the ray path. The phase path P (phase divided by the free space wavenumber $2\pi/\lambda_o = \omega/c$) is calculated by integrating

$$\frac{dP}{dP'} = \frac{c}{\omega}\left(k_r \frac{dr}{dP'} + k_\theta\, r \frac{d\theta}{dP'} + k_\varphi\, r \sin\theta \frac{d\varphi}{dP'}\right)$$

$$= -\frac{1}{\omega} \frac{k_r \frac{\partial H}{\partial k_r} + k_\theta \frac{\partial H}{\partial k_\theta} + k_\varphi \frac{\partial H}{\partial k_\varphi}}{\partial H/\partial \omega} \quad . \qquad (16)$$

If the absorption per wavelength is small (as it must be for this type of ray tracing to be valid), then an approximate formula can be integrated

to give the absorption in decibels

$$\frac{dA}{dP'} = -\frac{10}{\log_e 10}\,\frac{\omega}{c}\,\frac{\text{imag}\left(\frac{\omega^2}{c^2}\,n^2\right)}{k_r^2 + k_\theta^2 + k_\varphi^2}\,\frac{dP}{dP'}$$

$$= \frac{10}{\log_e 10}\,\frac{\text{imag}\left(\frac{\omega^2}{c^2}\,n^2\right)}{k_r^2 + k_\theta^2 + k_\varphi^2}\,\frac{k_r\frac{\partial H}{\partial k_r} + k_\theta\frac{\partial H}{\partial k_\theta} + k_\varphi\frac{\partial H}{\partial k_\varphi}}{c\;\partial H/\partial\omega}\,, \qquad (17)$$

where n is the (complex) phase refractive index. The geometrical path length of the ray can be calculated by integrating

$$\frac{ds}{dP'} = \sqrt{\left(\frac{dr}{dP'}\right)^2 + r^2\left(\frac{d\theta}{dP'}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dP'}\right)^2}$$

$$= -\frac{\sqrt{\left(\frac{\partial H}{\partial k_r}\right)^2 + \left(\frac{\partial H}{\partial k_\theta}\right)^2 + \left(\frac{\partial H}{\partial k_\varphi}\right)^2}}{c\;\partial H/\partial\omega}\,. \qquad (18)$$

The user can choose to have frequency shift, phase path, absorption, or path length calculated using equations (15), (16), (17), or (18) and printed by setting the appropriate value in the input W array. (W59, W57, W58, W60 in Table 2.)

If the user wants to add differential equations to the program, he can do so by modifying subroutine HAMLTN, which evaluates Hamilton's equations.

The Hamiltonian and its derivatives are calculated by one of the versions of subroutine RINDEX, which also calculates the phase refractive index and its derivatives.

4. CHOOSING AND CALCULATING THE HAMILTONIAN

Because Hamilton's equations guarantee that the Hamiltonian is constant along the ray path and because it is desirable to have the dispersion

relation satisfied at each point on the ray path, it is usual to write the dispersion relation in the form H = constant and choose that H as the Hamiltonian. Two problems arise. First, in a lossy medium the dispersion relation is complex, so that the resulting complex Hamiltonian gives ray paths having complex coordinates when used in Hamilton's equations. Second, in some cases some forms of the dispersion relation have computational advantages over others when used as a Hamiltonian.

Allowing the coordinates of the ray path to assume complex values is called ray tracing in complex space (Budden and Jull, 1964; Jones, 1970; Budden and Terry, 1971) which is the extension to three dimensions of the phase integral method (Budden, 1961). Ray tracing in complex space is necessary to calculate the propagation of LF radio waves in the D region of the ionosphere (Jones, 1970), and it may also be needed for some medium frequencies.

However, the effect of losses on the ray path of HF radio waves in the ionosphere is probably small, so that the only effect of losses is to attenuate the signal. For this case, then, it is desirable to find a prescription for calculating ray paths having real coordinates. Several methods exist for doing this, and except for computational difficulties, one is probably as good as another. One should recognize that along the ray path:

(1) the dispersion cannot be exactly satisfied, or

(2) Hamilton's equations cannot be satisfied, or

(3) both of the above.

In our program, we have chosen to keep Hamilton's equations and require only the real part of the dispersion relation to be satisfied, neglecting the imaginary part. Another approach (Suchy, 1972) is to alter Hamilton's equations so that the full complex dispersion relation is still satisfied along a ray path having real coordinates. We are

reasonably certain that for any situation in which Suchy's method gives significantly different answers from ours, neither method is valid; ray tracing in complex space or an equivalent method would then be required.

Three choices for the Hamiltonian illustrate the computational difficulties involved. Haselgrove (1954) used the following Hamiltonian

$$H = \frac{c}{\omega} \frac{(k_r^2 + k_\theta^2 + k_\varphi^2)^{\frac{1}{2}}}{\text{real}(n)} - 1 , \qquad (19)$$

which, except for the effects of errors in the numerical integration and the value of the independent variable, is equivalent to

$$H = 1 - \frac{\omega}{c} \frac{\text{real}(n)}{(k_r^2 + k_\theta^2 + k_\varphi^2)^{\frac{1}{2}}}$$

$$= \text{real}\left\{1 - \frac{\omega}{c} \frac{n}{(k_r^2 + k_\theta^2 + k_\varphi^2)^{\frac{1}{2}}}\right\} . \qquad (20)$$

There are eight versions of the subroutine RINDEX which calculate the Hamiltonian and its partial derivatives. (Eight versions allow the user to choose the Appleton-Hartree formula or the Sen-Wyller formula, and to include or ignore the earth's magnetic field and collisions.) Six of these versions (subroutines AHWFWC, AHWFNC, AHNFWC, AHNFNC, SWWF, and SWNF) use the following Hamiltonian:

$$H = \frac{1}{2}\left(\frac{c^2}{\omega^2}(k_r^2 + k_\theta^2 + k_\varphi^2) - \text{real}(n^2)\right)$$

$$= \text{real}\left\{\frac{1}{2}\left(\frac{c^2}{\omega^2}(k_r^2 + k_\theta^2 + k_\varphi^2) - n^2\right)\right\} , \qquad (21)$$

The other two versions (subroutines BQWFWC and BQWFNC) use as a Hamiltonian the real part of the quadratic equation whose solution is the Appleton-Hartree formula (Budden, 1961)

$$H = \text{real}\left\{\left[(U - X)\,U^2 - Y^2 U\right] c^4 k^4 + X(k \cdot Y)^2\, c^4 k^2 + \right.$$

$$+ \left[-2U(U - X)^2 + Y^2(2U - X)\right] c^2 k^2 \omega^2 - X(k \cdot Y)^2\, c^2 \omega^2 +$$

$$\left. + \left[(U - X)^2 - Y^2\right](U - X)\,\omega^4 \right\} \qquad (22)$$

except in or near free space (defined by $X < 0.1$) where they also use (21) as the Hamiltonian. In (22), $U = 1 - iZ$, and X, Y, and Z are the usual magnetoionic parameters.

In a lossy medium, the Hamiltonians in (20), (21), and (22) determine slightly different ray paths, but the differences are significant only when it is no longer valid to represent ray paths with coordinates that are real rather than complex. In fact, this is a weak criterion. The ray paths determined by these three Hamiltonians will become invalid before there are noticeable differences between the three ray paths. In a lossless medium, the above three Hamiltonians determine identical ray paths (except for integration errors).

For either a lossy or lossless medium, some of the above three Hamiltonians have computational difficulties. Special care must be taken in using (19) or (20) in an evanescent region (which is frequently necessary at or near vertical incidence because the numerical integration subroutine usually requires the evaluation of the differential equations not only on the ray path, but also at points near the ray path). For instance, in a lossless medium, real (n) is zero in an evanescent region, which leads to problems in (19) and (20). This problem will not arise in (21) because real (n^2) is well behaved in or at the boundary of an evanescent region, nor will it occur in using (22).

Neither (20) nor (21) (nor any other Hamiltonian based on the refractive index) will work for a ray passing through a spitze (Davies, 1965, p. 202) because the refractive index is indeterminate at a spitze, and

some of the derivatives of n diverge. So far, we have had no problems using (22) to calculate ray paths through a spitze with or without collisions.

However, the Hamiltonian in (22) will not work in or near free space because all of its derivatives are zero in free space. This problem is related to (22) not being able to distinguish between ordinary and extraordinary waves. To get started, the program uses (21) until the electron density is large enough that X is equal or greater than 1/10.

As far as we can tell, the AHWFWC (Appleton-Hartree, with field, with collisions) version of subroutine RINDEX has been made obsolete by the BQWFWC (Booker quartic, with field, with collisions) version. The latter will do everything the AHWFWC version will do and in addition it will calculate rays through spitzes. A few trial runs, however, indicate that AHWFWC runs about 30 percent faster than BQWFWC.

Similarly, the AHWFNC (Appleton-Hartree, with field, no collisions) version has been made obsolete by the BQWFNC version, which apparently runs just as fast as the AHWFNC version. We are continuing to include the AHWFNC version just in case there are undiscovered problems with the BQWFNC version.

In addition to the Appleton-Hartree formula, which is based on a constant collision frequency, the program also includes the generalized formula of Sen and Wyller (1960), which assumes a Maxwell-Boltzman distribution of electron energy and a collision frequency proportional to energy. Two versions of subroutine RINDEX use the Sen-Wyller formula for calculating the refractive index and the resulting Hamiltonian with its derivatives. These are SWWF, which includes the effects of the earth's magnetic field, and SWNF, which neglects the Earth's magnetic field. The SWWF version will probably not work for calculating rays through a spitze. It would be possible to make a version which used as

its Hamiltonian the quadratic equation whose solution is the Sen-Wyller formula for calculating rays through a spitze, but it is unlikely that we will ever do that.

The versions of subroutine RINDEX that use (21) for a Hamiltonian use the following formulas for calculating the derivatives of that Hamiltonian.

$$\frac{\partial H}{\partial t} = -n \frac{\partial n}{\partial t} \quad , \tag{23}$$

$$\frac{\partial H}{\partial r} = -n \frac{\partial n}{\partial r} \quad , \tag{24}$$

$$\frac{\partial H}{\partial \theta} = -n \frac{\partial n}{\partial \theta} \quad , \tag{25}$$

$$\frac{\partial H}{\partial \varphi} = -n \frac{\partial n}{\partial \varphi} \quad , \tag{26}$$

$$\frac{\partial H}{\partial \omega} = -\frac{n\, n'}{\omega} \quad , \tag{27}$$

$$\frac{\partial H}{\partial k_r} = \frac{c^2}{\omega^2} k_r - \frac{c}{\omega} n \frac{\partial n}{\partial V_r} \quad , \tag{28}$$

$$\frac{\partial H}{\partial k_\theta} = \frac{c^2}{\omega^2} k_\theta - \frac{c}{\omega} n \frac{\partial n}{\partial V_\theta} \quad , \tag{29}$$

$$\frac{\partial H}{\partial k_\varphi} = \frac{c^2}{\omega^2} k_\varphi - \frac{c}{\omega} n \frac{\partial n}{\partial V_\varphi} \quad , \tag{30}$$

$$\vec{k} \cdot \frac{\partial H}{\partial \vec{k}} = k_r \frac{\partial H}{\partial k_r} + k_\theta \frac{\partial H}{\partial k_\theta} + k_\varphi \frac{\partial H}{\partial k_\varphi} = n^2 \quad , \tag{31}$$

where n' is the group refractive index defined by

$$n' = n + f\frac{dn}{df} = n + \omega\frac{dn}{d\omega} \quad , \tag{32}$$

and V_r, V_θ, and V_φ are the components of the wave normal direction in the r, θ, and φ directions normalized so that

$$V_r^2 + V_\theta^2 + V_\varphi^2 = \text{Real}\{n^2\} \quad . \tag{33}$$

The derivatives of the Hamiltonian in (22) are given in section 5. 5.

5. REFRACTIVE INDEX EQUATIONS

The refractive index equations used in this ray tracing program are based either on the Appleton-Hartree formula (Budden, 1961) or on the generalized formula of Sen and Wyller (1960). There are eight versions of SUBROUTINE RINDEX, the subroutine that calculates the refractive index and its gradient:

(1) Appleton-Hartree formula with field, with collisions.

(2) Appleton-Hartree formula with field, no collisions.

(3) Appleton-Hartree formula with collisions, no field.

(4) Appleton-Hartree formula no field, no collisions.

(5) Booker quartic with field, with collisions.

(6) Booker quartic with field, no collisions.

(7) Sen-Wyller formula with field.

(8) Sen-Wyller formula, no field.

Each of these eight versions calculates n^2, nn', $n\,\partial n/\partial r$, $n\,\partial n/\partial\theta$, $n\,\partial n/\partial\varphi$, $n\,\partial n/\partial V_r$, $n\,\partial n/\partial V_\theta$, $n\,\partial n/\partial V_\varphi$, $n\,\partial n/\partial t$, and the polarization, where n is the complex phase refractive index; n' is the complex group refractive index; r, θ, and φ are the spherical polar coordinates of a point on the ray path, and V_r, V_θ, and V_φ are the components of the wave normal direction in the r, θ, and φ directions. The quantities