$$\text{II.}\quad \hat{B}^*_n = -\frac{1}{2}\left(\sum_i \langle\theta_i\rangle^2 L_i^{(2)} + O(\overline{\theta^4}) + 0\right) \doteq -\frac{1}{2}\sigma^{*2}_{\text{on-coh}}\,, \tag{A.3-19b}$$

which establishes (A.3-18) for the coherent LOBD, as expected. Similarly, for $(\text{LOBD})_{inc}$, (A.3-17d), we get

Incoherent Reception:

$$\text{I.}\quad \sum_{ij}\langle\theta_i\theta_j\rangle[\langle\ell_i\ell_j+\ell_i'\delta_{ij}\rangle_{1,\theta} - \langle\ell_i\ell_j+\ell_i'\delta_{ij}\rangle_{o,o}]$$

$$=0+\frac{1}{4}\sum_{ij}\langle\theta_i\theta_j\rangle^2\{(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}\}-0 \qquad \text{Eq. (A.2-21);} \tag{A.3-20a}$$

$$\sigma^{*2}_{\text{on-inc}} = \frac{1}{4}\sum_{ij}\langle\theta_i\theta_j\rangle^2\{(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}\},\ \text{Eq. (A.2-40),} \tag{A.3-20b}$$

so that I, (A.3-18), is clearly obeyed. We have also directly

$$\text{II.}\quad \hat{B}^*_n = -\frac{1}{2}\cdot\left(\frac{1}{4}\sum_{ij}\langle\theta_i\theta_j\rangle^2\{(L_i^{(4)}-2L_i^{(2)}L_j^{(2)})\delta_{ij}+2L_i^{(2)}L_j^{(2)}\}+0\right) = \frac{-\sigma^{*2}_{\text{on-inc}}}{2}\,, \tag{A.3-20c}$$

which again establishes the desired conditions (A.3-18) for the purely incoherent LOBD, (A.3-17d). Moreover, the proper bias, $\hat{B}^*_n$, in these cases is also equivalent to the average under H_0 of the next nonvanishing term after $H_n(\underset{\sim}{x})^*$, cf. (A.3-17) in the expansion of the original likelihood ratio (here $\log \Lambda_n$), as demonstrated in detail in Appendix A.1 above. In fact, this choice of bias was originally taken [14] to ensure consistency of the test (H_1 vs. H_0) as $n\to\infty$. We have shown above (and in Appendixes A.1, A.2)) that, with the appropriate "small-signal" condition on the input signal ($\bar{a}_0 > 0$) these biases are also the proper biases to insure the AO character of such LOBD's!

I. Detection of the Completely Coherent Signal (Case I):

We must distinguish two general cases of "composite" reception: Case I represents the situation where the signal to be detected is completely deterministic, i.e., entirely specified at the receiver; the only thing unknown to the detector is whether or not the signal is present in the accompanying noise. This means that $\langle\theta_i\rangle = \theta_i$; $\langle\theta_i^2\rangle = \langle\theta_i\rangle^2 = \theta_i^2$, etc., viz., $\langle\theta_i\theta_j\rangle = \langle\theta_i\rangle\langle\theta_j\rangle = \theta_i\theta_j$, etc. Reception is then fully coherent. Case II is the usual practical situation where the signal has random features vis-à-vis the detector, e.g., random fading amplitude, partial phase uncertainties, doppler, etc., so that $\langle\theta_i^2\rangle \neq \langle\theta_i\rangle^2 \neq \theta_i^2$, $\langle\theta_i\theta_j\rangle \neq \langle\theta_i\rangle\langle\theta_j\rangle$, etc. In the coherent detection cases signal epoch is still fully, or at least partially, known, but now the signal itself is only partially deterministic as seen at the detector.

Case I is rare in practice, while Case II represent essentially all practical applications. Nevertheless, before we can proceed to establish the LOB and AO conditions for the "composite" algorithms consisting of a suitable combination of purely coherent and incoherent LOBD's [cf. III, below), we must examine Case I for the two subcases (i), general nongauss noise, (ii), gauss noise.

For this purpose, let us use (A.1-4), (A.1-4a,b) modified, as usual, for independent (noise) samples (cf. Sec. A.1-2), to write first the general expansion of the optimum algorithm (A.1-4) (cf. (2.9) et seq., also)

$$\log \Lambda_n = \log \mu - \sum_i^n \ell_i\langle\theta_i\rangle + \frac{1}{2}\sum_{ij}^n [\langle\theta_i\theta_j\rangle - (\ell_i\ell_j + \ell_i'\delta_{ij}) - \langle\theta_i\rangle\langle\theta_j\rangle\ell_i\ell_j]$$

$$+\Theta_3 + \Theta_4 + t_n, \tag{A.3-21}$$

where explicitly,

$$\Theta_3 = -\frac{1}{3!}\left\{\left[\sum_i^n \langle\theta_i^3\rangle\left(\frac{w_{1i}^{(''')}}{w_{1i}}\right) + 3\sum_{ij}^{n'} \langle\theta_i\theta_j^2\rangle \ell_i(\ell_j^2+\ell_j') + \sum_{ijk}^{n('')=(i\neq j\neq k)} \langle\theta_i\theta_j\theta_k\rangle \ell_i\ell_j\ell_k\right]\right.$$

$$- \sum_{ijk}^n \langle\theta_i\rangle\langle\theta_j\theta_k\rangle \ell_i(\ell_j\ell_k+\ell_j'\delta_{jk}) + 2\sum_{ijk}^n \langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_k\rangle \ell_i\ell_j\ell_k \; ; \qquad \text{(A.3-21a)}$$

$$\Theta_4\Big|_{indep} = \text{Eq. (A.1-4b), Independent samples;} \qquad \text{(A.3-21b)}$$

and t_n is a remainder term.

Now, in the fully coherent (deterministic) Case I described above we may drop all the averages $\langle\ \rangle$ on θ_i, etc., in (A.3-21)-(A.3-21b). Clearly, Θ_3, Θ_4 do not vanish identically when the noise is nongaussian, i.e. $\ell_i \neq -x_i$, $\ell_i' = -1$. Consequently, for nongaussian noise the expansion (A.3-21) does not terminate (t_n being a series itself). Next, we use as our algorithm the first two terms of (A.3-21), ($\langle\theta_i\theta_j\rangle = \theta_i\theta_j$, etc.) with the bias B_n^* chosen as before (cf. Sec. A.1-3)

$$g^*_{n\text{-comp}}\Big|_{\text{Case I}} \equiv \log\mu - \sum_i^n \ell_i\theta_i + \frac{1}{2}\sum_{ij}^n \theta_i^2\ell_i' + [\langle\Theta_3\rangle_o \text{ or } \langle\Theta_4\rangle_o]_{\text{Case I}}, \qquad \text{(A.3-22)}$$

where the bias is established as the first non-vanishing term in the expansion of $\log\Lambda_n$ after the term $O(\langle\theta^2\rangle)$ when the average (over the data $\{x_i\}$) is taken with respect to the null hypothesis (H_0).

Let us evaluate $\langle\Theta_3\rangle_o$, (A.3-21a) here, without invoking the strictly deterministic conditions of Case I. Since $\langle\ell_i\rangle_o = 0$, $\langle\ell_i^2+\ell_i'\rangle_o = \langle w_1''/w_1\rangle_o = 0$; $\langle\ell_i^3\rangle_o = 0$, we see at once that each term of $\langle\Theta_3\rangle_o$, (A.3-21a), vanishes, so that $\langle\Theta_3\rangle_o = 0$, without recourse to the condition $\langle\theta_i\theta_j\theta_k\rangle = 0$, cf. (A.1-6) and footnote. Accordingly, the bias term is always $\langle\Theta_4\rangle_o$ ($\neq 0$), here, cf. Sec. A.1-3, Eqs. (A.1-20a,b). For Case I (non-gauss) here we get accordingly

$$g^*_{n\text{-comp}}\Big|_{\text{Case I}} = \log \mu + \{-\frac{1}{8}\sum_{ij} \theta_i^2\theta_j^2[(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}]\}$$

$$-\sum_i \theta_i \ell_i + \frac{1}{2}\sum_i \theta_i^2 \ell_i' \; . \qquad \text{(A.3-23)}$$

Our next step is to show that (A.3-23) does not satisfy the conditions (A.3-18) for LOBD and AODA's. To demonstrate this we evaluate $\langle g^*_{n\text{-Case I}}\rangle_{1,o}$ and $\text{var}_o g^*_{n\text{-Case I}}$. Accordingly, we have

$$\langle g^*_{n\text{-}I}\rangle_{1,\theta} = \log \mu + B^*_{N\text{-}I} - \sum_i \theta_i \langle \ell_i\rangle_{1,\theta} + \frac{1}{2}\sum_i \theta_i^2 \langle \ell_i'\rangle_{1,\theta} \; , \qquad \text{(A.3-24)}$$

where

$$\left\{\begin{array}{ll} \langle \ell_i\rangle_{1,\theta} = -\theta_i L_i^{(2)} - \theta_i^3 L_i^{(2,2)}; \; \langle \ell_i'\rangle_{1,\theta} & = \int_{-\infty}^{\infty} (\frac{w_{1i}''}{w_{1i}} - \ell_i^2) w_{1i}(x_i-\theta_i)dx_i \\ \langle \ell_i\rangle_o = 0; \quad \langle \ell_i'\rangle_o = -L_i^{(2)} \; ; & = \int_{-\infty}^{\infty} (\frac{w_{1i}''}{w_{1i}} - \ell_i^2)[w_{1i} - \theta_i w_{1i}' + \frac{\theta_i^2}{2} w_{1i}'' + ..]dx_i \\ & = \frac{\theta_i^2}{2} L_i^{(4)} - L_i^{(2)} - \frac{\theta_i^2}{2} L_i^{(2,2)} + .. \end{array}\right\}$$

(A.3-24a)

$$\therefore \langle g^*_{n\text{-}I}\rangle_{1,\theta} \doteq \log \mu + B^*_{n\text{-}I} + \sum_i^n \left\{\frac{\theta_i^2 L_i^{(2)}}{2} + \frac{\theta_i^4}{4}[L_i^{(4)} + 3L_i^{(2,2)}] + \ldots\right\} \; ; \text{ and} \qquad \text{(A.3-25a)}$$

$$\langle g^*_{n\text{-}I}\rangle_{o,o} = \log \mu + B^*_{n\text{-}I} - \sum_i \frac{\theta_i^2}{2} L_i^{(2)} \; . \qquad \text{(A.3-25b)}$$

Accordingly, Condition I., (A.3-18), becomes here

$$\langle g^*\rangle_{1,\theta}-\langle g^*\rangle_{o,o} = \sum_i \theta_i^2 L_i^{(2)} + \sum_i \frac{\theta_i^4}{4} (L_i^{(4)}+3L_i^{(2,2)})+O(\theta^6). \tag{A.3-26}$$

Next, we evaluate

$$\mathrm{var}_o g^*_{n-I} = \left\langle\left[\sum_i^n \left\langle\left(-\ell_i\theta_i + \frac{1}{2}\ell_i'\theta_i^2\right)\right]^2\right\rangle_o - \left\langle\sum_i (-\ell_i\theta_i + \frac{1}{2}\ell_i'\theta_i^2)\right\rangle_o^2 \tag{A.3-27}$$

$$= \sum_{ij}^n \left\langle(\ell_i\ell_j\theta_i\theta_j - \ell_i\ell_j'\theta_i\theta_j^2 + \frac{1}{4}\ell_i'\ell_j'\theta_i^2\theta_j^2)\right\rangle_o - \frac{1}{4}\left(\sum_i -\theta_i^2 L_i^{(2)}\right)^2. \tag{A.3-27a}$$

We observe that

$$\langle\ell_i\ell_j\rangle_o = L_i^{(2)}\delta_{ij} \;;\; \langle\ell_i\ell_j'\rangle_o = 0 \;;\; \langle\ell_i'\ell_j'\rangle_o = L_i^{(2)}L_j^{(2)}(1-\delta_{ij}) \;; \text{ or}$$

$$= \langle\ell_i'^2\rangle_o = \int_{-\infty}^{\infty}\left(\frac{w_1''}{w_1} - \ell^2\right)^2 w_1\,dx$$

$$[\langle\ell^4\rangle_o = \frac{L^{(2,2)}}{2}\text{, cf. (A.2-16a)]:} \qquad = L_i^{(4)} - 2L_i^{(2,2)} + \langle\ell_i^4\rangle_o,$$

$$= L_i^{(4)} - \frac{3}{2}L_i^{(2,2)}, \tag{A.3-28}$$

so that

$$\mathrm{var}_o g^*_{n-I} = \sum_i^n \theta_i^2 L_i^{(2)} + \left\{\frac{1}{4}\sum_i^n \theta_i^4[L_i^{(4)} - \frac{3}{2}L_i^{(2,2)} - L_i^{(2)^2}] = \sigma^{*2}_{on-I}\right. . \tag{A.3-29}$$

Comparing (A.3-29) and (A.3-26), $O(\theta^4)$, shows indeed that $\langle g^*\rangle_{1,\theta}-\langle g^*\rangle_{o,o} \neq \sigma^{*2}_{on-I}$, so that Condition I, (A.3-18), is violated. Moreover, Condition II

(A.3-18), is also clearly not obeyed, since from (A.3-25a,b) on removing $\log \mu(g^* \to \hat{g}^*)$ we get

$$-\frac{1}{2}\{\langle \hat{g}^*_{n-I}\rangle_{1,\theta} + \langle \hat{g}^*_{n-I}\rangle_{o}\} = -\frac{1}{2}\{2\hat{B}^*_{n-I} + \frac{1}{4}\sum_i^n \theta_i^4 (L_i^{(4)} + 3L_i^{(2,2)})\} \neq -\sigma^{*2}_{on-I}/2 \;, \tag{A.3-30}$$

where $\hat{B}^*_{n-I}$ is the bias term of (A.3-23), without $\log \mu$.

The upshot of the above is that for nongauss noise, under Case I (entirely coherent signal) conditions, we must follow the schema for the Composite LOBD's (III ff.): the proper composite form to use here is the sum of the (purely) coherent LOBD and a completely incoherent LOBD form, as for case II in practical situations.

On the other hand, when the noise is gaussian, the general expansion (A.3-21) terminates, e.g. $\Theta_3 = \Theta_4 \ldots, t_n = 0$, and, since $\ell_i = -x_i$, $\ell_i' = -1$ here, we get

$$\log \Lambda_n \Big|_{\text{det-Case I}} = \{\log \mu - \frac{1}{2}\sum_i^n \theta_i^2\} + \sum_i^n \theta_i x_i \;, \tag{A.3-31}$$

which is the well-known, exact result in this very special, limiting situation. The reason that we get different results in the two different noise situations (same completely specified signal) is that the signal and noise interact in a more complex fashion in the nongauss cases, so that adding the "incoherent" term (properly) increases the information at the detector relevant to the interaction and hence improves detection.

II. The General Composite ("On-off") LOBD: (Case II)

Our next step is to determine whether or not the composite, or "mixed" LOBD, (A.3-17b), also obeys the fundamental AO conditions (A.3-18), or (A.3-16). Here for this LOBD there is enough phase coherence at the receiver to obtain a coherent ($\langle\theta\rangle > 0$) as well as an incoherent contribution.

Let us begin with Condition I., (A.3-18), writing

$$\boxed{H^*_{comp} = \gamma^*_{coh}+\gamma^*_{inc}-\frac{\gamma^{*2}_{inc}}{2}} \tag{A.3-32}$$

$$\gamma^*_{coh} = \sum_i -\langle\theta_i\rangle \ell_i \;;\; \gamma^*_{inc} = \frac{1}{2}\sum_{ij} \langle\theta_i\theta_j\rangle(\ell_i\ell_j+\ell_i'\delta_{ij}) \;; \tag{A.3-32a}$$

$$\therefore\; \hat{g}^*_{comp} = \hat{B}^*_n + H^*_{comp} = \hat{B}^*_n+(\gamma^*_{coh}+[\gamma^*_{inc}-\gamma^{*2}_{inc}/2]) \;; \tag{A.3-33}$$

$$\therefore\; \mathrm{var}_{o,o}\hat{g}^*_{comp} = \langle H^{*2}_{comp}\rangle_{o,o}-\langle H^*_{comp}\rangle^2_{o,o} \;. \tag{A.3-34}$$

Expanding (A.3-34) gives

$$\mathrm{var}_{o,o}\hat{g}^*_{comp} = \mathrm{var}_o\gamma^*_{coh}+\mathrm{var}_o\gamma^*_{inc}+\{\tfrac{1}{4}\,\overset{①}{\mathrm{var}_o\gamma^{*2}_{coh}}+2(\langle\gamma^*_{coh}\gamma^*_{inc}\rangle_o-\langle\gamma^*_{coh}\rangle_o\langle\gamma^*_{inc}\rangle_o)$$

$$-(\langle\gamma^{*3}_{coh}\rangle_o-\langle\gamma_{coh}\rangle_o\langle\gamma^{*2}_{coh}\rangle_o)$$

$$-(\langle\underset{②}{\gamma^*_{inc}\gamma^{*2}_{coh}}\rangle-\langle\gamma^*_{inc}\rangle_o\langle\gamma^{*2}_{coh}\rangle_o)\}. \tag{A.3-35}$$

Proceeding, we have

$$\langle\gamma^*_{coh}\rangle_o = 0;\; \langle\gamma^{*2}_{coh}\rangle_o = \sum_i \langle\theta_i\rangle^2 L_i^{(2)}[= \mathrm{var}\;\hat{g}^*_{coh},\ \mathrm{cf.}\ (A.3\text{-}19a)]; \tag{A.3-36}$$

$$\left\langle \gamma^{*3}_{coh} \right\rangle_o = - \sum_{ijk}^{n} \langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_k\rangle\langle\ell_i\ell_j\ell_k\rangle_o = 0 \begin{cases} i\neq j\neq k \\ i=j(\neq k) \\ i=j=k:\ \text{odd} \end{cases} \tag{A.3-36b}$$

$$\left\langle \gamma^{*4}_{coh} \right\rangle_o = \sum_{ijk\ell}^{n} \langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_k\rangle\langle\theta_\ell\rangle\langle\ell_i\ell_j\ell_k\ell_\ell\rangle_o = 0 \text{ unless } i=j=k=\ell;$$

" $(i=j)\neq(k=\ell)$ in various combinations.

$$\text{i.e.} \begin{cases} 2x(i=j)\neq(k=\ell):\ (i=j,j=i)\equiv 2x(i=j) \text{ etc.} \\ 2x(i=k)\neq(j=\ell) \\ 2x(i=\ell)\neq(j=k) \end{cases}$$

$$= \sum_{i}^{n} \langle\theta_i\rangle^4 \langle\ell_i^4\rangle_o + 6 \sum_{ij}^{n}{}' \langle\theta_i\rangle^2\langle\theta_j\rangle^2 L_i^{(2)} L_j^{(2)} \ ; \begin{cases} \langle\ell_i^4\rangle_o = \frac{1}{2} L_i^{(2,2)}\ , \\ \text{cf. (A.2-16a)} \end{cases} \tag{A.3-36c}$$

$$\therefore \operatorname{var} \gamma^*_{coh} = \sum_i \langle\theta_i\rangle^2 L_i^{(2)} = \operatorname{var} \hat{g}^*_{coh}\ . \tag{A.3-36d}$$

$$\text{①} \therefore \operatorname{var} \gamma^{*2}_{coh} = \langle\gamma^{*4}_{coh}\rangle_o - \langle\gamma^{*2}_{coh}\rangle_o^2$$

$$= \sum_i^n \langle\theta_i\rangle^4 \frac{L_i^{(2,2)}}{2} + 6 \sum_{ij}^{n}{}' \langle\theta_i\rangle^2\langle\theta_j^2\rangle L_i^{(2)} L_j^{(2)} - \sum_{ij} \langle\theta_i\rangle^2\langle\theta_j\rangle^2 L_i^{(2)} L_j^{(2)}$$

$$= \sum_i^n \langle\theta_i\rangle^4 \left\{ \underbrace{\left(\frac{L_i^{(2,2)}}{2} - L_i^{(2)^2} \right)}_{= \operatorname{var}_o \ell^2} \right\} + 5 \sum_{ij}^{n}{}' \langle\theta_i\rangle^2\langle\theta_j^2\rangle L_i^{(2)} L_j^{(2)}. \tag{A.3-36e}$$

Also, we see that

② $$\left\langle \gamma^*_{inc}\gamma^{*2}_{coh}\right\rangle_o = \sum_{ijk\ell}\left\langle\theta_i\theta_j\right\rangle\left\langle(\ell_i\ell_j+\ell_i'\delta_{ij})\left\langle\theta_k\right\rangle\left\langle\theta_\ell\right\rangle\ell_k\ell_\ell\right\rangle_o = 0$$

$$\left.\begin{matrix} i \neq j \neq k \neq \ell \\ i = j\ (\neq k \neq \ell) \\ (i=j)\neq(k=\ell) \end{matrix}\right\}$$

$$= 4\sum_{ij}{}' \left\langle\theta_i\theta_j\right\rangle\left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle L_i^{(2)}L_j^{(2)} \neq 0;$$

$$\left.\begin{matrix} (i=k)\neq(j=\ell)\times 2 \\ (i=\ell)\neq(j=k)\times 2 \end{matrix}\right\} \qquad \text{(A.3-37a)}$$

and

$$\left\langle\gamma^*_{inc}\gamma^*_{coh}\right\rangle_o = -\sum_{ijk}\left\langle\theta_i\theta_j\right\rangle\left\langle(\ell_i\ell_j+\ell_i'\delta_{ij})\left\langle\theta_k\right\rangle\ell_k\right\rangle_o = 0; \qquad \text{(A.3-37b)}$$

$$\left\langle\gamma^*_{inc}\right\rangle_o = 0. \qquad \text{(A.3-37c)}$$

Consequently, (A.3-35) reduces to

$$\boxed{\begin{aligned} \text{var}_o\hat{g}^*_{comp} = {} & \text{var}_o\hat{g}^*_{coh}+\text{var}\ \hat{g}^*_{inc} + \{ \frac{1}{4}\sum_i^n \left\langle\theta_i\right\rangle^4 \text{var}_o\ell^2 \\ & + \frac{5}{4}\sum_{ij}^{n\,\prime} \left\langle\theta_i\right\rangle^2\left\langle\theta_j\right\rangle^2 L_i^{(2)}L_j^{(2)} - 4\sum_{ij}^{n\,\prime} \left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle\left\langle\theta_i\theta_j\right\rangle L_i^{(2)}L_j^{(2)} \}. \end{aligned}}$$

(A.3-38)

Next, we have

$$(\langle\hat{g}^*\rangle_{1,\theta}-\langle\hat{g}^*\rangle_{o,o\ \mathrm{comb}} = \langle\gamma^*_{coh}+\gamma^*_{inc}-\gamma^{*2}_{coh}/2\rangle_{1,\theta}-\langle\gamma^*_{coh}+\gamma^*_{inc}-\gamma^{*2}_{coh}/2\rangle_{o,o} \tag{A.3-39a}$$

$$= \underset{\substack{\downarrow\\ \text{Eq.(A.3-19a)}}}{\mathrm{var}_o\hat{g}^*_{coh}} + \underset{\substack{\downarrow\\ \text{Eqs.(A.3-20a,b)}}}{\mathrm{var}_o\hat{g}^*_{inc}} - \langle\gamma^{*2}_{coh}/2\rangle_{1,\theta}-0-0+\frac{1}{2}\,\mathrm{var}_o\hat{g}^*_{coh}. \tag{A.3-39b}$$

Also, we obtain

$$\langle\gamma^{*2}_{coh}\rangle_{1,\theta} = \sum_{ij}^{n}\langle\theta_i\rangle\langle\theta_j\rangle\langle\ell_i\ell_j\rangle_{1,\theta} = \sum_{i}^{n}\langle\theta_i\rangle^2\{L_i^{(2)}+\frac{\langle\theta_i^2\rangle}{2}L_i^{(2,2)}+\ldots\}$$

$$+\sum_{ij}^{n}{}'\langle\theta_i\rangle\langle\theta\rangle_j\langle\theta_i\theta_j\rangle L_i^{(2)}L_j^{(2)}+\ldots, \tag{A.3-39c}$$

(from (A.2-8).

Inserting this into (A.3-39b) and using (A.3-36d), we get finally

$$(\langle\hat{g}^*\rangle_{1,\theta}-\langle\hat{g}^*\rangle_{o,o})_{comb} = \mathrm{var}_o\hat{g}^*_{coh}+\mathrm{var}_o\hat{g}^*_{inc}$$

$$-\sum_{i}^{n}\frac{\langle\theta_i\rangle^2\langle\theta_i^2\rangle}{4e}L_i^{(2,2)}-\frac{1}{2}\sum_{ij}^{n}{}'\langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_i\theta_j\rangle L_i^{(2)}L_j^{(2)}$$

$$= \mathrm{var}_o\hat{g}_{coh}+\mathrm{var}\ \hat{g}^*_{inc}$$

$$-\frac{1}{2}\{\sum_{i}^{n}\langle\theta_i\rangle^4\mathrm{var}_o\ell^2+\sum_{ij}^{n}\langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_i\theta_j\rangle L_i^{(2)}L_j^{(2)}\}. \tag{A.3-40}$$

Clearly, comparing (A.3-38) and (A.3-40) shows at once that condition I, (A.3-18), is not obeyed here for the composite LOBD, (A.3-17b).

Moreover, Condition II, (A.3-18), is seen from (A.3-39a)-(A.3-39c) to be

$$\hat{B}_n^* = -\frac{1}{2}\left[\mathrm{var}_o\hat{g}_{coh}^* + \mathrm{var}_o\hat{g}_{inc}^* + -\sum_i^n \langle\theta_i\rangle^2 L_i^{(2)}\right.$$

$$\left. -\frac{1}{2}\left[\sum_i^n \langle\theta_i\rangle^4\ \mathrm{var}_o\ell^2 + \sum_{ij}^n \langle\theta_i\rangle\langle\theta_j\rangle\langle\theta_i\theta_j\rangle L_i^{(2)}L_j^{(2)}\right]\right], \qquad \text{(A.3-41)}$$

which is also not equal to $-\sigma_{on}^{*2}/2$ $(= -\frac{\mathrm{var}_o}{2}\hat{g}_{comp}^*)$, Eq. (A.3-27), unless $\bar{\theta}_i = 0$, (whereupon $\hat{g}_{coh} = 0$, of course).

Thus, we reach the important conclusion that when $\bar{\theta} > 0$ the general composite LOBD = $\hat{g}_{comp}^*$, (A.3-33), which includes the component $(-\gamma_{inc}^{*2}/2)$ in the incoherent position, is not an AODA as $n\to\infty$. Hence when $(\bar{\theta}_i > 0)$ it is always alternatively better to use the coherent LOBD alone, [without the full coherent term, $(\gamma_{inc}^* - \gamma_{coh}^{*2}/2)$ for small input signals, and hence large $n(>>1)$, for acceptably small error probabilities. However, as we note in III below, it is possible to find a composite LOBD which is better than either the $LOBD_{inc}$ or $LOBD_{coh}$ and the above general composite form (A.3-33):

III. The Composite ("On-off") LOBD: Case II

Although the general composite LOBD = g_{comp}^*, (A.3-33), which includes the term $(-\gamma_{coh}^{*2}/2)$ in the incoherent component, cf. (2.9), is not an AODA as we have shown above (I), we can easily find a composite LOBD which has the desired AO qualities and is better than either the coherent or incoherent LOBD's. This is accomplished immediately by setting $\langle\theta_i\rangle = 0$ in the incoherent portion of the algorithm, viz.

$$\hat{g}_{comp}^* = \hat{B}_{comp}^* + \gamma_{coh}^* + \gamma_{inc}^*, \qquad \text{(A.3-42)}$$

cf. (A.3-32,33). We call this composite LOBD a composite LOBD, or simply a composite or "mixed" LOBD, as distinct from the "general composite" LOBD discussed in I preceding. Accordingly, from (A.3-38) and (A.3-40), (A.3-41), we see that with $\langle\theta_i\rangle = 0$ in the $\gamma^{*2}_{coh}/2$ term (which then vanishes) that

$$\mathrm{var}_o \hat{g}^*_{comp} = \mathrm{var}_o \hat{g}^*_{coh} + \mathrm{var}_o \hat{g}^*_{inc} = (\langle\hat{g}^*\rangle_{1,\theta} - \langle\hat{g}^*\rangle_{o,o})_{comp}$$

$$= -2\hat{B}^*_{n-comp} = -2(\hat{B}^*_{n-coh} + \hat{B}^*_{n-inc}) \ . \qquad (A.3-43)$$

Thus, conditions I and II, (A.3-16), or (A.3-18) are fulfilled, and consequently $\hat{g}^*_{comp}$ is LOBD and AODA. Accordingly, this composite or mixed LOBD is simply the sum of the separate strictly coherent and strictly incoherent ($\bar{\theta}_1=0$) LOBD's of our principal analysis, with a composite bias which is the sum of the separate biases. Thus, this composite or "mixed" LOBD is specifically (in these "on-off" cases)

$$g^*_{n-comp} = \log\mu + [-\frac{1}{2}\sum_i^n \langle\theta_i\rangle^2 L_i^{(2)} - \frac{1}{8}\sum_{ij}^n \langle\theta_i\theta_j\rangle^2 \{(L_i^{(4)} - 2L_i^{(2)^2})\delta_{ij} + 2L_i^{(2)}L_j^{(2)}\}] - \sum_i^n \langle\theta_i\rangle \ell_i + \frac{1}{2}\sum_{ij}^n \langle\theta_i\theta_j\rangle[\ell_i\ell_j + \ell_i'\delta_{ij}] \ , \qquad (A.3-44)$$

$$= \log\mu + \hat{B}^*_{n-coh} + \hat{B}^*_{n-inc} + \sum_i^n -\ell_i\langle\theta_i\rangle + \frac{1}{2}\sum_{ij}^n [\ell_i\ell_j + \ell_i'\delta_{ij}]\langle\theta_i\theta_j\rangle \qquad (A.3-44a)$$

$$= \log\mu + \mathrm{LOBD}_{coh.} + \mathrm{LOBD}_{inc.} = \log\mu + \mathrm{LOBD}_{comp} \ . \qquad (A.3-44b)$$

These are the optimum canonical forms for the mixed threshold cases, for general signals and interference, which become AODA's as sample-size $n \to \infty$.

Several remarks are in order: (i), the "small-signal" condition here is essentially that which applies to the essential equality of the H_1 and H_0 variances; there is now no purely coherent or incoherent algorithm, cf. remarks ff. (6.79c); (ii), we can obtain the various (optimum) performance (i.e. error probability) measures [Sec. 6.1] directly, by appropriate use of the variance σ_{on}^{*2} ; (iii), extension to the binary signal cases is direct, cf. (6.12), (6.28), for $\sigma_{coh}^{(21)*2}$, $\sigma_{inc}^{(21)*2}$ used in (A.3-44). However the notions of processing gain and minimum detectable signal [Sec. 6.2] need to be redefined, a task we have briefly outlined in Sec. 6.5; (iv), for suboptimum systems, the conditions (A.3-16,18) are not obeyed, and these algorithms are neither LOBD's or AODA's, since $f_1^2 = f_1^*$, $f_2 \neq f_2^*$, cf. (A.3-38), i.e. they are not derived from the expansion of a likelihood ratio.

We note, also, that the composite results (A.3-44) apply, as well, for completely deterministic signals [with $\langle\theta_i\rangle = \theta_i$, $\langle\theta_i\theta_j\rangle = \theta_i\theta_j$, etc.] Case I, cf. I above, as long as the noise is nongaussian (which means that g_n^* is not the full expansion of $\log \Lambda_n$). In the gaussian situation (Case I), $\log \Lambda_n = g_n^*\big|_{gauss}$ terminates after the term $O(\theta^2)$ in the expansion, as required, cf. (A.3-31). The improvement gained in the Case I situations (when the noise is nongaussian) arises from the additional information relevant to signal and noise interaction in the composite LOBD form vis-à-vis the purely coherent LOBD form. For example, let us suppose that the noise is "Laplace" noise (A.4-50b); then for these Case I situations we have explicitly

$$\log \Lambda_n = \log \exp \{ \sqrt{2} \sum_i (|x_i| - |x_i - \theta_i|) \doteq g_{n\text{-comp}}^* , \qquad \text{(A.3-44c)}$$

and clearly the signal-noise "interaction" embodied in $|x_i - \theta_i|$, is not at all simple, resulting in a non-terminating series of the form (A.3-21).

Finally, we note that g^*_{n-comp} is never less effective (in performance) than g^*_{n-inc} and is always better than g^*_{n-coh} when coherent reception is possible at the receiver. This follows at once from the fact that, cf. (A.3-32):

$$\sigma^{*2}_{o-comp} = \sigma^{*2}_{o-coh} + \sigma^{*2}_{o-inc} \geq \sigma^{*2}_{o-coh} \text{ or } \sigma^{*2}_{o-inc} , \qquad \text{(A.3-33)}$$

where explicitly

("on-off"):

$$\boxed{\sigma^{*2}_{o-comp} = \sum_{i}^{n} \langle\theta_i\rangle^2 L_i^{(2)} + \frac{1}{4} \sum_{ij}^{n} \langle\theta_i \theta_j\rangle^2 \{(L_i^{(4)} - 2L_i^{(2)^2})\delta_{ij} + 2L_i^{(2)} L_j^{(2)}\},} \qquad \text{(A.3-34a)}$$

with bias

$$\hat{B}^*_{n-comp} = -\frac{1}{2}(\sigma^{*2}_{o-coh} + \sigma^{*2}_{o-inc}) = -\frac{1}{2}\,\sigma^{*2}_{o-mixed} = -\frac{1}{2} \cdot \text{Eq. (A.3-34a)}. \qquad \text{(A.3-34b)}$$

Using (A.3-33), (A.3-34a) in (6.2), (6.5), (and (6.5a), (6.5e) for binary signals), at once establishes the above statements.

IV. Binary Signals:

In the case of binary signals, we have at once from (6.12), (6.28), generally (binary):

$$
\left(\sigma^{(21)*}_{o\text{-}comp}\right)^2 = \left(\sigma^{(21)*}_{o\text{-}coh}\right)^2 + \left(\sigma^{(21)*}_{o\text{-}inc}\right)^2 = \sum_{i}^{n} L_i^{(2)} \left\{\left\langle a_{oi}^{(2)} s_i^{(2)}\right\rangle - \left\langle a_{oi}^{(1)} s_i^{(1)}\right\rangle\right\}^2
$$

$$
+ \frac{1}{4} \sum_{ij}^{n} \left(\left\{L_i^{(4)} - 2L_i^{(2)^2}\right\}\delta_{ij} + 2L_i^{(2)} L_j^{(2)}\right)
$$

$$
\cdot\left(\left\langle a_{oi}^{(2)} a_{oj}^{(2)} s_i^{(2)} s_j^{(2)}\right\rangle - \left\langle a_{oi}^{(1)} a_{oj}^{(1)} s_i^{(1)} s_j^{(1)}\right\rangle\right)^2 \quad .
$$

(A.3-35)

However, bias is now from (4.3a), (4.5a)

(binary):

$$
\hat{B}^{(21)*}_{n\text{-}comp} = \hat{B}^{*}_{n\text{-}coh} + \hat{B}^{*}_{n\text{-}inc} = -\frac{1}{2} \sum_{i}^{n} L_i^{(2)} \left\{\left\langle a_{oi}^{(2)} s_i^{(2)}\right\rangle^2 = \left\langle a_{oi}^{(1)} s_i^{(1)}\right\rangle^2\right\}
$$

$$
- \frac{1}{8} \sum_{ij}^{n} \left[\left(L_i^{(4)} - 2L_i^{(2)^2}\right)\delta_{ij} + 2L_i^{(2)} L_j^{(2)}\right]
$$

$$
\cdot\left[\left\langle a_{oi}^{(2)} a_{oj}^{(2)} s_i^{(2)} s_j^{(2)}\right\rangle^2 - \left\langle a_{oi}^{(1)} a_{oj}^{(1)} s_i^{(1)} s_j^{(1)}\right\rangle^2\right] , \qquad \text{(A.3-36)}
$$

which with the appropriate averages [$\langle\,\rangle_{1,\theta}$, etc.] over $\gamma^{(21)*}_{coh}$, $\gamma^{(21)*}_{inc}$, cf. (A.3-17), is required to give the correct variance (A.3-35) to this level of "small-signal" approximation, which insures that $\sigma^{(21)*}_{o\text{-}comp} \doteq \sigma^{(21)*}_{o\text{-}comp}$. The actual "small-signal" conditions are given by (A.2-15a), (A.2-42). However, we note again that the only condition here is that of equal variances; cf. remarks after (A.3-44). The LOBD (and AODA) in these binary cases is, of course, like (A.3-27) in the "on-off" situation,

$$g^{(21)*}_{n\text{-}comp} = \log \mu + LOBD^{(21)}_{coh} + LOBD^{(21)}_{inc}$$

$$= \log \mu + \hat{B}^{(21)*}_{n\text{-}comp} - \sum_i^n \ell_i \left(\left\langle a^{(2)}_{oi} s^{(2)}_i \right\rangle - \left\langle a^{(1)}_{oi} s^{(1)}_i \right\rangle \right)$$

$$+ \frac{1}{2} \sum_{ij}^n (\ell_i \ell_j + \ell_i' \delta_{ij}) \left[\left\langle a^{(2)}_{oi} a^{(2)}_{oj} s^{(2)}_i s^{(2)}_j \right\rangle - \left\langle a^{(1)}_{oi} a^{(1)}_{oj} s^{(1)}_i s^{(1)}_j \right\rangle \right]. \quad \text{(A.3-37)}$$

cf. (4.3),(4.5). (Optimum) performance, again, is obtained from (A.3-35) in (6.5a, 6.5e) directly. [For an example, see Part II, Section II, C of [1], and Figure 2 therein, in the specific binary case of narrow band signals with partially known RF phases.]

APPENDICES (cont'd):

Part II. Suboptimum Threshold Detectors

(David Middleton)

APPENDIX A-4

Canonical Formulations:

In this Appendix we shall derive both general and particular forms of suboptimum threshold signal detection algorithms, and their associated means and variances under (H_0,H_1), [or H_1,H_2) in the binary signal cases]. Again, we postulate independent noise samples, although our canonical approach is not in principle affected by this (not serious) constraint. In the following we first consider the canonical treatment of suboptimum receivers and then specialize the results to two particular limiting cases of suboptimum detectors, namely, clipper-correlators, using "super-clippers", and simple correlation detectors (i.e., without clipping). In these suboptimum cases we cannot, of course, expect the algorithms to be AODA's, [cf. Sec. A.3-3)], nor are they LOB optimum for any finite sample size ($n > \infty$). However, an exception to this arises when this particular class [cf. (A.4-1,2) ff.] of detectors is employed in the interference for which they <u>are</u> optimum, as we shall see in what follows, cf. Sec. A.4-1 ff.

A.4-1. A Class of Canonical Suboptimum Threshold Detection Algorithms:

Guided by the optimum canonical forms above [cf. (2.9) et seq., and in particular, (4.1) and (4.4)], we can specify a broad general class of generally suboptimum detection algorithms, defined essentially by their similar dependence on input signal structure [through $\langle\theta_i\rangle$, $\langle\theta_i\theta_j\rangle$], viz:

I. Coherent Detection:

$$g(\underset{\sim}{x})_{coh} = \log\mu + \hat{B}'_{coh} - \sum_i^n \langle\theta_i\rangle F(x_i) \; ; \qquad \text{(A.4-1)}$$

II. Incoherent Detection:

$$g(\underset{\sim}{x})_{inc} = \log \mu + \hat{B}'_{inc} + \frac{1}{2!}\sum_{ij}^{n} \langle \theta_i \theta_j \rangle H(x_i, x_j) \ , \tag{A.4-2}$$

where F, H are (real) functions of the data elements $\{x_i, x_j\}$, subject to appropriate constraints (to be discussed presently, cf. A.4-1,D) to insure that these algorithms do not produce singular results on finite sample sizes ($n<\infty$).

For the moment, the biases, $\hat{B}'_{(\ )}$, are arbitrary, while it is assumed that F and H are specified. It is desireable, however, that under appropriate circumstances these algorithms become LOBD's. This means, then, at once by direct comparison with the canonical LOBD forms (4.1), (4.4), that

$$\ell_{F_i} = F_i \ ; \text{ i.e., } \ell_i \text{ generally} \rightarrow F_i \ ; \therefore \ell'_i \rightarrow F'_{(\ )}, \text{ etc.,} \tag{A.4-3a}$$

$$H(x_i, x_j) \rightarrow \ell_{Fi}\ell_{Fj} + \ell'_{Fi}\delta_{ij} \rightarrow F_i F_j + F'_i \delta_{ij} . \tag{A.4-3b}$$

[A sometimes useful extension of this is $F_i \rightarrow F_{ij}\delta_{ij}$, cf. Sec. A.4-3 for an example. One simply replaces F_i by $F_{ij}\delta_{ij}$, etc. in the results below.] The bias is unspecified, and the algorithms contain no higher order terms in θ_i, so that we cannot apply the usual technique of the optimal cases of determining the biases by H_o-averaging of the next higher-order terms in θ.

However, our requirement that $g(\underset{\sim}{x})$ be optimal (all n) when the background noise has the pdf w_{1F} such that $F = \ell_F[\equiv (d/dx)\log w_{1F}(x)]$, i.e. derived from an appropriate $\log \Lambda_n$, suggests how to determine a bias, such that $g_F \rightarrow g^*_F$ is LOBD <u>and</u> AODA, cf. Sec. A.3-3. This is the observation that for symmetric channels ($\mu=1, K=1$)

$$\left\langle g^{(*)} \right\rangle_{1,\mu=1} = -\left\langle g^{(*)} \right\rangle_{o,\mu=1} \ , \tag{A.4-4}$$

and hence in the optimum cases P_e^* reduces to the canonical form (6.5). [We retain here only the leading terms in θ, of course.] Also, when $g_F \to g_F^*$, then $\hat{\sigma}_F^2 \to \sigma_F^{*2}$, i.e. for the noise pdf w_{1F}, these are now the optimum variances and biases. When the actual noise obeys $w_{1E} \neq w_{1F}$, the algorithms are suboptimum, including the biases. Consequently, to obtain $\hat{\sigma}_F^2$, and $\hat{B}'_{(\ )}$, we must use (A.4-3) in our previous calculations of the means and variances of g*, Appendix A.2 above, to obtain the new means and variances (which take the optimum canonical forms of the text (4.1), (4.4), etc.). We then use (A.4-4) to obtain a bias with the desired limiting optimal properties.

A. <u>Coherent Detection:</u>

Accordingly let us start with Sec. A.2-1, replacing $\ell(x)$ by F(x) in (A.2-1)-(A.2-4). We get

$$\left\langle g_{coh}\right\rangle_{1,\theta} = B'_{coh} - \sum_i^n \left\langle\theta_i\right\rangle \{\bar{F}_i - \left\langle\theta_i\right\rangle \int_{-\infty}^{\infty} F_i w'_{1i}\, dx_i + O(\overline{\theta^2})\}. \qquad \text{(A.4-5a)}$$

At this point and subsequently we restrict <u>F to be antisymmetrical</u>, e.g. $F(-x) = -F(x)$, and w_{1E} to be symmetrical. This is no real restriction, since we are using both positive and negative values of the amplitude data $(-\infty \le x \le \infty)$. Then, (A.4-5a) becomes

$$\left.\begin{aligned} \left\langle g_{coh}\right\rangle_{1,\theta} &= B'_{coh} - \sum_i^n \left\langle\theta_i\right\rangle^2 \left\langle F'_i\right\rangle_0 + O(\left\langle\theta^4\right\rangle); \\ \left\langle g_{coh}\right\rangle_{1,0} &= B'_{coh}\ , \ (\text{all } \theta). \end{aligned}\right\} \qquad \text{(A.4-5b)}$$

with $B'_{coh} \equiv B'_{coh} + \log\mu$; (A.4-6)

Similarly, we obtain the variances from (A.2-5)-(A.2-12). Since

$$\left\langle F_i\right\rangle_{1,\theta} = \left\langle\theta_i\right\rangle\left\langle F_i'\right\rangle_o + O(\overline{\theta_i^3}) \;;\; \left\langle F_i F_j\right\rangle_{1,\theta}\Big|_{i\neq j} = \left\langle\theta_i\theta_j\right\rangle\left\langle F_i'\right\rangle_o\left\langle F_j'\right\rangle_o + O(\overline{\theta^4}) \;;$$ (A.4-7a)

$$\left\langle F_i^2\right\rangle_{1,\theta} = \left\langle F_i^2\right\rangle_o + \left\langle\theta_i^2\right\rangle\left\langle(F_i'^2 + F_i F_i'')\right\rangle_o + O(\overline{\theta_i^4}) \;;\; \left\langle F_i\right\rangle_o = 0 \text{ by antisymmetry;}$$ (A.4-7b)

and setting

$$\therefore \left\langle F_i\right\rangle_o \equiv -L_{iF:E}^{(2)} \;;\; \left\langle F_i'^2 + F_i F_i''\right\rangle_o \equiv \frac{1}{2} L_{iF:E}^{(2,2)} \;;\; \left\langle F_i^2\right\rangle_o \equiv \hat{L}_{iF:E}^{(2)} \;; \qquad \text{(A.4-7c)}$$

we obtain the following suboptimum forms for (A.2-11),(A.2-12):

$$\hat{\sigma}_1^2 = \sum_i \left\langle\theta_i\right\rangle^2 \{\left\langle F_1^2\right\rangle_o + \frac{\left\langle\theta_i^2\right\rangle}{2} L_{iF:E}^{(2,2)} - \left\langle\theta_i\right\rangle^2\left\langle F_i'\right\rangle_o^2 + \ldots\}$$

$$+ \sum_{ij}{}' \left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle\{\left\langle\theta_i\theta_j\right\rangle - \left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle\left\langle F_i\right\rangle_o\left\langle F_i'\right\rangle_o\} + \ldots \qquad \text{(A.4-8)}$$

$$\therefore \quad \boxed{\hat{\sigma}_{o,(coh)}^2 = \sum_i \left\langle\theta_i\right\rangle^2 \hat{L}_{iF:E}^{(2)} .} \qquad \text{(A.4-9)}$$

The condition that $\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2$, i.e. "closeness" condition on the maximum size of the input signal (a_o) is

$$\boxed{\left|\sum_i \left\langle\theta_i\right\rangle^2 \left\{\frac{\left\langle\theta_i^2\right\rangle}{2} L_{if:E}^{(2,2)} - \left\langle\theta_i\right\rangle^2 L_{F:E}^{(2)2}\right\} + \sum_{ij}{}' \left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle(\left\langle\theta_i\theta_j\right\rangle - \left\langle\theta_i\right\rangle\left\langle\theta_j\right\rangle)\left\langle L_{iF:E}^{(2)}\right\rangle_o\left\langle L_{jF:E}^{(2)}\right\rangle_o\right| \ll \sum_i \left\langle\theta_i\right\rangle^2 L_{iF:E}^{(2)} .} \qquad \text{(A.4-10)}$$

cf. (A.2-15a), and (A.2-51),

From (A.4-5b), (A.4-6) we can write directly

$$\langle g_{coh}\rangle_1 - \langle g_{coh}\rangle_o \doteq -\sum_i \langle\theta_i\rangle^2 \langle F_i'\rangle_o \;\; (= \sum_i \langle\theta_i\rangle^2 L^{(2)}_{iF:E}) \;, \tag{A.4-11a}$$

and from A.4-4 now specify the bias (μ=1):

$$\boxed{\hat{B}'_{coh} = -\frac{1}{2}\sum_i \{\langle\theta_i\rangle^2 L^{(2)}_{F:E}\} \;,} \tag{A.4-11b}$$

where terms $O(\langle\theta^4\rangle)$ are omitted. Note that when $F'\to\ell'$, $\langle F_i'\rangle_o = -L^{(2)}_{F:E} = -L^{(2)}$, since

$$\int_{-\infty}^{\infty} \ell' w_{1E} dx = -\int_{-\infty}^{\infty} \ell w_1' dx = -\int_{-\infty}^{\infty} \ell^2 w_1 dx = -L^{(2)} \;,$$

cf. (A.1-15) and $\hat{B}'_{coh}\to\hat{B}^*_{coh} = -\frac{1}{2}\sum_i\langle\theta_i\rangle^2 L^{(2)} = -\sigma_o^{*2}/2$, as required, cf. Sec. A.3-3. The "distance" (A.4-10) becomes σ_o^{*2}, also as required, cf. (A.3-10). We observe, that although the bias ($\hat{B}'_{coh}$) does not appear in the "distance", it does show up implicitly when one sets the false-alarm probability $\alpha_F^{(*)}$, via (2.25).

Finally, let us observe that the resulting arguments of the eror functions in the probability measures of performance (2.31),(2.32) in this (coherent) suboptimum class, are here from (A.4-8), A.4-10)

$$\boxed{\frac{\langle g_{coh}\rangle_1 - \langle g_{coh}\rangle_o}{\sqrt{2}\,\hat{\sigma}_{o-coh}} \doteq \frac{\sum_i^n \langle\theta_i\rangle^2 L^{(2)}_{iF:E}}{\sqrt{2}\,(\sum_{i=1}^n \langle\theta_i\rangle^2 \hat{L}^{(2)}_{iF:E})^{1/2}} \equiv \frac{\sigma_{o-coh(F)}}{\sqrt{2}} \;;} \tag{A.4-12a}$$

$$\frac{\langle g_{coh}\rangle_1}{\sqrt{2}\hat{\sigma}_{coh}} \doteq \frac{-\langle g_{coh}\rangle_o}{\sqrt{2}\,\hat{\sigma}_{coh}} \doteq \frac{\sum_{i=1}^{n}\langle\theta_i\rangle^2}{2\sqrt{2}\,(\sum_{i=1}^{n}\langle\theta_i\rangle^2 L_{1F:E}^{(2)})^{1/2}} \equiv \frac{\sigma_{o\text{-}coh(F)}}{2\sqrt{2}} . \qquad (A.4\text{-}12b)$$

[Clearly, when $F \to \ell_F$, these reduce to $(L_i^{(2)}\langle\theta_i\rangle^2)^{1/2}/\sqrt{2} = \sigma_o^*/\sqrt{2}$, and $\sigma_o^*/2\sqrt{2}$, respectively, cf. (6.2), (6.5), as required, i.e. $\sigma_{o\text{-}coh} \to \sigma^*_{o\text{-}coh}$, cf. (A.2-14); generally, $\sigma_{o\text{-}coh} \neq \hat{\sigma}_{o\text{-}coh}$, (A.4-9).]

B. Incoherent Detection:

We proceed as above, according to (A.4-3) applied to Section A.2-2. Thus (A.2-19) becomes now

(i=j):

$$\langle \hat{F}_i^2+F_i'\rangle_{1,\theta} = \left\langle \int_{-\infty}^{\infty}(F_i^2+F_i')w_{1E}(x_i-\theta_i)dx_i\right\rangle_\theta \qquad (A.4\text{-}13a)$$

$$= \int_{-\infty}^{\infty}(F_i^2+F_i')[w_{1e}-\langle\theta_i\rangle w_{1E}'+\frac{\langle\theta_i^2\rangle}{2}w_{1E}''\ldots]dx_i;\ \langle\theta_i\rangle=0,\ \langle\theta_i^3\rangle=0,\text{ etc.},$$

$$= \langle F_i^2+F_i'\rangle_o + \frac{\langle\theta_i^2\rangle}{2}\langle(F_i^2+F_i')''\rangle_o + 0(\overline{\theta_i^4}) \qquad (A.4\text{-}13b)$$

$$= \langle F_i^2+F_i'\rangle_o + \frac{\langle\theta_i^2\rangle}{2}\int_{-\infty}^{\infty}(F_i^2+F_i')w_{1E}''(x_i)dx_i, \qquad (A.4\text{-}13c)$$

whichever is defined (e.g., $(F_i^2+F_i')''$ or w_{1E}''); w_{1E}'' is usually defined, $|<\infty|$.

Similarly, we have for (A.2-20) the general result

$\underline{(i \neq j):}$

$$\left\langle F_i F_j + F_i' \delta_{ij} \right\rangle_{1,\theta} = \left\langle F_i F_j \right\rangle_{1,\theta} = \left\langle \left\langle F_i \right\rangle_1 \left\langle F_j \right\rangle_1 \right\rangle_\theta = \left\langle \theta_i \theta_j \right\rangle \left\langle F_i' \right\rangle_o \left\langle F_j' \right\rangle_o + O(\overline{\theta^4}) \ , \tag{A.4-14}$$

so that combined with (A.4-13) in (A.4-15):

$$\left\langle g_{inc} \right\rangle_{1,\theta} = B_{inc}' + \frac{1}{2} \sum_{ij}^{n} \left\langle \theta_i \theta_j \right\rangle \left\langle F_i F_j + F_i' \delta_{ij} \right\rangle_{1,\theta}, \tag{A.4-15}$$

cf. (A.4-36) in (A.4-2). We get directly

$$\boxed{\left\langle g_{inc} \right\rangle_{1,\theta} = B_{inc}' + \frac{1}{2} \sum_{ij}^{n} \left\langle \theta_i \theta_j \right\rangle \{ L_{iE:F}^{(1)} \delta_{ij} + \left\langle \theta_i \theta_j \right\rangle \left(\frac{L_{iF:E}^{(4)}}{2} - \left\langle F_i' \right\rangle_o^2 \right) \delta_{ij} + \left\langle \theta_i \theta_j \right\rangle \left\langle F_i' \right\rangle_o \left\langle F_j' \right\rangle + O(\overline{\theta^4}) \} \ ,} \tag{A.4-16}$$

where

$$\hat{L}_{iF:E}^{(4)} \equiv \left\langle (F_i^2 + F_i')'' \right\rangle_o = \int_{-\infty}^{\infty} (F_i^2 + F_i') w_{1E}'' dx_i \ ; \ L_{1F:E}^{(1)} \equiv \left\langle F_i^2 + F_i' \right\rangle_o ; \tag{A.4-16b}$$

and $\hat{L}_{iE|E}^{(4)} = L_i^{(4)} = \left\langle (w_{1E}''/w_{1E})^2 \right\rangle_o$ = (A.1-19b) in the optimum cases ($F \to E$). Moreover, also, we have

$$\left\langle F_i^2 + F_i' \right\rangle_o \to \left\langle \ell_{Ei}^2 + \ell_{Ei}' \right\rangle_o = \int_{-\infty}^{\infty} w_{1E}'' dx_i = 0$$

here, with $\left\langle F_i^1 \right\rangle_o = -L_{1E:E}^{(2)} = -L_i^{(2)}$, cf. (A.4-11) et seq.

In the H_o-case, (A.4-16) from (A.4-13) reduces at once to

$$\langle g_{inc}\rangle_o = B'_{inc} + \frac{1}{2!}\sum_{ij}^{n}\langle\theta_i\theta_j\rangle L^{(1)}_{iF:E}\,\delta_{ij}\ . \qquad (A.4\text{-}17)$$

The bias is now chosen according to (A.4-4), (μ=1), which from (A.4-16), (A.4-17) becomes directly, with terms $O(\langle\theta^6\rangle)$ omitted:

$$\hat{B}'_{inc} = -\frac{1}{2}\sum_{i}^{n}\langle\theta_i^2\rangle L^{(1)}_{iF:E} - \frac{1}{8}\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2 \cdot[\{\hat{L}^{(4)}_{iF:E} - 2L^{(2)^2}_{iF:E}\}\delta_{ij} + 2\,L^{(2)}_{iF:E}\,L^{(2)}_{jF:E}]. \qquad (A.4\text{-}18)$$

From the optimal forms [cf. (A.4-16b) ff.] we see that (A.4-16) reduces at once to (A.1-20b), which is now exact since $B^{*\prime}_{inc}$ is calculated under H_o. The same observation holds for the coherent cases (A.4-11) under optimality ($F \rightarrow E$).

Our next step is to obtain the variances $\hat{\sigma}^2_{o,1}$, by appropriate modification of the results of Sec. A.2-2, according to the substitution (A.4-3) in (A.2-26)-(A.2-41). We indicate the results of (1)-(5) therein:

(1). ($i \neq k$):

$$\langle\langle(F_i^2+F_i')\rangle_1\langle(F_k^2+F_k')\rangle_1\rangle_\theta = \langle F_i^2+F_i'\rangle_o\langle F_k^2+F_k'\rangle_o + O(\langle\theta^4\rangle) \qquad (A.4\text{-}19a)$$

(2). ($i=k$):

$$\langle(F_i^2+F_i')^2\rangle_{1,\theta} = \langle(F_i^2+F_i')^2\rangle_o + \frac{\langle\theta_i^2\rangle}{2}\int_{-\infty}^{\infty}(F_i^2+F_i')^2 w''_{1E}\,dx_i + O(\overline{\theta^4})$$

$$= \langle(F_i^2+F_i')^2\rangle_o + \frac{\langle\theta_i^2\rangle}{2}L^{(6)}_{F:E} + O(\overline{\theta^4})\ , \qquad (A.9\text{-}19b)$$

where

$$L^{(4)}_{F:E} \equiv \left\langle (F^2+F')^2 \right\rangle_o \; ; \; L^{(6)}_{F:E} \equiv \int_{-\infty}^{\infty} (F^2+F')^2 w''_{1E} dx \; ; \quad \text{(A.4-19c)}$$

(3). $(i\neq j)\neq(k\neq\ell)$:

$$\left\langle F_i F_j F_k F_\ell \right\rangle_{1,\theta} = O(\overline{\theta^4}) \langle F'_i \rangle_o \langle F'_j \rangle_o \langle F'_k \rangle_o \langle F'_\ell \rangle_o . \quad \text{(A.4-19d)}$$

(4). $(i\neq j)$; $(k\neq\ell)$:

$$\text{(a).} \quad \left\langle F_i^2 F_j F_\ell \right\rangle_{1,\theta} = \hat{L}^{(2)}_{iF:E} \langle \theta_j \theta_\ell \rangle L^{(2)}_{jF:E} \, L^{(2)}_{\ell F:E} + O(\overline{\theta^4}) \; ; \quad \text{(A.4-20a)}$$

$$\text{(b).} \quad \left\langle F_i^2 F_j^2 \right\rangle_{1,\theta} = \hat{L}^{(2)}_{iF:E} \, \hat{L}^{(2)}_{jF:E} + O(\overline{\theta^4}) \; , \quad \text{(A.4-20b)}$$

where the number of terms is as indicated in (A.2-31), (A.2-32) above. We continue with (A.2-33)-(A.2-36):

(5a).

$$\left.\begin{aligned} \left\langle (F_i^2+F'_i) F_k F_\ell \right\rangle_{1,\theta} : \underline{k\neq\ell : i=k:} &= \langle \theta_i \theta_\ell \rangle \int_{-\infty}^{\infty} (F_i^2+F'_i) F_i w'_{1i} dx_i \int_{-\infty}^{\infty} F_\ell w'_{1\ell} dx_\ell + O(\overline{\theta^4}) \\ &= \langle \theta_i \theta_\ell \rangle \, L^{(2)}_{\ell F:E} \, \hat{L}^{(2,2)}_{iF:E} + O(\overline{\theta^4}) \; , \end{aligned}\right\} \quad \text{(A.4-21)}$$

where

$$\hat{L}^{(2,2)}_{iF:E} \equiv -\left\langle (F^3+F'F')' \right\rangle_o = -\left\langle 3F^2F' + F'^2 + FF'' \right\rangle_o = \int_{-\infty}^{\infty} (F_i^2+F'_i) F_i w'_{1i} dx_i . \quad \text{(A.4-21a)}$$

(5b). $k \neq \ell$: $i=\ell$:

$$\left\langle (F_i^2+F_i')F_iF_k \right\rangle_{1,\theta} = \left\langle \theta_i\theta_k \right\rangle L_k^{(2)}\hat{L}_{iF:E}^{(2,2)} + 0(\overline{\theta^4}) \ . \qquad \text{(A.4-22)}$$

(5c). $k \neq \ell$: $i \neq k$ ($\neq \ell$):

$$\left\langle (F_i^2+F_i')F_kF_\ell \right\rangle_{1,\theta} = \left\langle F_i^2 L_{iF:E}^{(1)} + F_i' \right\rangle_o \left\langle \theta_k\theta_\ell \right\rangle L_{k:F:E}^{(2)} L_{\ell F:E}^{(2)} \ , \ [\ \overline{\theta} = \overline{\theta^3} = 0]$$

$$+ \frac{\left\langle \theta_i^2\theta_k\theta_\ell \right\rangle}{2} \hat{L}_{iF:E}^{(4)} L_{k:F:E}^{(2)} L_{\ell F:E}^{(2)} \ . \qquad \text{(A.4-23)}$$

Now we combine (A.4-19a)-(A.4-23) according to the above and the "counting" of (A.2-28)-(A.2-36):

$$\sum_{ijk\ell}^{n} \left\langle F_{ij}F_{k\ell} \right\rangle_{1,\theta} = \sum_{ij}^{n} \left\langle \theta_i \right\rangle^2 \left\langle \theta_j \right\rangle^2 L_{iF:E}^{(1)} L_{jF:E}^{(1)} +$$

$$+ \sum_{ij}^{n} \left\langle \theta_i\theta_j \right\rangle^2 \{ (L_{iF:E}^{(4)} - 2\hat{L}_{iF:E}^{(2)^2})\delta_{ij} + 2\hat{L}_{iF:E}^{(2)} \hat{L}_{jF:E}^{(2)} \}$$

$$+ \sum_i \left\langle \theta_i^2 \right\rangle^3 \frac{L_{iF:E}^{(6)}}{2} + 6 \sum_{ij} \left\langle \theta_i^2 \right\rangle \left\langle \theta_i\theta_j \right\rangle (L_i^{(2)}\hat{L}_j^{(2,2)})_{F:E}$$

$$+ \sum_{ijk}^{i \neq j \neq k} [4 \left\langle \theta_i\theta_j \right\rangle \left\langle \theta_j\theta_k \right\rangle \left\langle \theta_k\theta_i \right\rangle - 2 \left\langle \theta_i^2 \right\rangle \left\langle \theta_j\theta_k \right\rangle^2]$$

$$\cdot (\hat{L}_i^{(2)} L_j^{(2)} L_k^{(2)})_{F:E} + 2 \sum_{ik\ell}^{i \neq k \neq \ell} \left\langle \theta_i^2 \right\rangle \left\langle \theta_k\theta_\ell \right\rangle^2$$

$$\cdot (L_i^{(1)} L_k^{(2)} L_\ell^{(2)})_{F:E} + 0(\overline{\theta^8}) \ . \qquad \text{(A.4-24)}$$

Similarly, we have

$$\sum_{ijk\ell}^{n}\langle F_{ij}\rangle_{1,\theta}\langle F_{k\ell}\rangle_{1,\theta} = \sum_{ij}^{n}\left\{\begin{matrix}\langle\theta_i\theta_j\rangle^2 L^{(2)}_{iF:E}L^{(2)}_{jF:E}(1-\delta_{ij}) \\ \langle\theta_i^2\rangle L^{(1)}_{iF:E}+\langle\theta_i^2\rangle^2 L^{(4)}_{iF:E}\delta_{ij}\end{matrix}\right\}$$

$$\cdot\sum_{k\ell}^{n}\left\{\begin{matrix}\langle\theta_k\theta_\ell\rangle^2 L^{(2)}_{k}L^{(2)}_{\ell})_{F:E}(1-\delta_{k\ell}) \\ \langle\theta_k\rangle^2 L^{(1)}_{k}+\langle\theta_k^2\rangle^2\hat{L}^{(4)}_{iF:E}\delta_{k\ell}\end{matrix}\right\} +0(\overline{\theta^6}) \,. \tag{A.4-25}$$

cf. (A.2-29). We need to investigate (A.4-13) for terms $0(\overline{\theta^4})$, in order to obtain terms $0(\overline{\theta^6})$ in (A.4-25). We have

$$\underline{0(\overline{\theta^4}):}\quad \frac{\langle\theta_i^4\rangle}{4!}\int_{-\infty}^{\infty}(F_i^2+F_i')w_{1E}^{('')}(x_i)dx_i \equiv \frac{\langle\theta_i^4\rangle}{4!}\hat{L}^{(6)}_{iFiE} \,, \tag{A.4-26}$$

so that the contributions of (A.4-25) become

$$\sum_{ijk\ell}^{n}\langle F_{ij}\rangle_{1,\theta}\langle F_{k\ell}\rangle_{1,\theta} = \sum_{ik}^{n}\langle\theta_i^2\rangle\langle\theta_k^2\{L_i^{(1)}L_k^{(1)}+2\langle\theta_i^2\rangle\hat{L}_i^{(4)}L_k^{(1)}+0(\overline{\theta^4})\}$$

$$+0(\overline{\theta^8})_{i\neq j,k\neq\ell} \,. \tag{A.4-27}$$

Accordingly, since

$$\text{var}_{1,\theta}g_{inc} = (\hat{\sigma}^2_{1-inc}) = \frac{1}{4}\sum_{ijk\ell}^{n}\{\langle F_{ij}F_{k\ell}\rangle_{1,\theta}-\langle F_{ij}\rangle_{1,\theta}\langle F_{k\ell}\rangle_{1,\theta}\},\ \text{cf. (A.2-26)},$$

(A.4-28a)

$$\hat{\sigma}^2_{1\text{-}inc} \doteq \frac{1}{4}\sum_{ij}^{n} \langle\theta_i\theta_j\rangle^2\{(L_i^{(4)}-2\hat{L}_i^{(2)^2})\delta_{ij}+2\hat{L}_i^{(2)}\hat{L}_j^{(2)}\}_{F:E}$$

$$+0(\overline{\theta^6}) \,, \qquad \text{(A.4-28b)}$$

and

$$\therefore\ \hat{\sigma}^2_{o\text{-}inc} = \frac{1}{4}\sum_{ij}^{n} \langle\theta_i\theta_j\rangle^2\{(L_i^{(4)}-2\hat{L}_i^{(2)^2})\delta_{ij}+2\hat{L}_i^{(2)}\hat{L}_j^{(2)}\}_{F:E}, \qquad \text{(A.4-29)}$$

exactly.

The "smallness" condition on the input signal (a_o) is determined by $\hat{\sigma}^2_{o\text{-}inc} \doteq \hat{\sigma}^2_{o\text{-}inc}$, which requires accordingly that terms $0(\overline{\theta^6})$ in $\hat{\sigma}^2_{1\text{-}inc}$ be much less than $\hat{\sigma}^2_{o\text{-}inc}$, as before, cf. (A.2-41). From (A.4-24), (A.4-27) we obtain specifically

$\hat{\sigma}^2_{1\text{-}inc} \doteq \hat{\sigma}^2_{o\text{-}inc}$:

$$|\{\sum_{ij} \langle\theta_i^2\rangle\langle\theta_i\theta_j\rangle^2(\{\frac{L_i^{(6)}}{2}\}\delta_{ij}+6L_i^{(2)}\hat{L}_j^{(2,2)})+\sum_{ijk}^{i\neq j\neq k}[4\langle\theta_i\theta_j\rangle\langle\theta_j\theta_k\rangle\langle\theta_k\theta_i\rangle$$

$$-2\langle\theta_i^2\rangle\langle\theta_j\theta_k\rangle^2]L_i^{(2)}L_j^{(2)}L_k^{(2)}\}_{F:E}+2\sum_{ijk}^{i\neq j\neq k}\{\langle\theta_j\theta_k\rangle^2\langle\theta_i^2\rangle$$

$$\cdot L_j^{(2)}L_k^{(2)}L_{ik}^{(1)}-\langle\theta_i^2\rangle\langle\theta_j^2\rangle\langle\theta_k^2\rangle\hat{L}_k^{(4)}L_i^{(1)}\delta_{jk}\}_{F:E}|$$

$$<< \sum_{ij}\langle\theta_i\theta_j\rangle^2[\{L_i^{(4)}-2\hat{L}_i^{(2)^2}\}\delta_{ij}+2\hat{L}_i^{(2)}\hat{L}_j^{(2)}]_{F:E}\,.$$

(A.4-30)

We can now parallel the coherent cases (A.4-12) and write

$$\frac{\langle g_{inc}\rangle_1-\langle g_{inc}\rangle_o}{\sqrt{2}\,\hat{\sigma}_{o-inc}} \doteq \frac{(\frac{1}{4})\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2(\{\hat{L}_i^{(4)}-2L_i^{(2)^2}\}\delta_{ij}+2L_i^{(2)})_{F:E}}{\sqrt{2}\,[(\frac{1}{4})\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2(\{L_i^{(4)}-2\hat{L}_i^{(2)^2}\}\delta_{ij}+2\hat{L}_i^{(2)}\hat{L}_j^{(2)}]_{F:E}^{1/2}}$$

$$\equiv \frac{\sigma_{o-inc(F)}}{\sqrt{2}}\,. \qquad \text{(A.4-31a)}$$

$(\mu=1)$:

$$\frac{\langle g_{inc}\rangle_1}{\sqrt{2}\,\hat{\sigma}_{o-inc}} = \frac{-\langle g_{inc}\rangle_o}{\sqrt{2}\,\hat{\sigma}_{o-inc}}$$

$$\doteq \frac{(\frac{1}{8})\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2(\{\hat{L}_i^{(4)}-2L_i^{(2)^2}\}\delta_{ij}+2L_i^{(2)}L_j^{(2)})_{F:E}}{\sqrt{2}\,[(\frac{1}{4})\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2(\{L_i^{(4)}-2\hat{L}_i^{(2)^2}\}\delta_{ij}+2\hat{L}_i^{(2)}\hat{L}_j^{(2)})_{F:E}]^{1/2}}$$

$$\equiv \frac{\sigma_{o-inc(F)}}{2\sqrt{2}}\,, \qquad \text{(A.4-31b)}$$

which defines σ_{o-inc} now. When $(F\to E)$, i.e. $F\to\ell_E$: the system is optimum, we have $\hat{L}_i^{(2)} = L_i^{(2)} = \langle\ell_o^2\rangle_o$; $\hat{L}^{(4)} = L_i^{(4)} = \langle(\ell_i^2+\ell_i')^2\rangle_o$; (also, $L_{E:E}^{(1)} = 0$), and $\sigma_{o-inc}\to\sigma^*_{o-inc}$, Eq. (A.2-40), cf. (A.4-39)-(A.4-36). Since $\hat{L}^{(2)}, \hat{L}^{(4)} \neq L^{(2)}, L^{(4)}$, etc., generally in the suboptimum cases, we have the more complicated but symmetrical forms above for σ_{o-inc} $(\neq \hat{\sigma}_{o-inc})$. For the usual stationary régimes we simply drop the (i,j) subscripts on the various L's, as before.

C. The Canonical Parameters $L^{(2)}, \hat{L}^{(2)}$, etc.: Robustness Formulations:

Our results above are quite general for these broad classes (A.4-1), (A.4-2) of suboptimum (threshold) detectors. They not only permit an examination of various linear and nonlinear detector elements. They also allow us to study the robustness [see, for example, [42]-[45]] of one type of detection algorithm, optimum in one class of interference, when employed against another, as noted earlier in Sec. 4.3 above.

Here we summarize the canonical parameters ($\hat{L}^{(2)}_{F:E}$, $L^{(4)}_{F:E}$, etc.) employed in both the structure of these algorithms and in the parameters of their performance measures, with an appropriate expansion of the notation to indicate the specific character of the suboptimum state involved. Thus, we write (for a particular $i^{\underline{th}}$ sample):

$L^{(\ )}_{F:D|E}$, etc: F refers to the basic detector data processing element, F(x), cf. (A.4-1,2); D|E denotes the D-class of noise parameters used in an E-class noise pdf. The E-class pdf represents the actual noise in which detection is taking place [cf. Sec. 4.3].

From A and B above we can write, remembering that $F(x) = -F(-x)$:

(Eq. (A.4-16b):

$$L^{(1)}_{F:D|E} \equiv \left\langle F^2+F' \right\rangle_o = \int_{-\infty}^{\infty} [F(x)^2+F'(x)] w_{1E}(x|D)_o \, dx \; ; \qquad \text{(A.4-32)}$$

Eq. (A.4-7e):

$$\left.\begin{aligned} L^{(2)}_{F:D|E} &\equiv -\left\langle F' \right\rangle_o = -\int_{-\infty}^{\infty} F(x)' w_{1E}(x|D)_o dx && \text{(A.4-33a)} \\ \hat{L}^{(2)}_{F:D|E} &\equiv \left\langle F^2 \right\rangle_o = \int_{-\infty}^{\infty} F(x)^2 w_{1E}(x|D)_o \, dx && \text{(A.4-33b)} \end{aligned}\right\} \; ;$$

Eq. (A.4-19c):

$$L^{(4)}_{F:D|E} \equiv \left\langle (F^2+F')^2 \right\rangle_o = \int_{-\infty}^{\infty} [F(x)^2+F'(x)]^2 w_{1E}(x|D)_o dx \; ; \qquad \text{(A.4-34a)}$$

Eq. (A.4-16b):

$$\hat{L}^{(4)}_{F:D|E} \equiv \left\langle (F^2+F')'' \right\rangle_o = \int_{-\infty}^{\infty} [F^2(x)+F'(x)] w''_{1E}(x|D)_o dx \, , \qquad \text{(A.4-34b)}$$

$$= \int_{-\infty}^{\infty} (F^2+F')'' w_{1E}(x|D)_o \; dx \; ;$$

Eq. (A.4-7c):

$$L^{(2,2)}_{F:D|E} \equiv 2\left\langle F'^2+FF'' \right\rangle_o = \int_{-\infty}^{\infty} F^2(x) w''_{1E}(x|D)_o dx \; ; \qquad \text{(A.4-35a)}$$

Eq. (A.4-21a):

$$\hat{L}^{(2,2)}_{F:D|E} \equiv -\left\langle (F^3+FF')' \right\rangle_o = \begin{pmatrix} -\int_{-\infty}^{\infty} [F(x)^3+F(x)F'(x)]' w_{1E}(x|D)_o dx \\ \int_{-\infty}^{\infty} (F^2+F') F w'_{1E} dx \end{pmatrix} ; \qquad \text{(A.4-35b)}$$

Eq. (A.4-19c):

$$L^{(6)}_{F:D|E} \equiv \left\langle (F^2+F')^{2''} \right\rangle_o = \int_{-\infty}^{\infty} [F^2(x)+F'(x)]^2 w''_{1E}(x|D)_o dx \, . \qquad \text{(A.4-36)}$$

To examine the "robustness" question, say, of using a detector algorithm which is optimum in Class A noise, when actually the interference is Class B and the Class A, B parameters are exact, for example, we have $F \to \ell_A(x)$, $E{:}w_{1B}$, etc., so that from (A.4-32), (A.4-33), etc.:

$$L^{(1)}_{F:D|E} \to L^{(1)}_{A:B} = \int_{-\infty}^{\infty} [\ell_A(x)^2+\ell'_A(x)] w_{1B}(x|B)_o dx \; ; \qquad \text{(A.4-37a)}$$

$$L^{(2)}_{F:D|E} \rightarrow L^{(2)}_{A:B} = \int_{-\infty}^{\infty} \ell_A'(x) w_{1B}(x|B)_o \, dx, \text{ etc.} \qquad \text{(A.4-37b)}$$

Another robustness problem of interest arises when the correct operator is used, say a Class A operator (in Class A noise), but only necessarily inaccurate estimates (A') of the true Class A parameters are employed. Then, we have $F \rightarrow A'|A$: $F(x) \rightarrow \ell_A(x|A')$, and $w_{1E} \rightarrow w_{1A}(x|A)_o$:

$$\therefore L^{(1)}_{A'|A:A|A} = L^{(1)}_{A'|A:A} = \int_{-\infty}^{\infty} [\ell_A(x|A')^2 + \ell_A'(x|A')] w_{1A}(x|A)_o dx \text{ , etc.} \qquad \text{(A.4-38)}$$

Still other possibilities can be constructed: Class A noise, with parameter estimates (A'), in Class B noise, with Class B parameter estimates (B'), e.g. $L^{(1)} \rightarrow L^{(1)}_{A'|A:B'|B}$, $F(x) \rightarrow \ell_A(x|A')$, $w_{1E} \rightarrow w_{1B}(x|B')_o$, etc. [Usually, however, we wish to refer the various suboptimum situations to the "true" or limiting population statistics, where the estimates A', B' become (some) "true" or limiting values.]

Finally, it is clear that when $F \rightarrow E$, i.e. $F(x) \rightarrow +\ell_E(x|E)$, for $w_{1E}(x|E)_o$, the above canonical parameters must reduce to the optimum (or LOBD) values. Thus, from (A.4-32)-(A.4-36) we get

$$L^{(1)}_{F:D\;E} \rightarrow L^{(1)}_{E:E} \equiv L^{(1)}_E = \int_{-\infty}^{\infty} [\ell_E(x)^2 + \ell_E'(x)] w_{1E} dx = \int_{-\infty}^{\infty} w_{1E}'' dx = w_{1E}' \Big|_{-\infty}^{\infty} = 0 \qquad \text{(A.4-39)}$$

$$\left\{ \begin{aligned} L^{(2)}_{F:D\;E} \rightarrow L^{(2)}_{E:E} &= -\int_{-\infty}^{\infty} \ell_E'(x) w_{1E} dx = -\int_{-\infty}^{\infty} \left(\frac{w_1'}{w_1}\right)_E' w_{1E} dx = -\int_{-\infty}^{\infty} \left(\frac{w_1''}{w_1} - \left(\frac{w_1'}{w_1^2}\right)^2\right)_E w_{1E} dx \\ &= 0 + \int_{-\infty}^{\infty} \ell_E^2 w_{1E}' dx = L_E^{(2)} = \left\langle \ell_E^2 \right\rangle_o ; \qquad \text{(A.4-40)} \\ \hat{L}^{(2)}_{F:D\;E} \rightarrow \hat{L}^{(2)}_{E:E} &= \int_{-\infty}^{\infty} \ell_E^2 w_1 dx = L_E^{(2)}; \qquad \text{(A.4-41)} \end{aligned} \right.$$

$$\begin{cases} L^{(4)}_{F:D|E} \to L^{(4)}_{E:E} = \int_{-\infty}^{\infty} (\ell_E^2 + \ell_E')^2 w_{1E} dx = \int_{-\infty}^{\infty} \left(\frac{w''_{1E}}{w_{1E}}\right)^2 w_{1E} dx \equiv L_E^{(4)}, \text{ cf. (A.1-19b)} & \text{(A.4-42)} \\ \hat{L}^{(4)}_{F:D|E} \to \hat{L}^{(4)}_{E:E} = \int_{-\infty}^{\infty} (\ell_E^2 + \ell_E') w''_{1E} dx = \int_{-\infty}^{\infty} \left(\frac{w''_{1E}}{w_1}\right)_E \left(\frac{w''_{1E}}{w_{1E}}\right) w_{1E} dx = L_E^{(4)} & \text{(A.4-43)} \end{cases}$$

$$\begin{cases} L^{(2,2)}_{F:D|E} \to L^{(2,2)}_{E:E} = \int_{-\infty}^{\infty} \ell_E^2 w''_{1E} dx = \int_{-\infty}^{\infty} \left(\frac{w_{1E}}{w_{1E}}\right)^2 \left(\frac{w''_{1E}}{w_{1E}}\right) w_{1E} dx = L_E^{(2,2)} = 2\langle \ell_E^4 \rangle, \\ \qquad \text{cf. (A.2-16a)}, & \text{(A.4-44)} \\ L^{(2,2)}_{F:D|E} \to L^{(2,2)}_{E:E} = \int_{-\infty}^{\infty} (\ell_E^2 + \ell_E') \ell_E w'_{1E} dx = \int_{-\infty}^{\infty} \left(\frac{w''_{1E}}{w_{1E}}\right) \left(\frac{w'_{1E}}{w_{1E}}\right)^2 w_{1E} dx = L_E^{(2,2)} = 2\langle \ell_E^4 \rangle, \end{cases}$$

(A.4-45)

$$L^{(6)}_{F:D|E} \to L^{(6)}_{E:E} = \int_{-\infty}^{\infty} \left(\frac{w''_1}{w_1}\right)_E^2 \left(\frac{w''_{1E}}{w_{1E}}\right) w_{1E} dx = \int_{-\infty}^{\infty} \left(\frac{w''_1}{w_1}\right)_E^3 w_{1E} dx = L_E^{(6)},$$

cf. (A.2-29b). (A.4-46)

D. Optimum Distributions for Specified Detector Nonlinearities:

The question here is, given a (threshold) detector structure (A.4-1,2,3), e.g., given $F(x)$, what is the pdf, $w_{1F}(x|F)_o$ for which these detector algorithms are optimum, i.e. are LOBD's and AODA's jointly. This is easily established formally from (A.4-3), since

$$F(x) = \ell_F(x) = \frac{d}{dx} \log w_{1F}(x|F)_o \ , \ (-\infty < x < \infty) \ , \tag{A.4-47}$$

which is readily integrated to

$$\boxed{w_{1F}(X|F)_o = Ae^{B\int F(x)dx} = AE^{BG(x)}} \ ; \ A = \int_{-\infty}^{\infty} e^{B\int F(x)dx} dx; \ (A, B > 0), \tag{A.4-48}$$

with A the normalizing constant, since w_{1F} is a proper pdf, e.g. $w_{1F} \geq 0$, $w_{1F}(\pm\infty) = 0$ (fast enough that) $\int_{-\infty}^{\infty} w_{1F} dx = 1$. The constant B is chosen to insure that $\overline{x^2} = 1$, i.e. x is normalized to the mean intensity $\langle X^2 \rangle$. We remember that $F(x) = -F(-x) = -F(|x|)$, $x < 0$, so that $\int F(x)dx \equiv G(x)\ [=G(-x)] \leq 0$, all $(-\infty < x < \infty)$, i.e. G is negative, even. We also require that $\lim_{|x|\to\infty} G(x) \to -\infty$ at least as fast as $\log |x|^{1+\eta}$, $\eta > 0+$, so that $\int_{-\infty}^{\infty} (\exp G)dx < \infty$, i.e. $(0<)A < \infty$.

Let us consider two simple but important examples:

(i). $F(x) = -x$: a (simple) correlation detector (A.4-49a)

(ii). $F(x) = -\sqrt{2}\ \mathrm{sgn}\ x$: a "super-clipper" [46] or hard limiter (normalized in accordance with (A.4-50b)). (A.4-49b)

[Other detector characteristics are handled in the same way, cf. [43], [44].] Applying (A.4-49a,b) to (A.4-48) gives directly

(Correlators):

$$w_{1F}(x|F)_0 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

; ($\overline{x^2}=1$ (by original normalization)); $A=1/\sqrt{2\pi}$, $B=1$; (A.4-50a)

("superclippers"):

$$w_{1F}(x|F)_0 = \frac{1}{\sqrt{2}} e^{-|x|\sqrt{2}}$$

($\overline{x^2}=1$, as required: $A = 1/\sqrt{2}$, $B = \sqrt{2}$). (A.4-50b)

As we expect, the optimum noise for correlators in threshold detection is gaussian, while for the "super-clipper" it turns out to be "Laplace" noise, cf. (A.4-50b), [a result obtained by the author about 1967 in ONR studies]. (Note that the addition of a gaussian component in Case (ii), (A.4-49b), destroys the optimality of the super-clipper.)

Finally, if F is not available but $H(x_i, s_j)$ is specified, we can find $F(x)$ from the fact that $H(x_i, x_i) \equiv h(x_i)^2$ and the consequent Riccati equation from (A.4-3b):

$$F^2(x) + F'(x) = h(x)^2 \; . \tag{A.4-51}$$

For F one solves the associated equation ([41], Sec. 2.15, p. 24)

$$u''(x) - h^2(x)u = 0; \text{ where } F = u'/u \; , \tag{A.4-52}$$

at all non-singular points (of u, u', u'') in $-\infty < x < \infty$.

A.4-2. Suboptimum Detectors, I: Simple Correlators and Energy Detectors:

In these important cases, which we have already shown to be LOBD when the interference is gaussian, cf. Sec. A.1-3, we see at once from (A.1-24,25) that in (A.4-1,2) we set

$$F(x) = -x \; ; \therefore F' = -1 \quad , \tag{A.4-53}$$

and accordingly from (A.4-32)-(A.4-36) we have for the associated structure and performance parameters (the L's):

$$\begin{aligned}
L^{(1)}_{F:E} &= \left\langle x^2-1 \right\rangle_o = \overline{x^2}-1 = 0, \text{ cf. (A.4-50a)} \; ; \\
L^{(2)}_{F:E} &= \left\langle 1 \right\rangle_o = 1 \; ; \\
\hat{L}^{(2)}_{F:E} &= \left\langle x^2 \right\rangle_o = 1 \; ; \\
L^{(4)}_{F:E} &= \left\langle (x^2-1)^2 \right\rangle_o = \overline{x^4} - 2\overline{x^2} + 1 = \overline{x^4}-1 \; ;
\end{aligned} \tag{A.4-54}$$

$$\hat{L}_{F:E}^{(4)} = \left\langle (x^2-1)'' \right\rangle_o = \left\langle 2 \right\rangle_o = 2 \; ;$$

$$L_{F:E}^{(2,2)} = 2\left\langle 1 \right\rangle_o = 2 \; ; \; \hat{L}_{F:E}^{(2,2)} = -\left\langle (-x^3+x)' \right\rangle_o = \left\langle 3x^2-1 \right\rangle_o = 2;$$

$$L_{F:E}^{(6)} = \left\langle (x^2-1)^{2''} \right\rangle_o = \left\langle 12x^2-4 \right\rangle_o = 8. \qquad \text{(A.4-54) (cont'd.)}$$

Substituting (A.4-53,54) into A.4-1,2; and [A.4-11, A.4-18 for the biases], we get at once

$$B'_{coh} = \log\mu - \frac{1}{2}\sum_i^n \left\langle \theta_i \right\rangle^2 \; ; \; B'_{inc} = \log\mu - \frac{1}{4}\sum_{ij}^n \left\langle \theta_i\theta_j \right\rangle^2 \; ; \qquad \text{(A.4-55)}$$

<u>cross-correlators:</u>

$$g(\underset{\sim}{x})_{coh} = \{\log\mu - \frac{1}{2}\sum_i^n \left\langle \theta_i \right\rangle^2\} + \sum_i^n \left\langle \theta_i \right\rangle x \; ; \qquad \text{(A.4-56a)}$$

<u>auto-correlators:</u>

$$g(\underset{\sim}{x})_{inc} = \log\mu - \frac{1}{4}\sum_{ij}^n \left\langle \theta_i\theta_j \right\rangle^2 + \frac{1}{2!}\sum_i^n (x_i x_j - \delta_{ij})\left\langle \theta_i\theta_j \right\rangle \; ,$$

$$= \{\log\mu - \frac{1}{2}\sum_i^n \left\langle \theta_i^2 \right\rangle - \frac{1}{4}\sum_{ij}^n \left\langle \theta_i\theta_j \right\rangle^2\} + \frac{1}{2!}\sum_{ij}^n \left\langle \theta_i\theta_j \right\rangle x_i x_j \; , \qquad \text{(A.4-56b)}$$

which are precisely our previously derived results (A.1-23), (A.1-24), respectively, for the LOBD's here in gaussian noise.

In the same way, we obtain $\hat{\sigma}_o$ from (A.4-9), (A.4-29), viz.:

$$\boxed{\hat{\sigma}_{o-coh}^2 \doteq \sum_i^n \left\langle \theta_i \right\rangle^2; \quad \hat{\sigma}_{o-inc}^2 \doteq \frac{1}{4}\sum_{ij}^n \left\langle \theta_i\theta_j \right\rangle^2 \{(\overline{x_i^4}-3)\delta_{ij}+2\},} \qquad \text{(A.4-57)}$$

and $\sigma_{o-(\,)}$ from (A.4-12), and (A.4-31), viz.:

$$\sigma_{o-coh} \doteq \{\sum_{i}^{n} \langle\theta_i\rangle^2\}^{1/2} ; \quad \text{(A.4-58a)}$$

$$\sigma_{o-inc} \doteq \sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2 / (\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2 [(\overline{x_i^4}-3)\delta_{ij}+2])^{1/2} . \quad \text{(A.4-58b)}$$

[Generally $\overline{x_i^4} \geqq 1$, so that all variances are positive as required.]

The conditions (A.4-10), (A.4-30) on the maximum "small" values of a_o (>0) permitted to insure $\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2$ are:

coherent: ($\hat{\sigma}_o^2=\hat{\sigma}_o^2$):

$$|\sum_{ij}^{n}\langle\theta_i\theta_j\rangle[\langle\theta_i\theta_j\rangle-\langle\theta_i\rangle\langle\theta_j\rangle] \,/\, \sum_{i}^{n}\langle\theta_i\rangle^2 << 1; \;= 0<<1;\; \langle\theta_i\theta_j\rangle=\langle\theta_i\rangle\langle\theta_j\rangle.$$

Incoherent:

$$|\sum_{ij}^{n}\langle\theta_i^2\rangle\langle\theta_i\theta_j\rangle^2+\sum_{ijk}^{i\neq j\neq k}\{4\langle\theta_i\theta_j\rangle\langle\theta_j\theta_k\rangle\langle\theta_k\theta_i\rangle-2\langle\theta_i^2\rangle\langle\theta_j\theta_k\rangle^2\}$$

$$<< \sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2\{(\overline{x_i^4}-3)\delta_{ij}+2\}.$$

A. The Energy Detector:

The energy detector is a special case of (A.4-2), where now we set

$$F_i \to F_{ij} = -x_i\delta_{ij} \quad \text{(A.4-60)}$$

in (A.4-2), since the energy detector is physically a quadratic device with no memory. We write from (A.4-3b), accordingly

$$g(\underset{\sim}{x})_{inc} = \log \mu + \hat{B}'_{inc} + \frac{1}{2!}\sum_i^n \left\langle\theta_i^2\right\rangle(x_i^2-1) \; , \tag{A.4-61a}$$

where the proper bias $\hat{B}'_{inc}$, is from (A.4-18) and (A.4-54), (j=i) now

$$\hat{B}'_{inc} \doteq -\frac{1}{4}\sum_i^n \left\langle\theta_i^2\right\rangle^2 \; , \tag{A.4-61b}$$

so that we can rewrite (A.4-61a) in the equivalent form

$$g(\underset{\sim}{x})_{inc}\Big|_{energy} = \{\log \mu - \frac{1}{2}\sum_i^n\left\langle\theta_i^2\right\rangle - \frac{1}{4}\sum_i^n \left\langle\theta_i^2\right\rangle^2\} + \frac{1}{2}\sum_i^n \left\langle\theta_i^2\right\rangle x_i^2 \; . \tag{A.4-61c}$$

The variances $\hat{\sigma}^2_{o-inc}$ and σ_o^2 are (from (A.4-57), (A.4-58b), on setting $2\to2\delta_{ij}$:

$$\hat{\sigma}^2_{o-inc} \doteq \frac{1}{4}\sum_i^n \left\langle\theta_i^2\right\rangle (\overline{x_i^4}-1) \; ; \; \sigma_{o-inc} \doteq \sum_i^n \left\langle\theta_i^2\right\rangle^2 / \{\sum_i^n \left\langle\theta_i^2\right\rangle^2(\overline{x_i^4}-1)\}^{1/2} \; . \tag{A.4-62}$$

The controlling condition on the maximum value of the input signal, for which $\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2$, cf. (A.4-30), becomes from (A.4-60) therein:

$$\underline{\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2}: \quad 4\sum_i^n \left\langle\theta_i^2\right\rangle^3 / \sum_i^n \left\langle\theta_i^2\right\rangle^2(\overline{x_i^4}-1) \ll 1. \tag{A.4-63}$$

Finally, we observe that for correlators (of which the energy detector is a special case) in the threshold régime, only the fourth-order moments $(\overline{x_i^4})$

(relative to the intensity $\langle x^2 \rangle$, e.g. $\overline{x^2} = 1$) are significant, because of the fundamentally second-order ($\sim x_i x_j$) nonlinearities of the detectors, cf. (A.4-61a). This is in sharp contrast with optimum detectors (LOBD's), which operate against the whole noise pdf (i.e. all moments, when they exist), via $F \to \ell_F(x)$. From the fact that only $\overline{x_i^4}$ appears in the argument ($\sim\hat{\sigma}_{o\text{-}inc}$) of the probability measures of performance, rather than the appropriate functional of the entire pdf, indicates that performance of correlation detectors can be very suboptimum vis-à-vis the LOBD's appropriate to the noise in question, as is, of course, well-known [cf. [1a,b], [13], [33], [34] for the original work, employing empirically established statistical-physical models of the real-world EMI environment, cf. Sec. 3.]

A.4-3 Suboptimum Detectors II: Hard Limiters ("Super-clippers" and "Clipper-Correlators"):

Here the detector characteristic is given by (A.4-49b), viz., $F = -\sqrt{2}\,\mathrm{sgn}\,x$, and $\therefore F' = -2\sqrt{2}\,\delta(x-o)$, where the factor 2 represents the weight (2) of the jump at x=0, for the superclipper. From (A.4-39)-(A.4-46) we obtain accordingly (remembering that F is odd and w_{1E} is even) when $F \neq E$:

$$L_{F:E}^{(1)} = \int_{-\infty}^{\infty} [2(\mathrm{sgn}\,x)^2 - \sqrt{2}\,\mathrm{sgn}'x] w_{1E}\,dx = \{2 - \int_{-\infty}^{\infty} 2\sqrt{2}\,\delta(x-o) w_{1E}\,dx\}$$

$$= 2(1-\sqrt{2}\,w_{1E}(0)) \quad ; \qquad \text{(A.4-64)}$$

and

$$L_{F:E}^{(2)} = \int_{-\infty}^{\infty} 2\sqrt{2}\,\delta(x-o) w_{1E}\,dx = 2\sqrt{2}\,w_{1E}(0) \; ; \; \hat{L}_{F:E}^{(2)} = \int_{-\infty}^{\infty} 2\,\mathrm{sgn}^2 x\, w_{1E}\,dx = 2 \; ; \qquad \text{(A.4-65a)}$$

$$L_{F:E}^{(4)} = \int_{-\infty}^{\infty} [2(\mathrm{sgn}\,x)^2 - 2\sqrt{2}\delta(x-o)]^2 w_{1E}\,dx = 4 + 8\delta(x-o) w_{1E}(0) \; ; \; (\mathrm{sgn}\,0 = 0); \qquad \text{(A.4-65b)}$$

$$\hat{L}_{F:E}^{(4)} = \int_{-\infty}^{\infty}[(\text{sgn } x)^2-2\sqrt{2}\ \delta(x-0)]w''_{1E}dx = 0-2\sqrt{2}\ w''_{1E}(0); \quad |<\infty| \quad ; \tag{A.4-65c}$$

$$L_{F:E}^{(2,2)} = \int_{-\infty}^{\infty} 2\ \text{sgn}^2 x\cdot w''_{1E}(x)dx = \int_{-\infty}^{\infty} w''_{1E}(x)dx = 0 \quad ; \tag{A.4-65d}$$

$$\hat{L}_{F:E}^{(2,2)} = -\int_{-\infty}^{\infty}\sqrt{2}[2\text{sgn}^2 x-2\sqrt{2}\ \delta(x-o)]\text{sgn } x\cdot w'_{1E}(x)dx = 4\sqrt{2}\int_0^{\infty} w'_{1E}(x)dx-0$$

$$= 4\sqrt{2}\ w_{1E}(0); \tag{A.4-65e}$$

$$L_{F:E}^{(6)} = \int_{-\infty}^{\infty}[2\ \text{sgn}^2 x-2\sqrt{2}\ \delta(x-o)]^2 w''_{1E}(x)dx = 0-0+8\delta(x-o)w''_{1E}(0)\ . \tag{A.4-65f}$$

[When F=E: i.e. $w_{1E}(x)$ is given by (A.4-50b), we have the optimum case in which the receiver is "matched" to the (Laplacian) noise, and we use (A.4-39)-(A.4-46) for the parameters.]

In this case we must discard the singular component of detector structure and of any $L_{F:E}$ [which occurs here when $x = 0$] when we apply the above results both to the detection algorithm and the evaluation of performance. This is to ensure that detection on a finite sample ($n<\infty$) is not perfect in the presence of finite (positive) noise intensity. Of course, physically, the "super-limiter" characteristic $F(x) = -\sqrt{2}\ \text{sgn } x$) is a mathematical idealization: in actual practice one uses a processing element where $|F'(x)| < \infty$, i.e., there are no infinite slopes, and hence no singularities in the structure or the assocated performance parameters. Accordingly, with the above in mind, we may substitute (A.4-64),(A.4-65) into (A.4-1,2), and (A.4-11,18) for the bias to get

$$B'_{coh} = \log\mu - \sqrt{2}\sum_i^n \langle\theta_i\rangle^2 w_{1E}(o)_i \quad ; \tag{A.4-66a}$$

$$B'_{inc} = \log\mu - \sum_i^n \langle\theta_i^2\rangle(1-\sqrt{2}\, w_{1E}(0)_i) - \frac{1}{4}\sum_{ij}^n \langle\theta_i\theta_j\rangle^2 [8w_{1E}(0)_i w_{1E}(0)_j$$

$$-\{\sqrt{2}\, w''_{1E}(0)_i + 8w_{1E}(0)_i^2\}\delta_{ij}] \, , \qquad \text{(A.4-66b)}$$

and the algorithms for coherent and incoherent detection, respectively, are thus

coherent: $$g(\underset{\sim}{x})_{coh} = \{\log\mu - \sqrt{2}\sum_i^n \langle\theta_i\rangle^2 w_{1E}(0)_i\} + \sqrt{2}\sum_i^n \langle\theta_i\rangle \,\mathrm{sgn}\, x_i \, , \qquad \text{(A.4-67a)}$$

incoherent: $$g(\underset{\sim}{x})_{inc} = B'_{inc}\Big|_{Eq.(A.4-66b)} + \sum_{ij}^n \langle\theta_i\theta_j\rangle \,\mathrm{sgn}\, x_i \,\mathrm{sgn}\, x_j \, , \qquad \text{(A.4-67b)}$$

this last where we have omitted the singular term $F'_i\delta_{ij} = -2\delta(x-o)$, which is zero (all $x \neq o$), for the reasons cited above. These algorithms (A.4-67a,b) represent "clipper-correlators": the former a clipper crosscorrelator, the latter, a clipper autocorrelator.

In a similar way we obtain the various $\hat{\sigma}_o$ [from (A.4-9,29)] and σ_o [from (A.4-12-31)], viz.:

$$\hat{\sigma}^2_{o-coh} \doteq 2\sum_i^n \langle\theta_i\rangle^2 \; ; \; \hat{\sigma}^2_{o-inc} \doteq \sum_{ij}^n \langle\theta_i\theta_j\rangle^2(2-\delta_{ij}) \, , \qquad \text{(A.4-68)}$$

and

$$\sigma_{o-coh} \doteq 2\sum_i^n \langle\theta_i\rangle^2 w_{1E}(0)_i / (\sum_i^n \langle\theta_i\rangle^2)^{1/2} \; ; \qquad \text{(A.4-69a)}$$

$$\sigma_{o\text{-}inc} \doteq \frac{\sum_{ij}^{n} \langle\theta_i\theta_j\rangle^2 \{8w_{1E}(0)_i w_{1E}(0)_j - [\sqrt{2}\, w''_{1E}(0)_i + 8w_{1E}(0)_i^2]\delta_{ij}\}}{2\{\sum_{ij}^{n} \langle\theta_i\theta_j\rangle^2 (2-\delta_{ij})\}^{1/2}} . \tag{A.4-69b}$$

[Since $w''_{1E}(0)_i \leq 0$, $w_{1E}(0)_i > 0$, we see that $\sigma_{o\text{-}coh}$, $\sigma_{o\text{-}inc}$ are always positive, as are $\hat{\sigma}_{o\text{-}coh}$, $\hat{\sigma}_{o\text{-}inc}$, as required for proper variances.]

The conditions (A.4-10), (A.4-30) on the maximum allowed values of the (small) input signal (a_o), to insure $\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2$ are specifically here

coherent:

$$4 \sum_i \langle\theta_i\rangle^4 w_{1i}(0)_E^2 \ll \sum_i \langle\theta_i\rangle^2, \tag{A.4-70a}$$

when $\langle\theta_i\theta_j\rangle = \langle\theta_i\rangle\langle\theta_j\rangle$ in these coherent cases. We have also

incoherent:

$$\Big|\Big\{\sum_{ij}' 6 \langle\theta_i^2\rangle\langle\theta_i\theta_j\rangle^2 w_{1E}(0)_i w_{1E}(0)_j + \sum_{ijk}^{i\neq j\neq k} [w_{1E}(0)_j w_{1E}(0)_k \langle\theta_j\theta_k\rangle$$

$$\cdot\{16\langle\theta_i\theta_j\rangle\langle\theta_k\theta_i\rangle + 16\langle\theta_j\theta_k\rangle\langle\theta_i^2\rangle\ (1-\sqrt{2}\, w_{1E}(0)_i)\}$$

$$+4\sqrt{2}\ \langle\theta_i^2\rangle\langle\theta_j^2\rangle\langle\theta_k^2\rangle w''_{1E}(0)\{1-\sqrt{2}\, w_{1E}(0)_i\}\delta_{ij}]\Big|$$

$$\ll\ 4\sum_{ij}^{n} \langle\theta_i\theta_j\rangle^2 (2-\delta_{ij}). \tag{A.4-70b}$$

Unlike the (suboptimum) correlation detectors of Sec. A.4-2 above, these clipper-correlators, (A.4-67), are considerably closer to the optimum [42]-[45], because much more than the (second and) fourth moments of the pdf of the interference is employed, viz. the "zero-crossings" of the noise (and signal) via the $\{\mathrm{sgn}\, x_i\}$. This fact is also exhibited in the arguments of the probability measures of performance, namely (A.4-69a,b) specifically when our real-world noise models (cf. Sec. 3) are employed.

A.4-4 Binary Signals:

The algorithms for binary signals employing suboptimum detectors of Class (A.4-1,2,3) above are readily obtained from the general relations (2.13)-(2.17), esp. (2.15), (2.16). These relations, in turn, are specialized to the important special subclasses of simple correlators [Sec. (A.4-21)] and clipper-correlators [Sec. A.4-3], as given in Sec. 4.2 above. In general, we replace $\langle\theta_i\rangle$ by $\Delta\theta_i^{(21)}$ $(= \langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle$, and $\langle\theta_i\theta_j\rangle$ by $\Delta^{(21)}\theta_i\theta_j = \langle(\theta_i\theta_j)^{(2)}\rangle - \langle(\theta_i\theta_j)^{(1)}\rangle \equiv \Delta\rho_{ij}^{(21)}$ etc., in the "on-off" results. Thus, we have, for these binary signal cases:

A. Simple Correlators:

Eq. (A.4-57):

$$\hat{\sigma}^{(21)^2}_{o\text{-}coh} \doteq \sum_i^n \left(\langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle\right)^2;\ \hat{\sigma}^2_{o\text{-}inc} \doteq \frac{1}{4}\sum_{ij}^n \left[\langle(\theta_i\theta_j)^{(2)}\rangle^2 - \langle(\theta_i\theta_j)^{(1)}\rangle^2\right]$$

$$\cdot\{(\overline{x_i^4}-3)\delta_{ij}+2\},\quad [\langle\theta_i\theta_j\rangle^{(2)} = \langle a_{oi}^{(2)}a_{oj}^{(2)}\rangle\rho_{ij}^{(2)},\ \text{etc.}]; \tag{A.4-71}$$

Eqs. (A.4-58):

$$\sigma^{(21)}_{o\text{-}coh} \doteq \left\{\sum_i^n \left(\langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle\right)^2\right\}^{1/2};$$

$$\sigma^{(21)}_{o\text{-}inc} \doteq \sum_{ij}^{n}[\langle(\theta_i\theta_j)^{(2)}\rangle-\langle(\theta_i\theta_j)^{(1)}\rangle]^2/(\sum_{ij}^{n}[\langle(\theta_i\theta_j)^{(2)}\rangle-\langle(\theta_i\theta_j)^{(1)}\rangle]^2$$

$$\cdot\{(\overline{x_i^4}-3)\delta_{ij}+2\})^{1/2} \ . \qquad \text{(A.4-72b)}$$

The "smallness" conditions on the input signals $(a_o^{(1)},a_o^{(2)})$, permitting $\hat{\sigma}_1^2 \doteq \hat{\sigma}_o^2$, are obtained directly from (A.4-59), on making the substitutions indicated above, viz. $\langle\theta_i\rangle \to \langle\Delta\theta_i\rangle = \langle\theta_i^{(2)}\rangle-\langle\theta_i^{(1)}\rangle$, $\langle\theta_i\theta_j\rangle \to \rho_{ij}^{(2)}-\rho_{ij}^{(1)} \equiv \Delta\rho_{ij}^{(21)}$, etc., with $\langle\theta_i^2\rangle \to \langle\theta_i^{(2)}\theta_j^{(2)}\rangle$, and $\langle\theta_i^{(1)}\theta_j^{(1)}\rangle$, $\langle\theta_k\theta_i\rangle \to \langle\theta_k^{(2)}\theta_i^{(2)}\rangle$ etc., in (A.4-59b), cf. (A.2-57).

B. ("Super") Clipper-Correlators:

Eq. (A.4-68):

$$\hat{\sigma}^{(21)^2}_{o\text{-}coh} \doteq 2\sum_{i}^{n}[\langle\theta_i^{(2)}\rangle - \langle\theta_1^{(1)}\rangle)^2; \ \hat{\sigma}^2_{o\text{-}inc} \doteq \sum_{ij}^{n}(\langle(\theta_i\theta_j)^{(2)}\rangle-\langle(\theta_i\theta_j)^{(1)}\rangle)^2\{2-\delta_{ij}\} \ ; \qquad \text{(A.4-73)}$$

Eq. (A.4-69):

$$\sigma^{(21)}_{o\text{-}coh} \doteq 2\sum_{i}^{n}[\langle\theta_i^{(2)}\rangle-\langle\theta_i^{(1)}\rangle]\, w_{1E}(0)_i/\{\sum_{i}^{n}(\langle\theta_i^{(2)}\rangle-\langle\theta_i^{(1)}\rangle)^2\}^{1/2} \ ; \qquad \text{(A.4-74a)}$$

$$\sigma^{(21)}_{o\text{-}inc} \doteq \frac{\sum_{ij}^{n}(\langle(\theta_i\theta_j)^{(2)}\rangle-\langle(\theta_i\theta_j)^{(1)}\rangle)^2\{8w_{1E}(0)_i w_{1E}(0)_j-[\sqrt{2}\,w''_{1E}(0)_i+8w_{1E}(0)_i^2]\delta_{ij}\}}{2\{\sum_{ij}^{n}(\langle(\theta_i\theta_j)^{(2)}\rangle-\langle(\theta_i\theta_j)^{(1)}\rangle)^2(2-\delta_{ij})\}^{1/2}} \ . \qquad \text{(A.4-74b)}$$

Again, for the "smallness" condition on the input signals $(a_o^{(1)},a_o^{(2)})$ we make the indicated substitutions, $\langle\theta_i\rangle \to \langle\theta_i^{(2)}\rangle-\langle\theta_j^{(1)}\rangle$, $\langle\theta_i\theta_j\rangle^2 \to \langle(\theta_i\theta_j)^{(2)}\rangle^2 - \langle(\theta_i\theta_j)^{(1)}\rangle^2$, etc., in (A.4-70a,b) above. [Specifically, for (A.4-70b) we replace $\langle\theta_i^2\rangle$ by $\langle\theta_i^{(2)}\theta_j^{(2)}\rangle$, $\langle\theta_i^{(1)}\theta_j^{(1)}\rangle$ and $\langle\theta_k\theta_i\rangle$ by $\langle\theta_k^{(2)}\theta_i^{(2)}\rangle$, $\langle\theta_k^{(1)}\theta_i^{(1)}\rangle$; $\langle\theta_i\theta_j\rangle^2 \to \Delta\rho_{ij}^{(21)2}$ etc., cf. (A.2-57).]

APPENDIX A5

$\hat{Q}_n^{(21)}$, $\hat{B}_n^{(21)*}$, $R_n^{(21)*}$ for Incoherent Reception with Binary Symmetric Channels

For binary symmetric channels ($a_o^{(2)}=a_o^{(1)}=a_o$; $p_1=p_2=1/2$) we need to evaluate $\hat{Q}_n^{(21)}$, which appears in the processing gain, and $\hat{B}_n^{(21)*}$, the associated bias, cf. Table 6.1b. We also need $R_n^{(21)*}$, (A.2-61b), to help establish the upper bounds on input signal size (and the equality of $\sigma_{1n}^{(21)*} \doteq \sigma_{on}^{(21)*}$). We shall do this for the basic type of common binary signals: sinusoids of different frequencies, cf. (7.3a), when there is either no fading or slow fading (e.g., $m_{ij} = 1$, cf. (7.7)), in the stationary noise regimes.

The quantities to be evaluated are:

$$\hat{Q}_n^{(21)}-1 \equiv n^{-1} \sum_{ij}^{n}{}' \left[\rho_{ij}^{(2)}-\rho_{ij}^{(1)}\right]^2 = \frac{1}{n} \sum_{ij}^{n} \left[\rho_{ij}^{(2)}-\rho_{ij}^{(1)}\right]^2 \; ; \tag{A.5-1}$$

$$\hat{B}_n^{(21)*} \equiv -\frac{\overline{a_o^2}^2}{8} \sum_{ij} \left(\rho_{ij}^{(2)^2}-\rho_{ij}^{(1)^2}\right)\left[\left(L^{(4)}-2L^{(2)^2}\right)\delta_{ij}+2L^{(2)^2}\right] , \tag{A.5-2}$$

where

$$\rho_{ij}^{(1),(2)} = \left\langle s_i^{(1),(2)} s_j^{(1),(2)} \right\rangle = \cos \omega_{o1,o2} (t_i-t_j). \tag{A.5-3}$$

Let us examine $\hat{Q}_n^{(21)}$ and use $T = n\Delta t$; $t_i = i\Delta t = x$, etc., so that we have to a good approximation:

$$\hat{Q}_n^{(21)}-1 \approx \frac{n}{T^2} \iint_0^T \left[\cos \omega_{o2}(x-y)-\cos\omega_{o1}(x-y)\right]^2 dxdy$$

$$= \frac{n}{4T^2} \iint_{-T}^{T} \left[\cos\omega_{o2}(x-y)-\cos \omega_{o1}(x-y)\right]^2 dxdy$$

$$= \frac{n}{2T^2} \int_0^{2T} (2T-z)(\cos \omega_{o2}z - \cos \omega_{o1}z)^2 dz \ , \tag{A.5-4}$$

where we have the identity

$$\iint_{-T}^{T} f(x-y)dxdy = 2 \int_0^{2T} (2T-z)f(z)dz \ . \tag{A.5-4a}$$

The evaluation of (A.5-4) proceeds directly:

$$\hat{Q}_n^{(21)} - 1 \cong \frac{n}{2T^2} \int_0^{2T} (2T-z)\{1 + \frac{1}{2}\cos \omega_{o2}z + \frac{1}{2}\cos \omega_{o1}z - \cos(\omega_{o2}-\omega_{o1})z$$

$$-\cos(\omega_{o2}+\omega_{o1})z\}dz \tag{A.5-5a}$$

$$\simeq n\{1 + \frac{1}{2T^2} \int_0^{2T} (2T-z) \ \{\frac{\cos \omega_{o2}z + \cos \omega_{o1}z}{2} - \cos(\omega_{o2}-\omega_{o1})z$$

$$-\cos(\omega_{o2}-\omega_{o1})z\}dz \quad , \tag{A.5-5b}$$

$$\therefore \quad \boxed{\hat{Q}_n^{(21)} - 1 \simeq n\{1 + O(1/\omega_{o1}T \text{ or } 1/\omega_{o2}T)\} \simeq n, \ (\omega_{o1,o2}T \sim n \gg 1) \ ,} \tag{A.5-5c}$$

as expected. Note that $\hat{Q}_n^{(21)}$ is twice Q_n, (A.2-42e), $n \gg 1$, which is to be expected, since here the binary signals have twice as much energy as the "on-off" cases. [With purely incoherent structure $\rho_{ij}^{(\ )} \approx \delta_{ij}$, $\therefore \hat{Q}_n^{(21)} \approx 1$, and and so $\hat{Q}^{(21)} - 1 \doteq 0$, which gives zero processing gain - cf. Table 6.1b. This is also to be expected, since now we have two indistinguishable, equal energy signals, with no coherent structure.]

We proceed similarly with the bias, (A.5-2), which we can write directly here for these symmetrical channels

$$\hat{B}_n^{(21)*} = -\frac{L^{(2)^2}}{4}\overline{a_o^2}^2 \sum_{ij}^{n}{}' [\rho_{ij}^{(2)^2} - \rho_{ij}^{(1)^2}]$$

$$= -\frac{L^{(2)^2}}{4}\overline{a_o^2}^2 \sum_{ij}^{n} [\cos^2\omega_{o2}(t_i-t_j) - \cos^2\omega_{o1}(t_i-t_j)] \qquad \text{(A.5-6a)}$$

$$= -\frac{L^{(2)^2}}{8}\overline{a_o^2}^2 \sum_{ij}^{n} [\cos 2\omega_{o2}(t_i-t_j) - \cos 2\omega_{o1}(t_i-t_j)] \qquad \text{(A.5-6b)}$$

$$\cong -\frac{L^{(2)^2}\overline{a_o^2}^2 n^2}{16T^2}\int_0^{2T}(2T-z)(\cos 2\omega_{o2}z - \cos 2\omega_{o1}z)dz \qquad \text{(A.5-6c)}$$

$$\therefore \hat{B}_n^{(21)*} \cong -\frac{L^{(2)^2}\overline{a_o^2}^2 n^2}{32}\left[\frac{\sin^2 2\omega_{o2}T}{(\omega_{o2}T)^2} - \frac{\sin^2 2\omega_{o1}T}{(\omega_{o1}T)^2}\right] = 0, \qquad \text{(A.5-7)}$$

when we sample such that $\omega_{o1}T = \omega_{o1}n\Delta t = k'2\pi$, $\omega_{o2}T = \omega_{o2}n\Delta t = k''2\pi$, or $\Delta t = (2\pi\lambda_1)/\omega_{o1} = (2\pi\lambda_2)/\omega_{o2}$; $\lambda_1 \equiv k'/n$, $\lambda_2 = k''/n$, where k', k", n (>>1) are integers. Thus, $(\lambda_1/\omega_{o1}) = (\lambda_2/\omega_{o2})$, or

$$\boxed{\frac{k'}{k''} = \frac{\omega_{o1}}{\omega_{o2}} = \text{ratio of integers}\quad .} \qquad \text{(A.5-8)}$$

<u>This means that one should choose the carrier frequencies f_{o1}, f_{o2}, such that (A.5-8) is satisfied.</u> Otherwise the bias $\hat{B}_n^{(21)*}$ is not strictly zero, although it can be quite small.

APPENDIX A6

Computer Software

In this appendix we simply list the computer programs used for the calculations given in the report and required for similar calculations. The programs are essentially self-explanatory via the comment statements, but some further explanation may be helpful.

The first program given, NORMB, is used to compute the normalization parameter, Ω. The "basic" Class B model is normalized to the rms level of the gaussian portion of the noise process since the 2nd (and other) moments do not exist for the Class B model. In NORMB, the parameter Ω is computed by truncating the Class B model, either at an envelope level of 80 db (on the original scale, gauss rms = 1) or at a level for which the probability of exceedance is 10^{-6}, whichever occurs first. In any particular case, the Ω would be computed by comparison of the Class B model with actual measured envelope data or by other appropriate means. The program NORMB integrates the truncated envelope distribution to obtain the rms level. Since the envelope power is twice the actual noise power, the proper corresponding normalization for the instantaneous amplitude is obtained by using 2Ω. For example, in (3.15a) the parameter Ω_B is given by 2Ω, Ω from NORMB. This program requires the subroutine CONHYP for the confluent hypergeometric function and the function routine GAMMA for the gamma function.

The next programs given are LOBDNA and LOBDNB. The routine LOBDNA computes the LOBD nonlinearity for Class A noise for both the canonical (3.13) and quasi-canonical (3.14) models (Figure 7.1). The routine LOBDNB computes the nonlinearity for Class B noise (3.15) (Figures 7.2a and 7.2b).

The three programs PC1, PC2, and PC3, compute the general performance results and probabilistic controls given on Figure 7.3-7.6. The programs require complementary error function and inverse error function routines given by the function routines CERF and ERFIN.

The programs PARA and PARB compute the detection parameters, processing gains, and bounds for input signal size, Figures 7.7-7.22, for Class A (PARA) and Class B (PARB) noise. The program PARA requires the subroutine FUN1 and FUN2 and the program PARB requires the subroutine FUN.