APPENDIX B: Numerical Analysis

The integral equation (1) or equivalently (A-13) is of the form of a linear Volterra integral equation of the second kind, i.e.,

$$f(x) = g(x) - c \int_{0}^{x} f(s)\, K(x, s)\, ds \qquad \text{(B-1)}$$

where $f(x)$ is the unknown attenuation function whose value is to be determined in the interval $0 \leq s \leq x$. The function $g(x)$ and $K(x, s)$ are known, and c is a constant. If $g(x)$ is bounded and continuous and if

$$\int_{0}^{x} |K(x, s)|\, ds \leq L < \infty \qquad \text{(B-2)}$$

then the solution will be unique and continuous (Wagner, 1953). This integral equation can be solved by a stepwise calculation that divides the interval x into subintervals of arbitrary width.

That is, consider the subdivision

$$f(x_n) = W(x_n) - (i/\lambda)^{\frac{1}{2}} \left\{ \int_{0}^{x_1} f(s)\, K(x_n, s)\, ds + \int_{x_1}^{x_2} f(s)\, K(x_n, s)\, ds + \cdots + \int_{x_{n-1}}^{x_n} f(s) K(x_n, s)\, ds \right\} \qquad \text{(B-3)}$$

The unknown function, $f(s)$, is fitted with a polynomial of the form

$$f(s) = a_0 + a_1 s + a_2 s^2 \qquad \text{(B-4)}$$

Increasing the degree of the polynomial to 3 or higher would result in even higher accuracy; however, the algebra becomes more complicated and sufficient accuracy can be obtained with the polynomial of degree 2. In some examples, the solution may become unstable for the higher degree polynomial and oscillate between the fitted points.

The solution of the integral equation requires special starting procedures. We suggest that the interpolating polynomial be of the form

$$f(s) = \alpha_0 + \alpha_1 s^{\frac{1}{2}} + \alpha_2 s + \alpha_3 s^{3/2} \quad , \quad 0 \le s \le x_3 \tag{B-5}$$

and to use (B-4) for $x_3 \le s \le x_n$. The choice in (B-5) is a logical one if we assume the terrain is flat in the immediate vicinity of the transmitting antenna. If the terrain is flat the exact answer for the attenuation function is then in fact a half-order power series in the numerical distance. Requiring the polynomial in (B-5) to pass through the first four consecutive points yields

$$\alpha_0 = 1.0 \tag{B-6a}$$

$$\alpha_1 = R_1 f(x_1) + R_2 f(x_2) + R_3 f(x_3) + R_4 \tag{B-6b}$$

$$\alpha_2 = R_5 f(x_1) + R_6 f(x_2) + R_7 f(x_3) + R_8 \tag{B-6c}$$

$$\alpha_3 = R_9 f(x_1) + R_{10} f(x_2) + R_{11} f(x_3) + R_{12} \tag{B-6d}$$

The constants in (B-4) are found by requiring the polynomial to pass through the points x_{i-2}, x_{i-1} and x_i. It is a simple exercise to show that

$$a_0 = R_{13} f(x_i) + R_{14} f(x_{i-1}) + R_{15} f(x_{i-2}) \tag{B-7a}$$

$$a_1 = R_{16} f(x_i) + R_{17} f(x_{i-1}) + R_{18} f(x_{i-2}) \tag{B-7b}$$

$$a_2 = R_{19} f(x_i) + R_{20} f(x_{i-1}) + R_{21} f(x_{i-2}) \tag{B-7c}$$

where the R's in (B-6) and (B-7) are defined as

$$D = (x_1 x_2 x_3)^{\frac{1}{2}} \left[x_1 (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) + x_2 (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) + x_3 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) \right] \quad \text{(B-8a)}$$

$$R_1 = x_2 x_3 (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) / D \quad \text{(B-8b)}$$

$$R_2 = x_1 x_3 (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) / D \quad \text{(B-8c)}$$

$$R_3 = x_1 x_2 (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) / D \quad \text{(B-8d)}$$

$$R_4 = \left[x_1 (x_3^{3/2} - x_2^{3/2}) + x_2 (x_1^{3/2} - x_3^{3/2}) + x_3 (x_2^{3/2} - x_1^{3/2}) \right] / D \quad \text{(B-8e)}$$

$$R_5 = (x_2 x_3)^{\frac{1}{2}} (x_2 - x_3) / D \quad \text{(B-8f)}$$

$$R_6 = (x_1 x_3)^{\frac{1}{2}} (x_3 - x_1) / D \quad \text{(B-8g)}$$

$$R_7 = (x_1 x_2)^{\frac{1}{2}} (x_1 - x_2) / D \quad \text{(B-8h)}$$

$$R_8 = \left[x_1^{\frac{1}{2}} (x_2^{3/2} - x_3^{3/2}) + x_2^{\frac{1}{2}} (x_3^{3/2} - x_1^{3/2}) + x_3^{\frac{1}{2}} (x_1^{3/2} - x_2^{3/2}) \right] / D \quad \text{(B-8i)}$$

$$R_9 = (x_2 x_3)^{\frac{1}{2}} (x_3^{\frac{1}{2}} - x_2^{\frac{1}{2}}) / D \quad \text{(B-8j)}$$

$$R_{10} = (x_1 x_3)^{\frac{1}{2}} (x_1^{\frac{1}{2}} - x_3^{\frac{1}{2}}) / D \quad \text{(B-8k)}$$

$$R_{11} = (x_1 x_2)^{\frac{1}{2}} (x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}}) / D \quad \text{(B-8}\ell\text{)}$$

$$R_{12} = \left[x_1^{\frac{1}{2}} (x_3 - x_2) + x_2^{\frac{1}{2}} (x_1 - x_3) + x_3^{\frac{1}{2}} (x_2 - x_1) \right] / D \quad \text{(B-8m)}$$

$$D_i = (x_{i-2} - x_{i-1}) \left[x_i^2 - x_i (x_{i-2} + x_{i-1}) + x_{i-2} x_{i-1} \right] \quad \text{(B-8n)}$$

$$R_{13} = x_{i-1} x_{i-2} (x_{i-2} - x_{i-1}) / D_i \quad \text{(B-8o)}$$

$$R_{14} = x_i x_{i-2} (x_i - x_{i-2}) / D_i \quad \text{(B-8p)}$$

$$R_{15} = x_i x_{i-1} (x_{i-1} - x_i) / D_i \quad \text{(B-8q)}$$

$$R_{16} = (x_{i-1}^2 - x_{i-2}^2) / D_i \quad \text{(B-8r)}$$

$$R_{17} = (x_{i-2}^2 - x_i^2) / D_i \quad \text{(B-8s)}$$

$$R_{18} = (x_i^2 - x_{i-1}^2) / D_i \quad \text{(B-8t)}$$

$$R_{19} = (x_{i-2} - x_{i-1}) / D_i \quad \text{(B-8u)}$$

$$R_{20} = (x_i - x_{i-2})/D_i \quad \text{(B-8v)}$$

$$R_{21} = (x_{i-1} - x_i)/D_i \quad \text{(B-8w)}$$

Using our polynomial interpolation formulas for $f(s)$ we find that the integrals in (B-3) all have the following generic form

$$p_\ell(x_i, x_j, x_k) = \int_{x_k}^{x_j} s^{\ell/2} K(x_i, s)\, ds \quad \text{(B-9)}$$

with

$$\begin{aligned} &0 \le k \le j-1 \\ &1 \le j \le i \\ &2 \le i \le n \\ &\ell = 0,\ 1,\ 2,\ 3,\ 4. \end{aligned} \quad \text{(B-10)}$$

These integrals are evaluated numerically using a five point Gaussian integration formula with special attention given to those integrals having singularities at either of the endpoints of integration.

Substituting (B-4) through (B-10) into (B-3) yields the following general expression for $f(x)$ at the i^{th} point

$$f(i)\left\{1 + (i/\lambda)^{\frac{1}{2}}\left[R_{13}(i)p_o(i,i,i-1) + R_{16}(i)p_2(i,i,i-1) + R_{19}(i)p_4(i,i,i-1)\right]\right\}$$

$$= W(i) - (i/\lambda)^{\frac{1}{2}}\left\{\sum_{j=1}^{3} p_0(i,j,j-1) + R_4 \sum_{j=1}^{3} p_1(i,j,j-1) + R_8 \sum_{j=1}^{3} p_2(i,j,j-1)\right.$$

$$+ R_{12} \sum_{j=1}^{3} p_3(i,j,j-1) + f(1)\left[R_1 \sum_{j=1}^{3} p_1(i,j,j-1) + R_5 \sum_{j=1}^{3} p_2(i,j,j-1)\right.$$

$$+R_9 \sum_{j=1}^{3} p_3 (i, j, j-1) \Big] + f(2) \Big[R_2 \sum_{j=1}^{3} p_1 (i, j, j-1) + R_6 \sum_{j=1}^{3} p_2 (i, j, j-1)$$

$$+R_{10} \sum_{j=1}^{3} p_3 (i, j, j-1) \; R_{15} (4) p_0 (i, 4, 3) + R_{18} (4) p_2 (i, 4, 3) + R_{21} (4) p_4 (i, 4, 3) \Big]$$

$$+ f(3) \Big[R_3 \sum_{j=1}^{3} p_1 (i, j, j-1) + R_7 \sum_{j=1}^{3} p_2 (i, j, j-1) + R_{11} \sum_{j=1}^{3} p_3 (i, j, j-1)$$

$$+R_{14} (4) p_0 (i, 4, 3) + R_{17} (4) p_2 (i, 4, 3) + R_{20} (4) p_4 (i, 4, 3) + R_{15} (5) p_0 (i, 5, 4)$$

$$+ R_{18} (5) p_2 (i, 5, 4) + R_{21} (5) p_4 (i, 5, 4) \Big] + \sum_{m=4}^{i-2} f(m) \Big[R_{13} (m) p_0 (i, m, m-1)$$

$$+R_{14} (m+1) p_0 (i, m+1, m) + R_{15} (m+2) p_0 (i, m+2, m+1)$$

$$+R_{16} (m) p_2 (i, m, m-1) + R_{17} (m+1) p_2 (i, m+1, m) + R_{18} (m+2) p_2 (i, m+2, m+1)$$

$$+R_{19}(m)p_4(i,m,m-1)+R_{20}(m+1)p_4(i,m+1,m)+R_{21}(m+2)p_4(i,m+2,m+1)\Big]$$

$$+f(i-1)\Big[R_{13}(i-1)p_0(i,i-1\ i-2)+R_{14}(i)p_0(i,i,i-1)+R_{16}(i-1)p_2(i,i-1,i-2)$$

$$+R_{17}(i)p_2(i,i,i-1)+R_{19}(i-1)p_4(i,i-1,i-2)+R_{20}(i)p_4(i,i,i-1)\Big]\Big\} \qquad \text{(B-11)}$$

Reference

(B-1) Wagner, Carl (1953), "On the numerical solution of Volterra integral equations," J. Math. and Phys. 32, pp. 289-401.