APPENDIX A: Derivation of the integral equation

Consider a solution, φ, of the wave equation

$$\text{(i)} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + k^2 \varphi = -2\pi \tau (x, y) \quad , \quad y > y(x)$$

which satisfies an impedance boundary condition of the form

$$\text{(ii)} \quad \frac{\partial \varphi}{\partial n} = \frac{ik\Delta \varphi}{\sqrt{1 + (y')^2}} \quad , \quad y = y(x)$$

where φ represents the vertical component of $\underline{E}$ for the case of vertical polarization or the vertical component of $\underline{H}$ for horizontal polarization. The time dependence is $\exp(i\omega t)$ and the normalized impedance, Δ, near grazing is

$$\Delta = \begin{cases} \dfrac{\sqrt{\eta - 1}}{\eta} \quad , & \text{vertical polarization} \\ \sqrt{\eta - 1} \quad , & \text{horizontal polarization} \end{cases}$$

with

$$\eta = \epsilon_r - \frac{i\sigma}{\omega \epsilon_o}$$

where ϵ_r is the dielectric constant, σ is the conductivity and ω the angular frequency.

The source distribution is $\tau(x, y)$. Let

$$\varphi = e^{-ikx} \psi(x, y)$$

and i) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2ik \frac{\partial \psi}{\partial x} = -2\pi\tau (x, y) \; e^{ikx}$$

Assuming that the fast variation with x occurs in $\exp(-ikx)$

$$\frac{\partial^2 \psi}{\partial x^2} \cong 0$$

or that $\partial^2 \psi / \partial x^2$ is small compared with remaining terms we find

$$\frac{\partial^2 \psi}{\partial y^2} - 2ik \frac{\partial \psi}{\partial x} = -2\pi\tau(x, y) \; e^{ikx} \qquad \text{(A-1)}$$

An elementary function for (A-1) is (Ott and Berry, 1970)

$$\sqrt{\frac{2ik}{\pi}} \, G(x, y; \xi, \eta) = \frac{e^{-ik(\eta - y)^2 / 2(\xi - x)}}{\sqrt{\xi - x}}$$

$$+ \frac{ik\Delta \, e^{-ik\Delta\eta}}{\sqrt{\xi - x}} \int_{\eta}^{\infty} \exp\{-ik(t-y)^2 / 2(\xi - x)\} e^{ik\Delta t} \, dt, \; x < \xi$$

$$= \frac{e^{-ik(\eta - y)^2 / 2(\xi - x)}}{\sqrt{\xi - x}} \quad W(x, \xi), \; x < \xi.$$

$$\sqrt{\frac{2ik}{\pi}} \, G(x, y; \xi, \eta) = 0, \; x > \xi .$$

The function satisfies

$$\frac{\partial^2 G}{\partial y^2} + 2ik\frac{\partial G}{\partial x} = -2\pi\delta(x-\xi,\ y-\eta) \qquad \text{(A-3)}$$

The constant on the left-hand-side of (A-2) comes from integrating both sides of (A-3) over the region $R = \{x, y: -\infty < x \leq \infty,\ y(x) < y < \infty\}$.

Multiplying (A-1) by G, (A-3) by ψ, and subtracting and integrating over the region R yields

$$\iint\limits_R \left(G\frac{\partial^2 \psi}{\partial y^2} - \psi\frac{\partial^2 G}{\partial y^2}\right) dx\, dy - 2ik \iint\limits_R \left(G\frac{\partial \psi}{\partial x} + \psi\frac{\partial G}{\partial x}\right) dx\, dy$$

$$= -2\pi \iint\limits_\Sigma e^{ikx}\, \tau\, G\, dx\, dy + \pi\psi(P) \qquad \text{(A-4)}$$

where P is the observation point (ξ, η), and Σ is a region around the source. The divergence theorem yields on the surface y(x)

$$\iint\limits_R \left(G\frac{\partial^2 \psi}{\partial y^2} - \psi\frac{\partial^2 G}{\partial y^2}\right) dx\, dy = \int\limits_C \left(G\frac{\partial \psi}{\partial y} - \psi\frac{\partial G}{\partial y}\right) \underline{e}_n \cdot \underline{e}_y\, dc \qquad \text{(A-5)}$$

where $\underline{e}_n$ is the outward directed normal (into the surface) and C is a contour enclosing R and

$$\underline{e}_n = -\frac{-y'\,\underline{e}_x + \underline{e}_y}{\sqrt{1+(y')^2}}$$

and along $y = y(x)$

$$dc = \sqrt{1 + (y')^2}\ dx$$

Also

$$2ik \iint_R \left(G \frac{\partial \psi}{\partial x} + \psi \frac{\partial G}{\partial x}\right) dx\, dy = 2ik \iint_R \frac{\partial}{\partial x} (G\psi)\, dx\, dy$$

$$= 2ik \int_C G\, \psi\, \underline{e}_n \cdot \underline{e}_x\, dc \qquad \text{(A-6)}$$

From (ii), and neglecting $\partial\psi / \partial x$ compared with other terms

$$\frac{\partial \psi}{\partial y} = ik\Delta\psi - iky'\psi \qquad \text{(A-7)}$$

and substituting (A-5), (A-6), and (A-7) into (A-4), and assuming $\psi = 0$, for $x \leq 0$, which means neglecting backscatter from the region $x \leq 0$, and all sources are in the region $x > 0$,

$$-\int_o^\xi \left[ik\Delta\psi G - iky'\psi G - \psi \frac{\partial G}{\partial y}\right] dx - 2ik \int_o^\xi G\, \psi\, y'\, dx$$

$$+ 2\pi \iint_\Sigma \tau\, e^{ikx} G\, dx\, dy = \pi\, \psi\, (P)$$

or

$$\int_0^{\xi} [-ik\Delta\psi G - iky'\psi G + \psi \frac{\partial G}{\partial y}]\, dx + 2\pi \iint_{\Sigma} \tau G e^{ikx}\, dx\, dy = \pi\psi(P) \qquad \text{(A-8)}$$

Substituting (Ott and Berry, 1970)

$$\sqrt{\frac{2ik}{\pi}}\, \frac{\partial G}{\partial y} = ik\Delta \sqrt{\frac{2ik}{\pi}}\, G + \frac{ik \exp\{-ik(\eta - y)^2/2(\xi - x)\}}{\sqrt{\xi - x}} \left[\frac{\eta - y}{\xi - x}\right] \qquad \text{(A-9)}$$

in (A-8) gives

$$-ik \int_0^{\xi} \{y'\psi G - \psi \frac{\exp\{-ik(\eta - y)^2/2(\xi - x)\}}{\sqrt{\xi - x}} \left[\frac{\eta - y}{\xi - x}\right]\}\, dx$$

$$+ 2\pi \iint_{\Sigma} \tau G e^{ikx}\, dx\, dy = \pi\psi(P)\ .$$

Reintroducing φ and defining $\hat{G} = G\, e^{-ik(\xi - x)}$ yields

$$-\frac{ik}{2\pi} \int_0^{\xi} \{y'\varphi\hat{G} - \varphi \frac{\exp -ik\{(\xi - x) + [(\eta - y)^2/2(\xi - x)]\}}{\sqrt{\xi - x}} \left[\frac{\eta - y}{\xi - x}\right]\}\, dx$$

$$+ \iint_{\Sigma} \tau \hat{G}\, dx\, dy = \frac{1}{2}\, \varphi(P) e^{-ik\xi} \qquad \text{(A-10)}$$

We assume that the antenna has a phase center where the source distribution, $\tau(x, y)$, is located. Then we write

$$\tau(P) = g(P)\left\{\exp\left[-ik\left(x + \frac{y^2}{2x}\right)\right] \Big/ \sqrt{x}\right\} \delta(x, y) \qquad \text{(A-11)}$$

where $(x + y^2 / 2x)$ is the first two terms in the binomial expansion for the distance between the source point O and the observation point at P. The function $g(P)$ represents the antenna gain or pattern factor. We also introduce an attenuation function $f(P)$ defined as

$$\varphi(P) = 2 f(P) \exp\left[-ik\left(x + \frac{y^2}{2x}\right)\right] \Big/ \sqrt{x} \qquad \text{(A-12)}$$

When these two equations are substituted into (A-10), we find (interchanging (ξ, η) with (x, y))

$$f(x) = g(x, y)\ W(x, o)$$

$$- \sqrt{\frac{ik}{2\pi}} \int_o^x f(\xi)\ e^{-ik\omega(x, \xi)} \left\{y'(\xi)\ W(x, \xi) - \frac{y-\eta}{x-\xi}\right\} \left[\frac{x}{\xi(x-\xi)}\right]^{\frac{1}{2}} d\xi$$

(A-13)

where

$$\omega(x, \xi) = \frac{(y-\eta)^2}{2(x-\xi)} + \frac{\eta^2}{2\xi} - \frac{y^2}{2x}$$

$$y = y(x)$$

$$\eta = y(\xi)$$

which differs slightly from the result in Ott and Berry (1970); see for example Ott (1971).

$$W(x,\xi) = 1 - i\sqrt{\pi p}\; e^{-u}\; \mathrm{erfc}(i u^{\frac{1}{2}})$$

$$p = \frac{-ik\Delta^2 (x-\xi)}{2} \qquad \text{(A-14)}$$

$$u = p\left\{1 - \frac{y-\eta}{\Delta(x-\xi)}\right\}^2 \quad , \quad \xi < x$$

References

(A-1) Ott, R. H. and Berry, L. A. (1970), "An alternative integral equation for propagation over irregular terrain," Radio Science, 5, No. 5, pp. 767-771.

(A-2) Ott, R. H. (1971), "An alternative integral equation for propagation over irregular terrain, Part II," to be published Radio Science, May.