$$n' = n + f\frac{dn}{df} = n + \omega\frac{dn}{d\omega} \quad , \tag{32}$$

and V_r, V_θ, and V_φ are the components of the wave normal direction in the r, θ, and φ directions normalized so that

$$V_r^2 + V_\theta^2 + V_\varphi^2 = \text{Real}\,\{n^2\} \quad . \tag{33}$$

The derivatives of the Hamiltonian in (22) are given in section 5. 5.

5. REFRACTIVE INDEX EQUATIONS

The refractive index equations used in this ray tracing program are based either on the Appleton-Hartree formula (Budden, 1961) or on the generalized formula of Sen and Wyller (1960). There are eight versions of SUBROUTINE RINDEX, the subroutine that calculates the refractive index and its gradient:

(1) Appleton-Hartree formula with field, with collisions.
(2) Appleton-Hartree formula with field, no collisions.
(3) Appleton-Hartree formula with collisions, no field.
(4) Appleton-Hartree formula no field, no collisions.
(5) Booker quartic with field, with collisions.
(6) Booker quartic with field, no collisions.
(7) Sen-Wyller formula with field.
(8) Sen-Wyller formula, no field.

Each of these eight versions calculates n^2, nn', $n\,\partial n/\partial r$, $n\,\partial n/\partial \theta$, $n\,\partial n/\partial \varphi$, $n\,\partial n/\partial V_r$, $n\,\partial n/\partial V_\theta$, $n\,\partial n/\partial V_\varphi$, $n\,\partial n/\partial t$, and the polarization, where n is the complex phase refractive index; n' is the complex group refractive index; r, θ, and φ are the spherical polar coordinates of a point on the ray path, and V_r, V_θ, and V_φ are the components of the wave normal direction in the r, θ, and φ directions. The quantities

X, $\partial X/\partial r$, $\partial X/\partial\theta$, $\partial X/\partial\varphi$, and $\partial X/\partial t$ are supplied by one of the versions of subroutine ELECTX which defines the electron density model. The quantities Y, $\partial Y/\partial r$, $\partial Y/\partial\theta$, $\partial Y/\partial\varphi$, Y_r, $\partial Y_r/\partial r$, $\partial Y_r/\partial\theta$, $\partial Y_r/\partial\varphi$, Y_θ, $\partial Y_\theta/\partial r$, $\partial Y_\theta/\partial\theta$, $\partial Y_\varphi/\partial\varphi$, Y_φ, $\partial Y_\varphi/\partial r$, $\partial Y_\varphi/\partial\theta$, and $\partial Y_\varphi/\partial\varphi$ are supplied by one of the versions of subroutine MAGY which defines the magnetic field model. The quantities Z, $\partial Z/\partial r$, $\partial Z/\partial\theta$, and $\partial Z/\partial\varphi$ are supplied by one of the versions of subroutine COLFRZ which defines the collision frequency model.

In our formulation, we have tried to avoid using multivalued functions, such as the square root or $\cos^{-1}$, wherever possible. Only twice do we use the square root. One instance is the square root in the Appleton-Hartree formula, unavoidable without adding more differential equations to the system. The second instance is a square root used to calculate polarization. This latter use is unimportant because the polarization is not used in the ray tracing equations.

It is desirable to avoid multivalued functions because, unless extreme care is used, the value of such a function can change discontinuously from one point on the ray path to the next. A particularly troublesome case occurs at reflection for vertical incidence. At that point, the real part of n^2 goes through zero, and n changes from approximately purely real to approximately purely imaginary. Since the numerical integration subroutine usually requires the evaluation of the differential equations not only on the ray path, but also at points near the ray path, it is necessary to be able to evaluate the differential equations above the reflection height, that is, in an evanescent region.

We have found that it is possible to regroup the variables in the equations to avoid this problem: we calculate the real part of n^2 and its derivatives instead of the real part of n and its derivatives. And we calculate nn' instead of n'.

It was not easy to avoid using multivalued functions, however. Many of the usual parameters used to compute the refractive index require the use of multivalued functions in their calculation. Thus, we couldn't calculate

$$V = \sqrt{V_r^2 + V_\theta^2 + V_\varphi^2}$$

nor cos ψ, nor sin ψ, where ψ is the angle between the wave normal direction and the earth's magnetic field. Thus, we also could not calculate

$$Y_L = Y \cos \psi$$

nor

$$Y_T = Y \sin \psi .$$

The most difficult part of avoiding the use of multivalued functions was in calculating the derivatives.

The following is a list of the equations calculated by the eight versions of subroutine RINDEX.

5.1 Appleton-Hartree Formula with Field, with Collisions

The square of the complex phase refractive index is given by

$$n^2 = 1 - 2X \frac{1-iZ-X}{2(1-iZ)(1-iZ-X) - Y_T^2 \pm \sqrt{Y_T^4 + 4Y_L^2(1-iZ-X)^2}} , \tag{34}$$

where

$$X = \frac{f_N^2}{f^2} , \tag{35}$$

$$Y = \frac{f_H}{f} , \tag{36}$$

$$Z = \frac{\nu}{2\pi f} , \tag{37}$$

$$Y_T = Y \sin \psi , \tag{38}$$

$$Y_L = Y \cos \psi , \tag{39}$$

f_N is the plasma frequency, f_H is the electron gyrofrequency, ν is the electron collision frequency, f is the wave frequency, and ψ is the angle between the wave normal direction and the earth's magnetic field.

The following equations parallel the formulas in this version of RINDEX.

$$V^2 = V_r^2 + V_\theta^2 + V_\varphi^2 \tag{40}$$

$$V \cdot Y = V_r Y_r + V_\theta Y_\theta + V_\varphi Y_\varphi \tag{41}$$

$$\frac{Y_L}{V} = \frac{V \cdot Y}{V^2} \tag{42}$$

$$Y_L^2 = \frac{(V \cdot Y)^2}{V^2} \tag{43}$$

$$Y_T^2 = Y^2 - Y_L^2 \tag{44}$$

$$Y_T^4 = \left(Y_T^2\right)^2 \tag{45}$$

$$U = 1 - iZ \tag{46}$$

$$RAD = \pm\sqrt{Y_T^4 + 4Y_L^2 \ (U-X)^2} \tag{47}$$

$$D = 2U(U-X) - Y_T^2 + RAD \tag{48}$$

$$n^2 = 1 - \frac{2X(U-X)}{D} \tag{49}$$

$$\frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} = \frac{2X(U-X)\left(-1 + \dfrac{Y_T^2 - 2(U-X)^2}{RAD}\right)}{D^2} \tag{50}$$

$$Y_T Y_L \frac{\partial \psi}{\partial r} = \frac{Y_L^2}{Y} \frac{\partial Y}{\partial r} - \left(V_r \frac{\partial Y_r}{\partial r} + V_\theta \frac{\partial Y_\theta}{\partial r} + V_\varphi \frac{\partial Y_\varphi}{\partial r} \right) \left(\frac{Y_L}{V} \right) \tag{51}$$

$$Y_T Y_L \frac{\partial \psi}{\partial \theta} = \frac{Y_L^2}{Y} \frac{\partial Y}{\partial \theta} - \left(V_r \frac{\partial Y_r}{\partial \theta} + V_\theta \frac{\partial Y_\theta}{\partial \theta} + V_\varphi \frac{\partial Y_\varphi}{\partial \theta} \right) \left(\frac{Y_L}{V} \right) \tag{52}$$

$$Y_T Y_L \frac{\partial \psi}{\partial \varphi} = \frac{Y_L^2}{Y} \frac{\partial Y}{\partial \varphi} - \left(V_r \frac{\partial Y_r}{\partial \varphi} + V_\theta \frac{\partial Y_\theta}{\partial \varphi} + V_\varphi \frac{\partial Y_\varphi}{\partial \varphi} \right) \left(\frac{Y_L}{V} \right) \tag{53}$$

$$n \frac{\partial n}{\partial X} = - \frac{\left(2U(U-X)^2 - Y_T^2 (U-2X) + \frac{Y_T^4 (U-2X) + 4Y_L^2 (U-X)^3}{RAD} \right)}{D^2} \tag{54}$$

$$n \frac{\partial n}{\partial Y} = \frac{2X(U-X)}{D^2\ Y} \left(-Y_T^2 + \frac{Y_T^4 + 2Y_L^2 (U-X)^2}{RAD} \right) \tag{55}$$

$$n \frac{\partial n}{\partial Z} = \frac{iX}{D^2} \left(-2(U-X)^2 - Y_T^2 + \frac{Y_T^4}{RAD} \right) \tag{56}$$

$$n \frac{\partial n}{\partial r} = n \frac{\partial n}{\partial X} \frac{\partial X}{\partial r} + n \frac{\partial n}{\partial Y} \frac{\partial Y}{\partial r} + n \frac{\partial n}{\partial Z} \frac{\partial Z}{\partial r} + \frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} Y_L Y_T \frac{\partial \psi}{\partial r} \tag{57}$$

$$n \frac{\partial n}{\partial \theta} = n \frac{\partial n}{\partial X} \frac{\partial X}{\partial \theta} + n \frac{\partial n}{\partial Y} \frac{\partial Y}{\partial \theta} + n \frac{\partial n}{\partial Z} \frac{\partial Z}{\partial \theta} + \frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} Y_T Y_L \frac{\partial \psi}{\partial \theta} \tag{58}$$

$$n \frac{\partial n}{\partial \varphi} = n \frac{\partial n}{\partial X} \frac{\partial X}{\partial \varphi} + n \frac{\partial n}{\partial Y} \frac{\partial Y}{\partial \varphi} + n \frac{\partial n}{\partial Z} \frac{\partial Z}{\partial \varphi} + \frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} Y_T Y_L \frac{\partial \psi}{\partial \varphi} \tag{59}$$

$$n \frac{\partial n}{\partial V_r} = \frac{n}{Y_T Y_L} \frac{\partial n}{\partial \psi} \left(\frac{V_r Y_L^2}{V^2} - \left(\frac{Y_L}{V} \right) Y_r \right) \tag{60}$$

$$n \frac{\partial n}{\partial V_\theta} = \frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} \left(\frac{V_\theta Y_L^2}{V^2} - \left(\frac{Y_L}{V} \right) Y_\theta \right) \tag{61}$$

$$n \frac{\partial n}{\partial V_\varphi} = \frac{n}{Y_L Y_T} \frac{\partial n}{\partial \psi} \left(\frac{V_\varphi Y_L^2}{V^2} - \left(\frac{Y_L}{V} \right) Y_\varphi \right) \tag{62}$$

$$nn' = n^2 - \left(2Xn \frac{\partial n}{\partial X} + Y\, n \frac{\partial n}{\partial Y} + Z\, n \frac{\partial n}{\partial Z} \right) \tag{63}$$

$$\text{Polarization} = \rho = -i \frac{\left(-Y_T^2 + RAD\right)\sqrt{V^2}}{2(U-X)\, V \cdot Y} \tag{64a}$$

(Budden, 1961, page 49).

$$\text{Longitudinal polarization} = \frac{iX\sqrt{Y_T^2}}{(U-X)\left(U+i \dfrac{V \cdot Y}{\sqrt{V^2}}\ \rho\right)} \tag{64b}$$

(Budden, 1961, page 54).

$$n \frac{\partial n}{\partial t} = n \frac{\partial n}{\partial X} \frac{\partial X}{\partial t} \tag{65}$$

5.2 Appleton-Hartree Formula with Field, no Collisions

The equations are the same as with field with collisions except for:

$$Z = \frac{\partial Z}{\partial r} = \frac{\partial Z}{\partial \theta} = \frac{\partial Z}{\partial \varphi} = 0, \tag{66}$$

$$U = 1. \tag{67}$$

5.3 Appleton-Hartree Formula no Field, With Collisions

$$U = 1 - iZ \tag{68}$$

$$n^2 = 1 - \frac{X}{U} \tag{69}$$

$$n\frac{\partial n}{\partial X} = -\frac{.5}{U} \tag{70}$$

$$n\frac{\partial n}{\partial Z} = -\frac{.5iX}{U^2} \tag{71}$$

$$nn' = n^2 - \left(2Xn\frac{\partial n}{\partial X} + Zn\frac{\partial n}{\partial Z}\right) \tag{72}$$

$$n\frac{\partial n}{\partial r} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial r} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial r} \tag{73}$$

$$n\frac{\partial n}{\partial \theta} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial \theta} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial \theta} \tag{74}$$

$$n\frac{\partial n}{\partial \varphi} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial \varphi} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial \varphi} \tag{75}$$

$$n\frac{\partial n}{\partial V_r} = 0 \tag{76}$$

$$n\frac{\partial n}{\partial V_\theta} = 0 \tag{77}$$

$$n\frac{\partial n}{\partial V_\varphi} = 0 \tag{78}$$

$$\text{Polarization} = i \tag{79a}$$

$$\text{Longitudinal polarization} = 0 \tag{79b}$$

$$n\frac{\partial n}{\partial t} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial t} \tag{80}$$

5.4 Appleton-Hartree Formula no Field, no Collisions

$$n^2 = 1 - X \tag{81}$$

$$n\frac{\partial n}{\partial X} = -.5 \tag{82}$$

$$n\frac{\partial n}{\partial r} = \left(n\frac{\partial n}{\partial X}\right)\left(\frac{\partial X}{\partial r}\right) \tag{83}$$

$$n\frac{\partial n}{\partial \theta} = \left(n\frac{\partial n}{\partial X}\right)\frac{\partial X}{\partial \theta} \tag{84}$$

$$n\frac{\partial n}{\partial \varphi} = \left(n\frac{\partial n}{\partial X}\right)\frac{\partial X}{\partial \varphi} \tag{85}$$

$$n\frac{\partial n}{\partial V_r} = 0 \tag{86}$$

$$n\frac{\partial n}{\partial V_\theta} = 0 \tag{87}$$

$$n\frac{\partial n}{\partial V_\varphi} = 0 \tag{88}$$

$$nn' = 1 \tag{89}$$

$$\text{Polarization} = i \tag{90a}$$

$$\text{Longitudinal polarization} = 0 \tag{90b}$$

$$n\frac{\partial n}{\partial t} = n\frac{\partial n}{\partial X}\,\frac{\partial X}{\partial t} \tag{91}$$

5.5 Booker Quartic with Field, with Collisions

The form of the dispersion relation used for the Hamiltonian in this version of subroutine RINDEX is the quadratic equation whose solution is the Appleton-Hartree formula. This Hamiltonian, given by

(22), is also a special case of the Booker quartic for S = 0 and C = 1 (Budden, 1961). This version uses the Hamiltonian in (22) only when the electron density is large enough that X is greater than or equal to 1/10. For X less than 1/10, the Hamiltonian in (21) is used.

Below are the equations for the derivatives of the Hamiltonian in (22). The equations for the derivatives of the Hamiltonian in (21) are the same as those in section 4.1.

$$k^2 = k_r^2 + k_\theta^2 + k_\varphi^2 \ , \qquad (92)$$

$$k \cdot Y = k_r Y_r + k_\theta Y_\theta + k_\varphi Y_\varphi \ , \qquad (93)$$

$$U = 1 - iZ \ , \qquad (94)$$

$$A = (U - X)\, U^2 - UY^2 \ , \qquad (95)$$

$$B = -2U(U - X)^2 + Y^2(2U - X) \ , \qquad (96)$$

$$\alpha = A\, c^4 k^4 + X(k \cdot Y)^2\, c^4 k^2 \ , \qquad (97)$$

$$\beta = B\, c^2 k^2 \omega^2 - X(k \cdot Y)^2\, c^2 \omega^2 \ , \qquad (98)$$

$$\gamma = ((U - X)^2 - Y^2)(U - X)\, \omega^4 \ , \qquad (99)$$

$$H = \alpha + \beta + \gamma \ , \qquad (100)$$

$$\frac{\partial H}{\partial X} = -U^2 c^4 k^4 + (k \cdot Y)^2 c^4 k^2 + (4U(U - X) - Y^2) c^2 k^2 \omega^2 +$$

$$-(k \cdot Y)^2 c^2 \omega^2 + (-3(U - X)^2 + Y^2)\, \omega^4 \ , \qquad (101)$$

$$\frac{\partial H}{\partial(Y^2)} = -U\, c^4k^4 + (2U - X)\, c^2k^2\omega^2 - (U - X)\,\omega^4 \quad , \tag{102}$$

$$\frac{\partial H}{\partial((k\cdot Y)^2)} = X\, c^2(c^2k^2 - \omega^2) \quad , \tag{103}$$

$$\begin{aligned}\frac{\partial H}{\partial U} &= (2\,U(U - X) + U^2 - Y^2)\, c^4k^4 + \\ &\quad + 2(Y^2 - (U - X)^2 - 2U(U - X))\, c^2k^2\omega^2 + \\ &\quad + (3(U - X)^2 - Y^2)\,\omega^4 \quad ,\end{aligned} \tag{104}$$

$$\frac{\partial H}{\partial Z} = -i\,\frac{\partial H}{\partial U} \quad , \tag{105}$$

$$\frac{\partial H}{\partial(k^2)} = 2\,A\, c^4k^2 + X(k\cdot Y)^2\, c^4 + Bc^2\omega^2 \quad , \tag{106}$$

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial X}\,\frac{\partial X}{\partial t} \quad , \tag{107}$$

$$\begin{aligned}\frac{\partial H}{\partial r} &= \frac{\partial H}{\partial X}\,\frac{\partial X}{\partial r} + 2\,\frac{\partial H}{\partial(Y^2)}\; Y\,\frac{\partial Y}{\partial r} + \frac{\partial H}{\partial Z}\,\frac{\partial Z}{\partial r} + \\ &\quad + 2\,\frac{\partial H}{\partial((k\cdot Y)^2)}\,(k\cdot Y)\left(k_r\,\frac{\partial Y_r}{\partial r} + k_\theta\,\frac{\partial Y_\theta}{\partial r} + k_\varphi\,\frac{\partial Y_\varphi}{\partial r}\right) \quad ,\end{aligned} \tag{108}$$

$$\begin{aligned}\frac{\partial H}{\partial \theta} &= \frac{\partial H}{\partial X}\,\frac{\partial X}{\partial \theta} + 2\,\frac{\partial H}{\partial(Y^2)}\; Y\,\frac{\partial Y}{\partial \theta} + \frac{\partial H}{\partial Z}\,\frac{\partial Z}{\partial \theta} + \\ &\quad + 2\,\frac{\partial H}{\partial((k\cdot Y)^2)}\,(k\cdot Y)\left(k_r\,\frac{\partial Y_r}{\partial \theta} + k_\theta\,\frac{\partial Y_\theta}{\partial \theta} + k_\varphi\,\frac{\partial Y_\varphi}{\partial \theta}\right) \quad ,\end{aligned} \tag{109}$$

$$\frac{\partial H}{\partial \varphi} = \frac{\partial H}{\partial X}\frac{\partial X}{\partial \varphi} + 2\frac{\partial H}{\partial (Y^2)} Y \frac{\partial Y}{\partial \varphi} + \frac{\partial H}{\partial Z}\frac{\partial Z}{\partial \varphi} +$$

$$+ 2\frac{\partial H}{\partial ((k \cdot Y)^2)}(k \cdot Y)\left(k_r \frac{\partial Y_r}{\partial \varphi} + k_\theta \frac{\partial Y_\theta}{\partial \varphi} + k_\varphi \frac{\partial Y_\varphi}{\partial \varphi}\right) , \quad (110)$$

$$\frac{\partial H}{\partial \omega} = (2\beta + 4\gamma)/\omega - 2\frac{\partial H}{\partial X}\frac{X}{\omega} - 2\frac{\partial H}{\partial (Y^2)}\frac{Y^2}{\omega} +$$

$$- 2\frac{\partial H}{\partial ((k \cdot Y)^2)}\frac{(k \cdot Y)^2}{\omega} - \frac{\partial H}{\partial Z}\frac{Z}{\omega} , \quad (111)$$

$$\frac{\partial H}{\partial k_r} = 2\frac{\partial H}{\partial (k^2)} k_r + 2(k \cdot Y)\frac{\partial H}{\partial ((k \cdot Y)^2)} Y_r , \quad (112)$$

$$\frac{\partial H}{\partial k_\theta} = 2\frac{\partial H}{\partial (k^2)} k_\theta + 2(k \cdot Y)\frac{\partial H}{\partial ((k \cdot Y)^2)} Y_\theta , \quad (113)$$

$$\frac{\partial H}{\partial k_\varphi} = 2\frac{\partial H}{\partial (k^2)} k_\varphi + 2(k \cdot Y)\frac{\partial H}{\partial ((k \cdot Y)^2)} Y_\varphi , \quad (114)$$

$$k^2(\text{calculated}) = k^2 \; \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} , \quad (115)$$

$$k \cdot \frac{\partial H}{\partial k} = k_r \frac{\partial H}{\partial k_r} + k_\theta \frac{\partial H}{\partial k_\theta} + k_\varphi \frac{\partial H}{\partial k_\varphi} = 4\alpha + 2\beta , \quad (116)$$

$$\text{Polarization} = i\,\frac{\sqrt{k^2}}{k \cdot Y}\left(U + \frac{X\omega^2}{c^2 k^2_{\text{calculated}} - \omega^2}\right) , \quad (117)$$

$$\begin{array}{l}\text{Longitudinal} \\ \text{Polarization}\end{array} = i\,\frac{\sqrt{Y^2 - \frac{(k \cdot Y)^2}{k^2}}}{U - X}\left(1 - \frac{c^2 k^2_{\text{calculated}}}{\omega^2}\right) . \quad (118)$$

5.6 Booker Quartic with Field, no Collisions

All the equations here are the same as for the Booker quartic version with collisions (section 5.5) except for:

$$Z = \frac{\partial Z}{\partial r} = \frac{\partial Z}{\partial \theta} = \frac{\partial Z}{\partial \varphi} = 0 \ , \qquad (119)$$

$$U = 1. \qquad (120)$$

All the variables except polarization are real; polarization is pure imaginary.

5.7 Sen-Wyller Formula with Field

It is possible to write the generalized Appleton-Hartree formula of Sen and Wyller (1960) in the following form:

$$n^2 = 1 - \frac{2X(U-X) + 2\ AUX\ \sin^2\psi}{2U(U-X)(1+A) + 2\ AUX\ \sin^2\psi - U((1-BC)U + A(U+X))\ \sin^2\psi + RAD} \ , \qquad (121)$$

where

$$RAD = \pm\sqrt{U^2((1-BC)U + A(U+X))^2\ \sin^4\psi + U^2(U-X)^2(C-B)^2\ \cos^2\psi} \ , \qquad (122)$$

$$A = \frac{C+B}{2} - 1 \ , \qquad (123)$$

$$B = \frac{F\left(\frac{1}{Z}\right)}{F\left(\frac{1-Y}{Z}\right)} \ , \qquad (124)$$

$$C = \frac{F\left(\frac{1}{Z}\right)}{F\left(\frac{1+Y}{Z}\right)} , \tag{125}$$

$$U = \frac{Z}{F\left(\frac{1}{Z}\right)} \tag{126}$$

$$F(w) = w\, C_{3/2}(w) + i\,\frac{5}{2}\, C_{5/2}(w) = \frac{1}{\frac{3}{2}\,!} \int_{0}^{\infty} \frac{t^{3/2} e^{-t} dt}{w-it} , \tag{127}$$

$$X = \frac{f_N^2}{f^2} , \tag{128}$$

$$Y = \frac{f_H}{f} , \tag{129}$$

$$Z = \frac{\nu_m}{2\pi f} , \tag{130}$$

f_N is the plasma frequency, f_H is the electron gyrofrequency, ν_m is the mean electron collision frequency, f is the wave frequency, and ψ is the angle between the wave normal direction and the earth's magnetic field. (Note that if we would use

$$F(w) = \frac{1}{w-i} , \tag{131}$$

then (121) would reduce to the usual Appleton-Hartree formula.)

This version of RINDEX calls subroutine FSW. The first argument in the calling sequence is the argument w of F(w). The second argument in the calling sequence is the value of the function F(w) calculated by FSW. The third argument in the calling sequence is the derivative of the function, F'(w).

The following equations parallel the formulas in this version of RINDEX.

$$\alpha = F\left(\frac{1}{Z}\right) \tag{132}$$

$$\alpha' = F'\left(\frac{1}{Z}\right) \tag{133}$$

$$\beta = F\left(\frac{1-Y}{Z}\right) \tag{134}$$

$$\beta' = F'\left(\frac{1-Y}{Z}\right) \tag{135}$$

$$\gamma = F\left(\frac{1+Y}{Z}\right) \tag{136}$$

$$\gamma' = F'\left(\frac{1+Y}{Z}\right) \tag{137}$$

$$U = \frac{Z}{\alpha} \tag{138}$$

$$\frac{\partial U}{\partial Z} = \left(1 + \frac{\alpha'}{\alpha Z}\right) \Big/ \alpha \tag{139}$$

$$B = \frac{\alpha}{\beta} \tag{140}$$

$$\frac{\partial B}{\partial Y} = \frac{B}{Z} \ \frac{\beta'}{\beta} \tag{141}$$

$$\frac{\partial B}{\partial Z} = -\frac{B}{Z^2}\left(\frac{\alpha'}{\alpha} - (1-Y)\ \frac{\beta'}{\beta}\right) \tag{142}$$

$$C = \frac{\alpha}{\gamma} \tag{143}$$

$$\frac{\partial C}{\partial Y} = - \frac{C}{Z} \frac{\gamma'}{\gamma} \tag{144}$$

$$\frac{\partial C}{\partial Z} = - \frac{C}{Z^2} \left(\frac{\alpha'}{\alpha} - (1+Y) \frac{\gamma'}{\gamma} \right) \tag{145}$$

$$A = .5(B+C) - 1 \tag{146}$$

$$\frac{\partial A}{\partial Y} = .5 \left(\frac{\partial B}{\partial Y} + \frac{\partial C}{\partial Y} \right) \tag{147}$$

$$\frac{\partial A}{\partial Z} = .5 \left(\frac{\partial B}{\partial Z} + \frac{\partial C}{\partial Z} \right) \tag{148}$$

$$V^2 = V_r^2 + V_\theta^2 + V_\varphi^2 \tag{149}$$

$$V \cdot Y = V_r Y_r + V_\theta Y_\theta + V_\varphi Y_\varphi \tag{150}$$

$$Y_L^2 = \frac{(V \cdot Y)^2}{V^2} \tag{151}$$

$$Y_T^2 = Y^2 - Y_L^2 \tag{152}$$

$$\sin^2 \psi = \frac{Y_T^2}{Y^2} \tag{153}$$

$$\cos^2 \psi = \frac{Y_L^2}{Y^2} \tag{154}$$

$$T_1 = \left[(1-BC)\,U^2 + A\,U(U+X)\right] \sin^2\psi \qquad (155)$$

$$\frac{\partial T_1}{\partial X} = +AU \sin^2\psi \qquad (156)$$

$$\frac{\partial T_1}{\partial Y} = \left(U(U+X)\frac{\partial A}{\partial Y} - U^2 \left(B\frac{\partial C}{\partial Y} + C\frac{\partial B}{\partial Y} \right)\right) \sin^2\psi \qquad (157)$$

$$\frac{\partial T_1}{\partial Z} = \left(2U\frac{\partial U}{\partial Z}(1-BC+A) + A\,X\frac{\partial U}{\partial Z} - U^2\left(B\frac{\partial C}{\partial Z} + C\frac{\partial B}{\partial Z}\right) + U(U+X)\frac{\partial A}{\partial Z}\right) \sin^2\psi \qquad (158)$$

$$\frac{1}{Y_L Y_T}\,\frac{\partial T_1}{\partial \psi} = \frac{2T_1}{Y_T^2} \qquad (159)$$

$$T_2 = U^2\,(C-B)^2\,(U-X)^2\,\cos^2\psi \qquad (160)$$

$$\frac{\partial T_2}{\partial X} = -2(U-X)\,U^2\,(C-B)^2\cos^2\psi \qquad (161)$$

$$\frac{\partial T_2}{\partial Y} = 2U^2(U-X)^2\,\cos^2\psi\,(C-B)\left(\frac{\partial C}{\partial Y} - \frac{\partial B}{\partial Y}\right) \qquad (162)$$

$$\frac{\partial T_2}{\partial Z} = 2U^2\,(U-X)^2(C-B)\left(\frac{\partial C}{\partial Z} - \frac{\partial B}{\partial Z}\right)\cos^2\psi + 2T_2\left(\frac{1}{U} + \frac{1}{U-X}\,\frac{\partial U}{\partial Z}\right) \qquad (163)$$

$$\frac{1}{Y_L Y_T}\,\frac{\partial T_2}{\partial \psi} = -\,\frac{2T_2}{Y_L^2} \qquad (164)$$

$$RAD = \pm\sqrt{T_1^2 + T_2} \qquad (165)$$

$$\frac{\partial RAD}{\partial X} = \frac{T_1 \frac{\partial T_1}{\partial X} + \frac{1}{2} \frac{\partial T_2}{\partial X}}{RAD} \tag{166}$$

$$\frac{\partial RAD}{\partial Y} = \frac{T_1 \frac{\partial T_1}{\partial Y} + \frac{1}{2} \frac{\partial T_2}{\partial Y}}{RAD} \tag{167}$$

$$\frac{\partial RAD}{\partial Z} = \frac{T_1 \frac{\partial T_1}{\partial Z} + \frac{1}{2} \frac{\partial T_2}{\partial Z}}{RAD} \tag{168}$$

$$\frac{1}{Y_L Y_T} \frac{\partial RAD}{\partial \psi} = \frac{T_1 \left(\frac{1}{Y_L Y_T} \frac{\partial T_1}{\partial \psi} \right) + \frac{1}{2} \left(\frac{1}{Y_L Y_T} \frac{\partial T_2}{\partial \psi} \right)}{RAD} \tag{169}$$

$$D = 2U(U-X)(1+A) - T_1 + RAD + 2AUX \sin^2 \psi \tag{170}$$

$$\frac{\partial D}{\partial X} = -2U - \frac{\partial T_1}{\partial X} + \frac{\partial RAD}{\partial X} + 2AU \sin^2 \psi \tag{171}$$

$$\frac{\partial D}{\partial Y} = 2U(U-X)\frac{\partial A}{\partial Y} - \frac{\partial T_1}{\partial Y} + \frac{\partial RAD}{\partial Y} + 2U \sin^2 \psi \frac{\partial A}{\partial Y} \tag{172}$$

$$\frac{\partial D}{\partial Z} = 2(1+A)\frac{\partial U}{\partial Z}(2U-X) + 2U(U-X)\frac{\partial A}{\partial Z} - \frac{\partial T_1}{\partial Z} + \frac{\partial RAD}{\partial Z} + \\ + 2AX \sin^2 \psi \frac{\partial U}{\partial Z} + 2UX \sin^2 \psi \frac{\partial A}{\partial Z} \tag{173}$$

$$\frac{1}{Y_L Y_T} \frac{\partial D}{\partial \psi} = -\left(\frac{1}{Y_L Y_T} \frac{\partial T_1}{\partial \psi} \right) + \left(\frac{1}{Y_L Y_T} \frac{\partial RAD}{\partial \psi} \right) + 2AUX/Y^2 \tag{174}$$

$$n^2 - 1 = \frac{-2X}{D}\left((U-X) + UA\sin^2\psi \right) \tag{175}$$

$$n^2 = 1 + (n^2 - 1) \tag{176}$$

$$n\frac{\partial n}{\partial X} = \frac{1}{2}(n^2 - 1)\left(\frac{1}{X} - \frac{1}{D}\frac{\partial D}{\partial X}\right) + \frac{X}{D} \tag{177}$$

$$n\frac{\partial n}{\partial Y} = -\frac{XU\sin^2\psi}{D}\frac{\partial A}{\partial Y} - \frac{(n^2-1)}{2D}\frac{\partial D}{\partial Y} \tag{178}$$

$$n\frac{\partial n}{\partial Z} = -\frac{X}{D}(1+A\sin^2\psi)\frac{\partial U}{\partial Z} - \frac{XU}{D}\sin^2\psi\frac{\partial A}{\partial Z} - \frac{(n^2-1)}{2D}\frac{\partial D}{\partial Z} \tag{179}$$

$$\frac{n}{Y_T Y_L}\frac{\partial n}{\partial\psi} = -\frac{XUA}{Y^2 D} - \frac{(n^2-1)}{2D}\left(\frac{1}{Y_L Y_T}\frac{\partial D}{\partial\psi}\right) \tag{180}$$

$$Y_L Y_T\frac{\partial\psi}{\partial r} = \frac{Y_L^2}{Y}\frac{\partial Y}{\partial r} - \left(V_r\frac{\partial Y_r}{\partial r} + V_\theta\frac{\partial Y_\theta}{\partial r} + V_\varphi\frac{\partial Y_\varphi}{\partial r}\right)\frac{V\cdot Y}{V^2} \tag{181}$$

$$Y_L Y_T\frac{\partial\psi}{\partial\theta} = \frac{Y_L^2}{Y}\frac{\partial Y}{\partial\theta} - \left(V_r\frac{\partial Y_r}{\partial\theta} + V_\theta\frac{\partial Y_\theta}{\partial\theta} + V_\varphi\frac{\partial Y_\varphi}{\partial\theta}\right)\frac{V\cdot Y}{V^2} \tag{182}$$

$$Y_L Y_T\frac{\partial\psi}{\partial\varphi} = \frac{Y_L^2}{Y}\frac{\partial Y}{\partial\varphi} - \left(V_r\frac{\partial Y_r}{\partial\varphi} + V_\theta\frac{\partial Y_\theta}{\partial\varphi} + V_\varphi\frac{\partial Y_\varphi}{\partial\varphi}\right)\frac{V\cdot Y}{V^2} \tag{183}$$

$$n\frac{\partial n}{\partial r} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial r} + n\frac{\partial n}{\partial Y}\frac{\partial Y}{\partial r} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial r} + \left(\frac{n}{Y_L Y_T}\frac{\partial n}{\partial\psi}\right)\left(Y_L Y_T\frac{\partial\psi}{\partial r}\right) \tag{184}$$

$$n\frac{\partial n}{\partial\theta} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial\theta} + n\frac{\partial n}{\partial Y}\frac{\partial Y}{\partial\theta} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial\theta} + \left(\frac{n}{Y_L Y_T}\frac{\partial n}{\partial\psi}\right)\left(Y_L Y_T\frac{\partial\psi}{\partial\theta}\right) \tag{185}$$

$$n\frac{\partial n}{\partial\varphi} = n\frac{\partial n}{\partial X}\frac{\partial X}{\partial\varphi} + n\frac{\partial n}{\partial Y}\frac{\partial Y}{\partial\varphi} + n\frac{\partial n}{\partial Z}\frac{\partial Z}{\partial\varphi} + \left(\frac{n}{Y_L Y_T}\frac{\partial n}{\partial\psi}\right)\left(Y_L Y_T\frac{\partial\psi}{\partial\varphi}\right) \tag{186}$$

$$n\frac{\partial n}{\partial V_r} = \left(\frac{n}{Y_L Y_T}\ \frac{\partial n}{\partial \psi}\right)\left(V_r Y_L^2 - (V\cdot Y)Y_r\right) \Big/ V^2 \tag{187}$$

$$n\frac{\partial n}{\partial V_\theta} = \left(\frac{n}{Y_L Y_T}\ \frac{\partial n}{\partial \psi}\right)\left(V_\theta Y_L^2 - (V\cdot Y)Y_\theta\right) \Big/ V^2 \tag{188}$$

$$n\frac{\partial n}{\partial V_\varphi} = \left(\frac{n}{Y_L Y_T}\ \frac{\partial n}{\partial \psi}\right)\left(V_\varphi Y_L^2 - (V\cdot Y)Y_\varphi\right) \Big/ V^2 \tag{189}$$

$$nn' = n^2 - \left(2X\left(n\frac{\partial n}{\partial X}\right) + Y\left(n\frac{\partial n}{\partial Y}\right) + Z\left(n\frac{\partial n}{\partial Z}\right)\right) \tag{190}$$

$$\text{polarization} = \rho = \frac{i\,(T_1 - RAD)\,Y\sqrt{V^2}}{U(U-X)(C-B)(V\cdot Y)} \tag{191a}$$

$$\text{longitudinal polarization} = \frac{X(.5i\,(C-B)\,\rho + A\cos\psi\,)\sqrt{\sin^2\psi}}{\rho((U-X)(1+.5i(C-B)\rho\cos\psi)+A(U-X\cos^2\psi))} \tag{191b}$$

where

$$\cos\psi = \frac{V\cdot Y}{Y\sqrt{V^2}} \tag{192}$$

$$n\frac{\partial n}{\partial t} = n\frac{\partial n}{\partial X}\ \frac{\partial X}{\partial t} \tag{193}$$

5.8 Sen-Wyller Formula no Field

This subroutine uses the function G which is related to the function F (defined in the previous section) as follows:

$$G(w) = wF(w)\ . \tag{194}$$

The following equations parallel the formulas used in this version of RINDEX.

$$F_1 = F\left(\frac{1}{Z}\right) \tag{195}$$

$$F_1' = F'\left(\frac{1}{Z}\right) \tag{196}$$

$$G_1 = G\left(\frac{1}{Z}\right) \tag{197}$$

$$G_1' = G'\left(\frac{1}{Z}\right) \tag{198}$$

$$n^2 = 1 - X\,G_1 \tag{199}$$

$$n\frac{\partial n}{\partial X} = -\frac{1}{2}\,G_1 \tag{200}$$

$$n\frac{\partial n}{\partial Z} = \frac{X\,G_1'}{2\,Z^2} \tag{201}$$

$$n\frac{\partial n}{\partial r} = \left(n\frac{\partial n}{\partial X}\right)\frac{\partial X}{\partial r} + \left(n\frac{\partial n}{\partial Z}\right)\frac{\partial Z}{\partial r} \tag{202}$$

$$n\frac{\partial n}{\partial \theta} = \left(n\frac{\partial n}{\partial X}\right)\frac{\partial X}{\partial \theta} + \left(n\frac{\partial n}{\partial Z}\right)\frac{\partial Z}{\partial \theta} \tag{203}$$

$$n\frac{\partial n}{\partial \varphi} = \left(n\frac{\partial n}{\partial X}\right)\frac{\partial X}{\partial \varphi} + \left(n\frac{\partial n}{\partial Z}\right)\frac{\partial Z}{\partial \varphi} \tag{204}$$

$$nn' = n^2 - \left(2\,X\left(n\frac{\partial n}{\partial X}\right) + Z\left(n\frac{\partial n}{\partial Z}\right)\right) \tag{205}$$

$$\text{polarization} = i \tag{206a}$$

$$\text{longitudinal polarization} = 0 \tag{206b}$$

$$n\frac{\partial n}{\partial t} = n\frac{\partial n}{\partial X}\,\frac{\partial X}{\partial t} \tag{207}$$