4. Circulation and vorticity

4.1 Two-dimensional flows

This chapter is ultimately concerned with vorticity, a flow property that is hard to overestimate as an aid to understanding fluid dynamics. To provide some context, we begin by trying to classify *all* different kinds of motion in a *two*-dimensional velocity field. According to a theorem of Helmholtz, any two-dimensional flow $\boldsymbol{V}(x, y)$ can be decomposed as

$$\boldsymbol{V} = \hat{z} \times \nabla\psi + \nabla\chi, \tag{4.1}$$

where the "streamfunction" $\psi(x, y)$ and "velocity potential" $\chi(x, y)$ are scalar functions. The first part of the decomposition is non-divergent and the second part is irrotational.

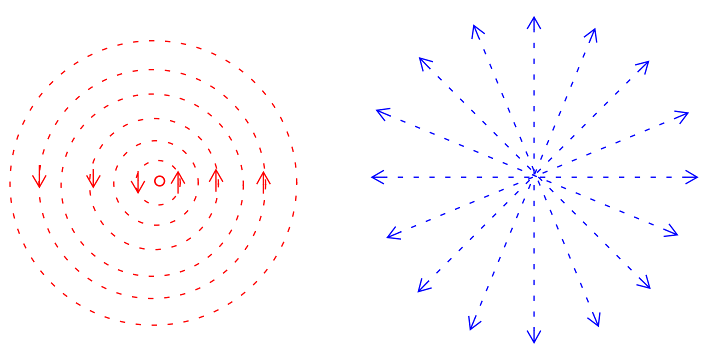
If we write $\boldsymbol{V} = u\hat{\boldsymbol{x}} + v\hat{\boldsymbol{y}}$ for the flow in the horizontal plane, then 4.1 says that $u = -\partial\psi/\partial y + \partial\chi/\partial x$ and $v = \partial\psi/\partial x + \partial\chi/\partial y$. We define the *vertical vorticity* ζ and the *divergence D* as follows:

$$\zeta = \hat{z} \cdot (\nabla \times \boldsymbol{V}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2\psi, \tag{4.2}$$

$$D = \nabla \cdot \boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla^2\chi. \tag{4.3}$$

Determination of the velocity field given ζ and D requires boundary conditions on ψ and χ. Contours of ψ are called "streamlines".

The vorticity is proportional to the local angular velocity about $\hat{z}$. It is known entirely from $\psi(x, y)$ or the first term on the rhs of 4.1. As seen in chapter 2, solid-body rotation, or $\boldsymbol{V} = \Omega\hat{z} \times \boldsymbol{r}$, has $\zeta = 2\Omega$. Hence the angular velocity of the flow at a point is $\zeta/2$. The divergence is equal to the local fractional change in area per unit time. It is known entirely from $\chi(x, y)$ or the second term on the rhs of 4.1. For a circular region, the fractional change in area a is $a^{-1}da/dt = (2/r)dr/dt$, where r is radial distance. Hence the radial derivative of the average radial velocity at a point is $D/2$.



$\zeta > 0$ $\qquad\qquad$ $D > 0$

The remaining types of motion are *translation* and *strain*. These satisfy both $\nabla^2\psi = 0$ (irrotational) and $\nabla^2\chi = 0$ (non-divergent). Flows satisfying the second condition are called "solenoidal" flows. Flows satisfying both conditions are called "potential" flows. The Helmholtz decomposition would be unique except for translation and strain.

Translation is simply $\boldsymbol{V} = \mathrm{const}$. That is, ψ and χ are linear functions of *x* and *y*.

Strain is determined by the remaining second partial derivatives of Ψ and χ:

$$S_1 \equiv \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -2\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\chi}{\partial x^2} - \frac{\partial^2\chi}{\partial y^2}, \tag{4.4}$$

$$S_2 \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\frac{\partial^2\chi}{\partial x \partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}. \tag{4.5}$$

The decomposition between ψ and χ is not unique when strain or translation exists. Strain is the tendency to stretch and squash material regions while preserving area. If δx and δy are the dimensions of a rectangular area element, it can be shown that $S_1 = (1/\mu)d\mu/dt$, where $\mu = \delta x/\delta y$, the *aspect ratio* of the element. If $S_1 > 0$, the flow is *confluent* along the *y*-axis and *diffluent* along the *x*-axis. Area is preserved through a balance of confluence and diffluence.

Similarly, $S_2 = (1/\mu')d\mu'/dt$, where $\mu' = \delta x'/\delta y'$ and the primed axes are rotated 45 deg counterclockwise from *x*-*y*. Pure strain along the *x*- or *y*-axis (due to S_1) causes no stretching in the x' or y' directions, whereas pure strain along the x'- or y'-axis (due to S_2) involves no stretching along *x* or *y*. We can now ask what is the strain in an arbitrary direction. Simply by calculating S_1 in a rotated coordinate system, one finds that, in the direction that makes an angle φ with the *x*-axis, the strain is

$$S(\varphi) = S_1\cos(2\varphi) + S_2\sin(2\varphi) = \bar{S}\cos[2(\varphi - \bar{\varphi})], \tag{4.6}$$

where $\bar{S} \equiv \sqrt{S_1^2 + S_2^2}$ (total strain) and $\bar{\varphi} = (1/2)\arctan(S_2/S_1)$ (orientation of the "principal axis of deformation"). Specifying S_1 and S_2 is equivalent to providing boundary conditions for the problem $\zeta(x, y), D(x, y) \to \psi(x, y), \chi(x, y)$.

A more elegant approach: Recall from chapter 1 that the evolution of a material line element in three dimensions is given by $(d/dt)\delta\boldsymbol{l} = (\delta\boldsymbol{l} \cdot \nabla)\boldsymbol{V}$, or

$$\frac{d}{dt}\{\delta l_i\} = \left\{\frac{\partial V_i}{\partial x_j}\right\}\{\delta l_j\}. \tag{4.7}$$

The antisymmetric part of the matrix $\mathbf{D} \equiv \{\partial V_i/\partial x_j\}$ is $\mathbf{D}^a = \frac{1}{2}\{\partial V_i/\partial x_j - \partial V_j/\partial x_i\}$. This part causes rotations around each of the three coordinate axes. For example, in two dimensions,

$$\mathbf{D}^a = \begin{bmatrix} 0 & -\zeta/2 \\ \zeta/2 & 0 \end{bmatrix}, \tag{4.8}$$

where $\zeta = v_x - u_y$, twice the angular velocity about $\hat{z}$. Now the symmetric part of $\mathbf{D}$ in two dimensions is

$$\mathbf{D}^s = \begin{bmatrix} u_x & \dfrac{u_y + v_x}{2} \\ \dfrac{u_y + v_x}{2} & v_y \end{bmatrix}. \tag{4.9}$$

This contains strain and divergence in the *x-y* plane. To isolate the strain, let $u_x + v_y = 0$, so that

$$\mathbf{D}^s = \begin{bmatrix} S_1/2 & S_2/2 \\ S_2/2 & -S_1/2 \end{bmatrix}. \tag{4.10}$$

The resulting negative-definite matrix has the two eigenvalues, $\pm\bar{S}/2$, where $\bar{S}$ is the total strain, obtained previously. The corresponding eigenvectors are $\boldsymbol{e}_1 = \begin{bmatrix}\cos\bar{\varphi} & \sin\bar{\varphi}\end{bmatrix}$ and $\boldsymbol{e}_2 = \hat{z} \times \boldsymbol{e}_1$, where $\bar{\varphi}$ is the angle beween the *x*-axis and the principal axis of deformation, also given previously in terms of S_1 and S_2.

4.2 Point vortices and point mass sources

We return to the issue of using 4.2 and 4.3 to find ψ and χ given full distributions of ζ and *D*. There is some conceptual and mathematical benefit in using the method of Green's functions. Thus, consider a “point vortex”,

$$\zeta = \zeta_0 \delta(r), \tag{4.11}$$

where $\delta(r)$ is the 2-d Dirac delta function and ζ_0 has dimensions of vorticity times area. The solution of 4.2 for this source, subject to a boundedness condition at infinity, is

$$\psi = \zeta_0 \psi_G(r) = \frac{\zeta_0}{2\pi} \log r, \tag{4.12}$$

where ψ_G is, by definition, the Green's function for this problem. For an axisymmetric flow, $\nabla^2\psi = (1/r)(r\psi_r)_r$.

From 4.1 and 4.12, the flow induced by a point vortex of strength ζ_0 is $\boldsymbol{V} = \zeta_0 (2\pi r)^{-1} \hat{z} \times \hat{\boldsymbol{r}}$. Note that the velocity falls off as 1/*r*. There is curvature in this flow but no vorticity away from the origin.

We can also consider a point “mass” source via $D = D_0 \delta(r)$. The solution of 4.3 is $\chi = D_0 (2\pi)^{-1} \log r$ and $\boldsymbol{V} = D_0 (2\pi r)^{-1} \hat{\boldsymbol{r}}$. There is no divergence away from the origin, but there is a steady, integrable source of “mass” at the origin.

By using Green's theorem and the above expression for ψ_G, we find that the solution for

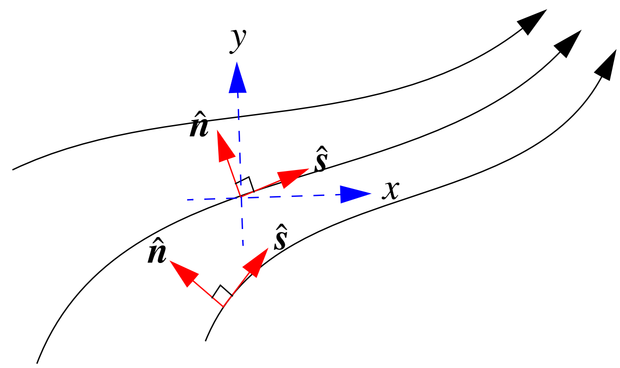
ψ given an arbitrary distribution of ζ is

$$\psi(x, y) = \frac{1}{4\pi}\int_{\sigma}\zeta(x', y')\log[(x - x')^2 + (y - y')^2]d\sigma\,, \tag{4.13}$$

where σ denotes area and the integral covers all places where $\zeta \neq 0$. There is an analogous result for χ given a distribution of D.

4.3 Natural coordinates

A pattern of non-intersecting 2-D trajectories determines a special orthogonal coordinate system (s, n) called *natural coordinates*. The unit vectors for this system are defined $\hat{\boldsymbol{s}} \equiv \boldsymbol{V}/|\boldsymbol{V}|$ and $\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}$. (If $\boldsymbol{V}$ is defined everywhere at the same time, the lines parameterized by s are called "streak lines".)



Since $\boldsymbol{V} = V\hat{\boldsymbol{s}}$, with $\boldsymbol{V} \equiv |\boldsymbol{V}|$, we find that

$$\zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{V} = (\hat{\boldsymbol{z}} \times \nabla V) \cdot \hat{\boldsymbol{s}} + V\hat{\boldsymbol{z}} \cdot \nabla \times \hat{\boldsymbol{s}}\,, \tag{4.14}$$

But $\hat{\boldsymbol{z}} \cdot \nabla \times \hat{\boldsymbol{s}} = \nabla\beta \cdot \hat{\boldsymbol{s}}$, where β is the angle between $\hat{\boldsymbol{s}}$ and the x-axis. Also, $(\hat{\boldsymbol{z}} \times \nabla V) \cdot \hat{\boldsymbol{s}} = -\nabla V \cdot (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}) = -\nabla V \cdot \hat{\boldsymbol{n}}$. Therefore,

$$\zeta = -\partial V/\partial n + V\partial\beta/\partial s\,. \tag{4.15}$$

The first term on the rhs is due to *shear*. The second term is due to *curvature* and may be written as V/R_T, where R_T is the "radius of curvature". In irrotational flow (*e.g.*, around a point vortex), the two terms are equal and opposite. In solid-body rotation, the two terms are equal.

The expression for divergence in natural coordinates is

$$D = \partial v/\partial s + V\partial\beta/\partial n\,. \tag{4.16}$$

The first term on the rhs is due to "longitudinal" compression or expansion. The second term is due to confluence or diffluence. The two parts must be equal and opposite in non-divergent flows.

4.4 Absolute circulation

Definition of *absolute circulation*:

$$C_a = \oint_\Gamma \boldsymbol{V}_a \cdot \boldsymbol{ds} \tag{4.17}$$

where Γ is a closed, *material* circuit, and $\boldsymbol{V}_a$ is the velocity in an inertial frame. By convention, the path direction is chosen to keep the interior of the loop on the left.

Since the integrand in 4.17 is the velocity component along the circuit, we expect a nice simplification when we consider the time derivative of C_a:

$$\frac{dC_a}{dt} = \oint_\Gamma \frac{d\mathrm{V}_a}{dt} \cdot \boldsymbol{ds} + \oint_\Gamma \boldsymbol{V}_a \cdot d\boldsymbol{V}_a . \tag{4.18}$$

Indeed, since the second integrand is a perfect differential, it contributes nothing to the integral. For the first integral we substitute from the momentum equation to reach

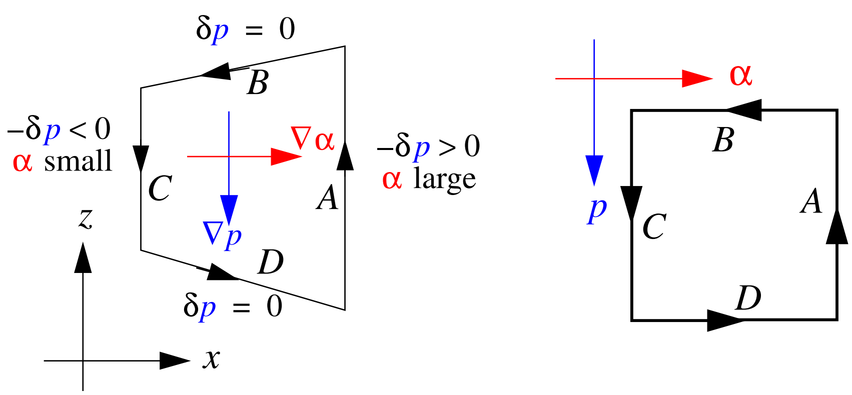
$$\frac{dC_a}{dt} = -\oint_\Gamma \frac{\nabla p}{\rho} \cdot \boldsymbol{ds} = -\oint_\Gamma \frac{dp}{\rho} . \tag{4.19}$$

This is called Bjerknes's Circulation Theorem. Notice that gravity -- or any other conservative force -- cannot contribute to the closed line integral.

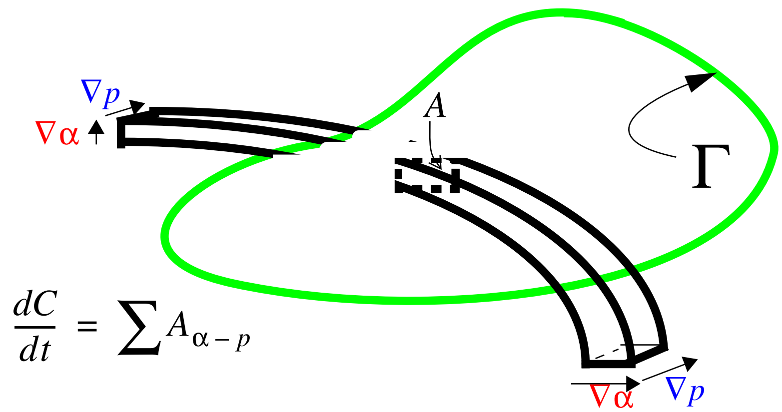
The rhs of 4.19 is called the *solenoidal term*. Using the ideal-gas equation of state, we can also write it as

$$-\oint_\Gamma \frac{dp}{\rho} = -\oint_\Gamma \alpha dp = -R\oint_\Gamma T d\log p \tag{4.20}$$

Graphically, the solenoidal term is proportional to the "area" inside Γ when Γ is drawn in α-p space or (for an ideal gas) T-$\log p$ space. If the sense of the integration path changes, so does the sign of the "area".



Tubes perpendicular to both ∇a and ∇p are called *solenoids*. The change in circulation on Γ is determined by the sum of the α-p cross-sectional areas of the solenoids passing through the loop.



Kelvin's Circulation Theorem: If there are no solenoids, $\frac{dC_a}{dt} = 0$. (Example: if Γ lies in an isobaric surface.)

For yet another way to express the solenoidal term, we use Stokes' Theorem:

$$\oint_{\Gamma} \boldsymbol{B} \cdot d\mathbf{s} = \iint_{\sigma} \nabla \times \boldsymbol{B} \cdot d\sigma' \tag{4.21}$$

Here $\boldsymbol{d\sigma}$ denotes the surface element (area element times unit normal vector to the surface). If we put $\boldsymbol{B} = -\alpha\nabla p$, the identity becomes

$$\int_{\Gamma} (-\alpha\nabla p) \cdot \boldsymbol{ds} = \iint_{\sigma} \boldsymbol{S} \cdot \boldsymbol{d\sigma}'. \tag{4.22}$$

The lhs is the solenoidal term. On the rhs, $\boldsymbol{S}$ is defined

$$\boldsymbol{S} \equiv \nabla \times (-\alpha\nabla p) = -\nabla\alpha \times \nabla p \tag{4.23}$$

Note that $\boldsymbol{S}$ is directed along a solenoid.

Definitions: a fluid is *barotropic* if $\boldsymbol{S} = 0$ everywhere; otherwise it is *baroclinic* and $\boldsymbol{S}$ is called the *baroclinicity vector.* Examples of barotropic fluids: (1) α constant, (2) Θ constant, (3) T constant, (4) α surfaces parallel to p surfaces.

In pressure coordinates, the horizontal pressure-gradient force is "conservative". As a result, there is no explicit baroclinicity term in p-coordinates. This is also true, strictly speaking, of the Boussinesq equations. However, in that model, $\boldsymbol{S}$ should be redefined to include the curl of the the buoyancy force. This may have non-zero horizontal components.

4.5 Absolute vorticity

The definition of absolute vorticity, ω_a is

$$\omega_a \equiv \nabla \times \boldsymbol{V}_a . \tag{4.24}$$

To get an equation for ω_a, start with the momentum equation in the form

$$\frac{\partial \boldsymbol{V}_a}{\partial t} = -\nabla \frac{\boldsymbol{V}_a \cdot \boldsymbol{V}_a}{2} + \boldsymbol{V}_a \times \omega_a - \nabla \phi_g - \alpha \nabla p . \tag{4.25}$$

Taking the curl, we obtain

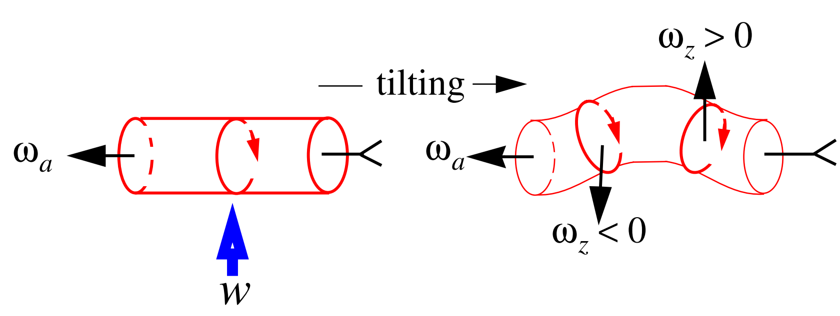
$$\frac{\partial \omega_a}{\partial t} = \begin{cases} -\boldsymbol{V}_a \cdot \nabla \omega_a \\ -\omega_a \nabla \cdot \boldsymbol{V}_a \\ +(\omega_a \cdot \nabla)\boldsymbol{V}_a \\ +\boldsymbol{S}. \end{cases} \tag{4.26}$$

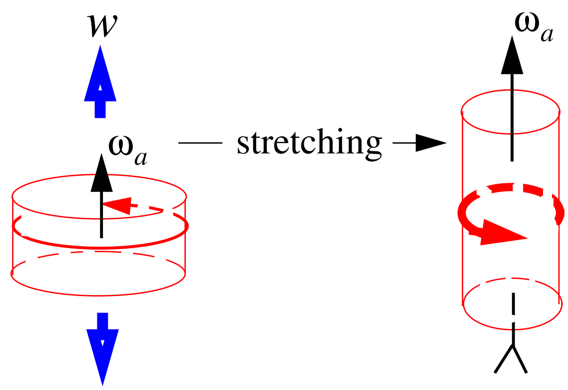
The first and second terms on the rhs are *advection* and *convergence*. The fourth is the solenoidal term.

The third term on the rhs of 4.26 is identical to the evolution of a material line element, as discussed in chapter 1 (below Eq. 1.6). Its vertical component (for instance) is

$$\hat{z} \cdot (\omega_a \cdot \nabla \boldsymbol{V}_a) = \omega_{ax}\frac{\partial w}{\partial x} + \omega_{ay}\frac{\partial w}{\partial y} + \omega_{az}\frac{\partial w}{\partial z} . \tag{4.27}$$

The first two terms on the rhs side describe *tilting* while the third describes *stretching*. These are illustrated in the diagrams below. Stretching combined with convergence is entirely due to the





"horizontal" part of the convergence. Thus,

$$\hat{z}\cdot(\text{stretch}+\text{converge}) = \omega_{az}\frac{\partial w}{\partial z}+(-\omega_{az}\nabla\cdot\boldsymbol{V}_a) = -\omega_{az}\left(\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right). \tag{4.28}$$

The first three terms in 4.26 can be written as a pure flux divergence. By combining the first two and exploiting the non-divergence of ω_a to rewrite the third, we get

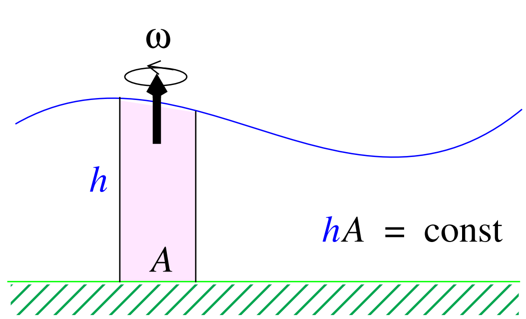
$$\frac{\partial}{\partial t}\omega_i = \frac{\partial}{\partial x_j}(\omega_j V_i - V_j\omega_i)+\boldsymbol{S}_i, \tag{4.29}$$

where the vorticity and velocity components are still in the inertial frame. The most important conclusion from 4.29 is that, in the absence of solenoids (and friction), the volume-integrated vorticity cannot be altered except by boundary fluxes. This is a *global* conservation statement about vorticity.

By invoking mass continuity, we can write 4.26 as

$$\rho\frac{d}{dt}\frac{\omega_a}{\rho} = (\omega_a\cdot\nabla)\boldsymbol{V}_a+\boldsymbol{S}. \tag{4.30}$$

For a flow in hydrostatic balance, w is removed from the expression for ω_a (and the first term on the rhs of 4.25). For the shallow-water model, illustrated below, we also have $\boldsymbol{S} = 0$ and $u_z = v_z = 0$. The latter constraint removes the remaining tilting terms in the vorticity equation.



The rhs of 4.30 is then the same as 4.28, i.e., just $\omega_{az}(dw/dz)$, or $\omega_{az}\Delta w/h$, where h is the depth of the fluid, $\omega_{az} = \partial v/\partial x - \partial u/\partial y$ and Δw is the difference in vertical velocity between top and bottom. Since $\Delta w = dh/dt$, it follows that

$$\frac{d}{dt}\left(\frac{\omega}{h}\right) = 0 \tag{4.31}$$

in the shallow-water model. This is similar to conservation of circulation except that the reciprocal of the *depth* now appears instead of the area.

Definitions: A *vortex line* is a line that is everywhere parallel to ω. A *vortex tube* is a closed surface consisting entirely of vortex lines.

Theorem: Vortex lines are conserved in a barotropic flow. That is, they are "frozen" into the fluid.

Proof. In general,

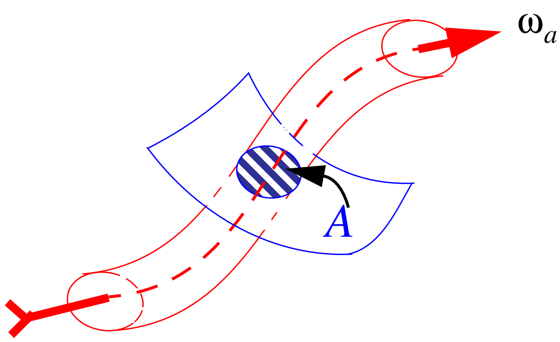
$$d(\omega \times \delta \boldsymbol{s})/dt = d\omega/dt \times \delta \boldsymbol{s} - d\delta \boldsymbol{s}/dt \times \omega$$

for any line element $\delta \boldsymbol{s}$. Let $\delta \boldsymbol{s}$ be a material line element, so that $(d/dt)\delta \boldsymbol{s} = \delta \boldsymbol{V} = \nabla \boldsymbol{V} \cdot \delta \boldsymbol{s}$. If it is initially a *vortex* line element, then $\delta \boldsymbol{s} = \varepsilon\omega$. Using these constraints together with the barotropic version of 4.26,

$$\frac{d\omega}{dt} = -\omega \nabla \cdot \boldsymbol{V} + (\omega \cdot \nabla)\boldsymbol{V},$$

one can show that the rhs of the first equation vanishes. Hence the condition $\omega \times \delta \boldsymbol{s} = 0$, which determines a vortex line element, is maintained. It follows that the vortex line is a material line.

This theorem can be used to extend Kelvin's Circulation Theorem to an entire vortex tube. Think of the vector ω_a as the "flow" in the tube. For any slice through the tube, consider the product $\omega_n A \equiv C$, where ω_n is the area average of $\omega_a \cdot \boldsymbol{d\sigma}/|\boldsymbol{d\sigma}|$ over the slice, $\boldsymbol{d\sigma}$ is the surface element and A is the area of the slice.



Since $\nabla \cdot \omega_a = 0$ (the "flow" is nondivergent), we know that C is the same for all slices. We may interpret it as the "intensity" of the tube. Furthermore, since material lines remain on the surface of the tube, Kelvin's Theorem applies. We may therefore conclude that the intensity C of the tube is constant in time.

4.6 Relative circulation

Definition of *relative circulation*:

$$C \equiv \oint \boldsymbol{V} \cdot d\boldsymbol{s} \tag{4.32}$$

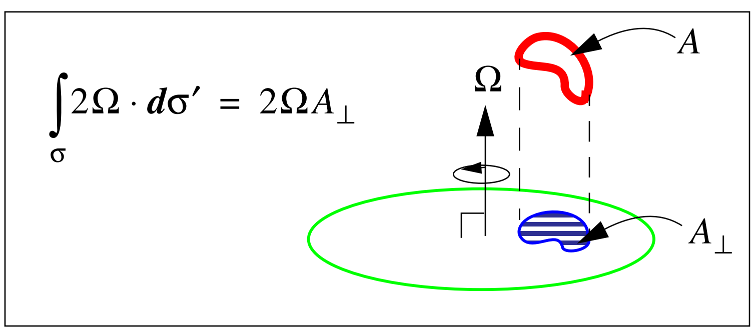
where $\boldsymbol{V}$ is the velocity in the rotating frame. Hence,

$$C = C_a - \oint (\Omega \times \boldsymbol{r}) \cdot d\boldsymbol{s} \,. \tag{4.33}$$

By Stokes' Theorem, the second term on the rhs becomes $\iint_\sigma \nabla \times (\Omega \times \boldsymbol{r}) \cdot \boldsymbol{d}\sigma'$. But since $\nabla \times (\Omega \times \boldsymbol{r}) = 2\Omega$, we have

$$C = C_a - 2\Omega A_\perp \,, \tag{4.34}$$

where $A_\perp \equiv \iint_\sigma \cos\varphi \, d\sigma'$ and φ is the angle between $\boldsymbol{d}\sigma'$ and Ω. Thus, $A_\perp$ is the area of the *projection* of the enclosed domain onto the equatorial plane (or any plane perpendicular to Ω).



Using 4.19, we can write

$$\frac{dC}{dt} = -\oint \alpha dp - 2\Omega \frac{dA_\perp}{dt} \,. \tag{4.35}$$

It follows that, in a barotropic atmosphere or ocean, $C + 2\Omega A_\perp$ is conserved on a material circuit. Notice that $A_\perp$ can change through latitudinal displacement as well as locally through tilting or contraction/expansion.

Definition of *relative vorticity*:

$$\omega = \nabla \times \boldsymbol{V}. \tag{4.36}$$

Since $\boldsymbol{V} = \boldsymbol{V}_a - \Omega \times \boldsymbol{r}$, we have

$$\omega = \omega_a - 2\Omega, \tag{4.37}$$

which should be compared to 4.34.

The vertical component of ω_a is $\omega_z + 2\Omega\sin\theta \equiv \zeta + f$, where ζ is the relative vorticity about the vertical axis and *f* is the Coriolis parameter. To see what form $dA_{\perp}/dt$ takes in the vorticity equation, we derive an equation for ζ.

First put the equations of motion in the form:

$$\frac{\partial u}{\partial t} = v(\zeta + f) - \frac{\partial}{\partial x}\left(\frac{u^2 + v^2}{2}\right) - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} - F_x, \tag{4.38}$$

$$\frac{\partial v}{\partial t} = -u(\zeta + f) - \frac{\partial}{\partial y}\left(\frac{u^2 + v^2}{2}\right) - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - F_y, \tag{4.39}$$

where (F_x, F_y) is frictional forcing. Then by cross-differentiating, we obtain

$$\frac{\partial \zeta}{\partial t} = \left\{\begin{array}{cc} -\boldsymbol{V}\cdot\nabla\zeta & \text{advection} \\ -v(df/dy) & \text{"beta" effect} \\ -(\zeta + f)D & \text{convergence + stretching} \\ +\dfrac{\partial v}{\partial z}\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\dfrac{\partial w}{\partial y} & \text{tilting} \\ -\dfrac{\partial}{\partial x}\left(\alpha\dfrac{\partial p}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(\alpha\dfrac{\partial p}{\partial x}\right) & \text{baroclinicity} \\ +\dfrac{\partial F_x}{\partial y} - \dfrac{\partial F_y}{\partial x} & \text{friction} \end{array}\right\}. \tag{4.40}$$

Most of the terms are familiar. Recall that $D \equiv \partial u/\partial x + \partial v/\partial y$. The second term on the rhs, normally written $-v\beta$, represents the so-called "beta effect". It corresponds to changes in relative circulation due to latitudinal displacements: $\partial A_{\perp}/\partial t = -Av(d/dy)\sin\theta$.

For the shallow-water model, the vorticity equation 4.40 reduces to

$$\frac{d(\zeta + f)}{dt} = -(\zeta + f)D \tag{4.41}$$

in the absence of friction. Invoking mass (volume) conservation and integrating over depth, we get

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0, \tag{4.42}$$

which may be compared to 4.31.

4.7 Potential Vorticity

Circulation theorems describe a relationship between velocity and circumference or, implicitly, between vorticity and area. There is a more powerful vorticity theorem that removes both the convergence and solenoidal sources of vorticity by combining the circulation principle with mass and entropy conservation. The use of mass conservation to remove the convergence effect was illustrated for the shallow-water model in 4.31.

What does it take to make the solenoidal term disappear? The baroclinicity vector is

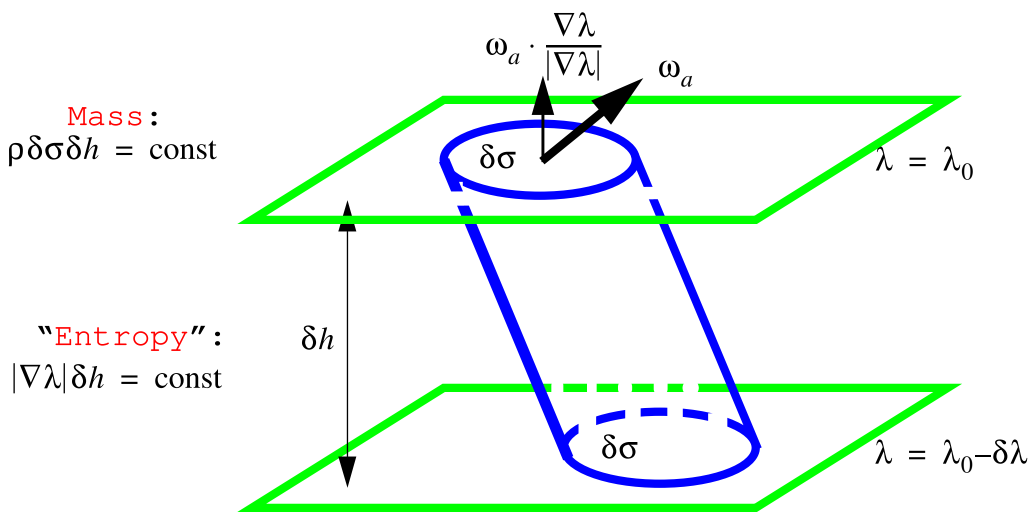
$$\boldsymbol{S} = -\nabla\alpha \times \nabla p = -\nabla s \times \nabla T \tag{4.43}$$

(check the second equality by assuming an ideal gas). We choose one of the quantities α, p, s and T and call it λ. Suppose that, for some reason, $d\lambda/dt = 0$. Then, since there are no solenoids in a λ-surface ($\boldsymbol{S}$ is tangent to it), the circulation around a closed loop in the surface is constant. Then for the material tube illustrated below (not necessarily a vortex tube), we have

$$\omega_a \cdot \frac{\nabla\lambda}{|\nabla\lambda|}\delta\sigma = \text{const}. \tag{4.44}$$

Here $\delta\sigma$ is a scalar element representing the average cross-sectional area of the tube between the two λ-surfaces. We also have the other two constraints noted in the diagram, so that finally

$$\frac{\omega_a \cdot \nabla\lambda}{\rho} = \text{const}. \tag{4.45}$$



This result expresses the conservation of *potential vorticity*. It is important to realize that potential vorticity is conserved even if the fluid is baroclinic.

More generally, we have Ertel's potential vorticity equation,

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega_a} \cdot \nabla\lambda}{\rho}\right) = \frac{\omega_a}{\rho} \cdot \nabla\frac{d\lambda}{dt} + \frac{1}{\rho}\nabla\lambda \cdot \boldsymbol{S}. \tag{4.46}$$

This can be obtained by forming the scalar product of 4.26 with $\nabla\lambda$. The rhs of 4.46 vanishes if $d\lambda/dt = 0$ and $\nabla\lambda \cdot \boldsymbol{S} = 0$. We then have conservation of Ertel's potential vorticity, usually denoted q. In incompressible flow, one gets $dq/dt = 0$ by putting $\lambda \equiv \rho$. In adiabatic flow of an ideal gas, $\lambda \equiv \Theta$ does the trick.

For large-scale disturbances in the atmosphere, we estimate:

	vertical	horizontal
$\lvert\omega_a\rvert$	$10^{-4}\mathrm{s}^{-1}$	$10^{-3}\mathrm{s}^{-1}$
$\lvert\nabla\Theta/\Theta\rvert$	$10^{-5}\mathrm{m}^{-1}$	$10^{-8}\mathrm{m}^{-1}$

Hence 4.45 may be approximated by

$$\frac{\omega_3}{\rho}\frac{\partial\Theta}{\partial z} = \text{const.} \tag{4.47}$$

This is analogous to the quantity ω/h in 4.31 and 4.42.

Problems

4.1 Show that the 2-dimensional flow field in the vicinity of a point can be determined if the velocity, divergence, vorticity and deformation at that point are known.

4.2 Show that the "parallel shear flow", $(u, v) = (0, ax)$, is the sum of uniform vorticity and uniform strain by decomposing the streamfunction into two parts.

4.3 Verify Eq. 4.6. Hint: use the definitions 4.4-4.5 and the transformations for rotations, $\boldsymbol{V'} = \mathbf{R}_\alpha \boldsymbol{V}$ and $\nabla' = \mathbf{R}_\alpha \nabla$, where

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

(column vectors assumed).

4.4 Using Eq. 4.12, decide where, on the unit circle, you would place a point vortex in order to maximize/minimize S_1 at the center of the circle. What about a point mass source?

4.5 A cylindrical column of air at 30 deg latitude with radius 100 km expands horizontally to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after expansion?

4.6 An air column at 60 deg N with $\zeta = 0$ initially reaches from the surface to a fixed tropopause at 10 km height. If the air column moves across a mountain 2.5-km high at 45 deg N, what is its absolute vorticity and relative vorticity as it passes the mountaintop?

4.7 Compute the rate of change of circulation about a square in the x-y plane with sides of 1000-km length if temperature increases eastward at a rate of 1 deg C per 200 km and pressure increases northward at a rate of 1 mb per 200 km. The average pressure is 1000 mb.

4.8 A homogeneous fluid is in solid-body rotation inside a cylindrical tank. By how much does the relative vorticity change in a column of fluid that is moved from the center of the tank to a distance 50 cm from the center? The fluid is rotating at the rate of 20 revolutions per minute and the depth at the center is 10 cm.