$$\left.\begin{aligned} &w_1(x)_o \equiv w_1(x|H_o) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \;:\therefore\; \ell = -x \;;\; \ell' = -1; \\ &L^{(2)} = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \overline{x^2} = 1;\; L^{(4)} = \int_{-\infty}^{\infty} (x^2-1)^2 w_1(x)_o\,dx = \overline{x^4} - 2\overline{x^2} + 1 = 3-2+1 = 2 \end{aligned}\right\} . \tag{A.1-22}$$

Additional quantities needed later (cf. Appendixes 2, 4) are (for the gauss pdf (A.1-22))

<u>gauss</u>

$$\left\{\begin{aligned} &L^{(2,2)} = 2\langle \ell^4 \rangle_o = 2\int_{-\infty}^{\infty} x^4 w_1(x)\,dx = 2\overline{x^4}\Big|_{gauss} = 6\overline{x^2} = 6 \;,\; (\overline{x^2} = 1); \\ &L^{(6)} = \left\langle \left(\frac{w_1''}{w_1}\right)^3 \right\rangle_o = \left\langle (x^2-1)^3 \right\rangle_o = \overline{x^6} - 3\overline{x^4} + 3\overline{x^2} - 1\Big|_{gauss} = 15 - 3\cdot 3 + 3 - 1 = 8. \end{aligned}\right. \tag{A.1-22a}$$

Consequently, we have

$$\boxed{L^{(4)} - 2L^{(2)^2}\Big|_{gauss} = 2 - 2\cdot 1^2 = 0 \;,} \tag{A.1-23}$$

so that (A.1-17) and (A.1-21) reduce now to

$$g^*_{coh}\Big|_{gauss} = [\log \mu - \sum_i^n \frac{\bar{\theta}_o^2}{2}] + \sum_i^n \bar{\theta}_i x_i \;;\quad \theta_i = a_{oi} s_i \;; \tag{A.1-24}$$

$$g^*_{inc}\Big|_{gauss} = [\log \mu - \frac{1}{2}\sum_i^n \langle \theta_i^2 \rangle - \frac{1}{4}\sum_{ij}^n \langle \theta_i \theta_j \rangle^2] + \frac{1}{2!}\sum_{ij}^n \langle \theta_i \theta_j \rangle x_i x_j \;. \tag{A.1-25}$$

These results demonstrate that the LOBD's for coherent and incoherent reception in gauss noise are, respectively, the cross-correlator $\sum_i^n \bar{\theta}_i x_i$, and the autocorrelator, $\sum_{ij}^n \langle \theta_i \theta_j \rangle x_i x_j$, specifically here for independent noise samples. (With correlated noise samples the corresponding structures are given in Sec. 2.3 above.) These results also agree precisely with the earlier developments (20.72), (20.81) or (20.11) of [12], when $\underset{\sim}{k}_N^{-1} = \delta_{ij}$ therein (independent noise samples). Note that these results apply for non-stationary as well as stationary noise processes: provided $w_1(x_i)$ is normalized to the mean intensity of the i^{th} sample, so that $L^{(2)}$, $L^{(4)}$ are then invariant of i. If a fixed normalization (over the observation period) is used, then $w_1 \rightarrow w_{1i}$, and we must explicitly account for the scale of the i^{th} sample. In the following analysis we shall, in the nonstationary cases, generally assume that the latter convention is chosen, so that the $L^{(2)}$, etc., must be indexed, e.g., $L_i^{(2)}$, etc., as distinct from the stationary cases.

APPENDIX A-2

Means and Variances of the Optimum Threshold Dectection Algorithm:

Here we calculate the first and second moments of the LOBD's g^*_{coh}, g^*_{inc}, in order to obtain the desired performance measures (P^*_D, P^*_e), as described generally in Section 2.4, for these threshold detection régimes. Again, independent noise samples are postulated, cf. Sec. A.1-2. We begin with the "on-off" cases (H_1 vs. H_0) in the coherent detection mode.

A.2-1: Coherent Detection

Let us consider the H_1-average, $\langle\ \rangle_{1,\theta}$ of g^*_{coh}, (A.1-17), for independent samples, viz:

$$\langle g^*_{coh}\rangle_{1,\theta} = \left\langle \int_{-\infty,(\underset{\sim}{x})}^{\infty} \prod_{i=1}^{n} w_1(x_i-\theta_i)_N g^*_{coh}(\underset{\sim}{x})d\underset{\sim}{x}\right\rangle_\theta$$

$$= B^*_{n-c} - \sum_{i=1}^{n}\langle\theta_i\rangle\left\langle\int_{-\infty}^{\infty}\ell(x_i)w_1(x_i-\theta_i)dx_i\right\rangle_\theta \ . \qquad \text{(A.2-1)}$$

Expanding w_1 about θ_i, we see that now for symmetrical pdf's, w_1,

$$\left\langle\int_{-\infty}^{\infty}\ell w_1 dx\right\rangle_\theta\Bigg|_i = \left(\left\langle\int_{-\infty}^{\infty}\frac{w_1'}{w_1}\left[w_1-\theta w_1'+\frac{\theta^2}{2!}w_1^{('')}-\frac{\theta^3}{3!}w_1^{(''')}+\ldots\right]dx\right\rangle_\theta\right)_i \qquad \text{(A.2-2a)}$$

$$= 0-\langle\theta_i\rangle\left(\int_{-\infty}^{\infty}\left(\frac{w_1'}{w_1}\right)^2 w_1 dx\right)_i+0-\frac{\langle\theta_i^3\rangle}{3!}\left(\int_{-\infty}^{\infty}\frac{w_1'w_1^{(''')}}{w_1}dx\right)_i + O(\langle\theta_i^5\rangle),$$

(A.2-2b)

since if w_1 is symmetric (about x=0), w_1', $w_1^{(''')}$, etc. are anti-symmetric, while $w_1^{('')}$, $w_1^{(4)}$, etc. remain symmetric. We have for (A.2-1), accordingly

$$\left\langle g^{*}_{coh}\right\rangle_{1,\theta} = B^{*}_{n-c} + \sum_{i=1}^{n} \{\langle\theta_i\rangle^2 L_i^{(2)} + \frac{\langle\theta_i^3\rangle\langle\theta_i\rangle}{3!} L_i^{(1,3)} + 0(\langle\theta_i^5\rangle)\}$$

$$\doteq \boxed{B^{*}_{n-c} + \sum_{i}^{n} \langle\theta_i\rangle^2 L_i^{(2)} + 0(\overline{\theta^4}) ,} \qquad \text{(A.2-3)}$$

where

$$L^{(1,3)} \equiv \int_{-\infty}^{\infty} \frac{w_1'}{w_1} \cdot \frac{w_1^{(''')}}{w_1} w_1 dx \;(\neq 0), \text{ etc.}$$

The H_o-average, $\langle\ \rangle_{o,\theta=0}$, of (A.1-17 follows at once from (A.2-2a) on setting $\theta=0$ therein (before $\langle\ \rangle_\theta$), e.g.

$$\boxed{\left\langle g^{*}_{coh}\right\rangle_{o} = B^{*}_{n-c}} \quad , \quad (\text{ all } \theta). \qquad \text{(A.2-4)}$$

We proceed in the same fashion for <u>the second moment:</u>

$$\left\langle (g^{*}_{coh})^2\right\rangle_{1,\theta} = \left\langle B^{*2}_{n-c} - 2B^{*}_{n-c} \sum_{i} \langle\theta_i\rangle \ell(x_i) + \sum_{ij} \langle\theta_i\rangle\langle\theta_j\rangle \ell_i \ell_j\right\rangle_{1,\theta}. \qquad \text{(A.2-5)}$$

Equation (A.2-2) gives us $\langle\ell\rangle_{1,\theta}$. For $\langle\ell_i\ell_j\rangle_{1,\theta}$ we have

<u>(i=j):</u>

$$\langle\ell_i^2\rangle_{1,\theta} = \left\langle\int_{-\infty}^{\infty} \left(\frac{w_1'}{w_1}\right)^2 [w_1 - \theta w_1' + \frac{\theta^2}{2!} w_1'' \ldots] dx\right\rangle_{\theta}\Bigg|_i$$

$$= L_i^{(2)} + \frac{\langle\theta_i^2\rangle}{2} L_i^{(2,2)} + 0(\langle\theta_i^4\rangle) ; \quad \boxed{L_i^{(2,2)} \equiv \int_{-\infty}^{\infty} \left(\frac{w_1'}{w_1}\right)^2 \left(\frac{w_1''}{w_1}\right) w_1 dx)_i ;}$$

(A.2-6)

$\underline{(i\neq j):}$

$$\langle \ell_i \ell_j \rangle_{1,\theta} = \left\langle \langle \ell_i \rangle_1 \langle \ell_j \rangle_1 \right\rangle_\theta = \left\langle (-\theta_i L_i^{(2)} - \frac{\theta_i^3}{3!} L_i^{(1,3)} \ldots)(-\theta_j L_j^{(2)} - \frac{\theta_j^3}{3!} L_j^{(1,3)} \ldots) \right\rangle_\theta \tag{A.2-7a}$$

$$\therefore \langle \ell_i \ell_j \rangle_{1,\theta} = \langle \theta_i \theta_j \rangle L_i^{(2)} L_j^{(2)} + \frac{1}{3!} \langle \theta_i^3 \theta_j \rangle L_i^{(1,2)} L_j^{(2)} + \frac{1}{3!} \langle \theta_i \theta_j^3 \rangle L_j^{(1,3)} L_i^{(2)} + \ldots ; \quad (i\neq j) . \tag{A.2-7b}$$

The result for the last term of (A.2-5) is

$$\sum_{ij} \langle \theta_i \rangle \langle \theta_j \rangle \langle \ell_i \ell_j \rangle_{1,\theta} = \sum_i \langle \theta_i \rangle^2 [L_i^{(2)} + \frac{\langle \theta_i^2 \rangle}{2} L_i^{(2,2)} + \ldots]$$

$$+ \sum_{ij}{}' \langle \theta_i \rangle \langle \theta_j \rangle [\langle \theta_i \theta_j \rangle L_i^{(2)} L_j^{(2)} + \ldots]. \tag{A.2-8}$$

Since we ultimately want the variance, $\mathrm{var}_{1,\theta} g_c^*$, rather than the second moment alone, we can write

$$\boxed{\mathrm{var}_{1,\theta} g_c^* = \langle g_c^{*2} \rangle_{1,\theta} - \langle g_c^* \rangle_{1,\theta}^2 = \sum_{ij} \langle \theta_i \rangle \langle \theta_j \rangle [\langle \ell_i \ell_j \rangle_{1,\theta} - \langle \ell_i \rangle_{1,\theta} \langle \ell_j \rangle_{1,\theta}],} \tag{A.2-9}$$

a simpler result, independent of the bias B_{n-c}^*, as expected. Since from (A.2-2b)

$$\langle \ell_i \rangle_{1,\theta} = -\langle \theta_i \rangle L_i^{(2)} - \frac{\langle \theta_i^3 \rangle}{3!} L_i^{(1,3)} \ldots , \tag{A.2-10}$$

we obtain from (A.2-8), (A.2-10), in (A.2-9)

$$(\sigma_1^*)^2 \equiv \mathrm{var}_{1,\theta} g_c^* = \sum_i \langle\theta_i\rangle^2 (L_i^{(2)} + \frac{\langle\theta_i^2\rangle}{2} L_i^{(2,2)} + \ldots - L_i^{(2)^2} \langle\theta_i\rangle^2 \ldots)$$

$$+ \sum_{ij}{}' \langle\theta_i\rangle\langle\theta_j\rangle (\langle\theta_i\theta_j\rangle L_i^{(2)} L_j^{(2)} + 1 \ldots - \langle\theta_i\rangle\langle\theta_j\rangle L_i^{(2)} L_j^{(2)} \ldots). \tag{A.2-11a}$$

In a similar way we obtain

$$\mathrm{var}_{o,o} g_c^* = \langle g_c^{*2}\rangle_{o,o} - \langle g_c^*\rangle_{o,o}^2 = \sum_{ij}^{n} \langle\theta_i\rangle\langle\theta_j\rangle [\langle \ell_i \ell_j\rangle_{o,o} - \langle \ell_i\rangle_{o,o} \langle \ell_j\rangle_{o,o}]. \tag{A.2-12}$$

From (A.2-2) $\langle \ell_i \rangle_{o,o} = 0$ and

$$\langle \ell_i \ell_j \rangle_{o,o} = \langle \ell_i^2 \rangle_{o,o} = L_i^{(2)} \delta_{ij} \; ; = \langle \ell_i\rangle_o \langle \ell_j\rangle_o = 0, \; i \neq j \, , \tag{A.2-13}$$

so that

$$(\sigma_o^*)^2 \equiv \mathrm{var}_{o,o} g_c^* = \sum_i^n \langle\theta_i\rangle^2 L_i^{(2)} = -2\hat{B}_{n-c}^*, \text{ cf. (A.1-16)} \, , \tag{A.2-14}$$

exactly.

From a comparison of (A.2-11) and (A.2-14) we see at once that because of the consistency condition on the threshold expansion by which the bias is determined [cf. Sec. (2.4)], which also requires that $\sigma_1^{*2} \doteq \sigma_o^{*2}$, we have specifically the requirement on input signal level $\langle\theta\rangle$, or $\langle\theta\rangle^2$, that

$\underline{\sigma^{*2}_{1c} \doteq \sigma^{*2}_{oc} :}$

$$\left| \sum_i \langle\theta_i\rangle^2 [\langle\theta_i^2\rangle \frac{L_i^{(2,2)}}{2} - \langle\theta_i\rangle^2 L_i^{(2)^2} + \sum_{ij}{}' \langle\theta_i\rangle\langle\theta_j\rangle L_i^{(2)} L_j^{(2)} (\langle\theta_i\theta_j\rangle - \langle\theta_i\rangle\langle\theta_j\rangle) \right| << \sigma_o^{*2} = \sum_i \langle\bar{\theta}_i\rangle^2 L_i^{(2)} .$$

(A.2-15a)

This reduces in the stationary régimes [where now $\langle\theta_i\rangle = \overline{a_o}\,\overline{s_i} = \overline{a_o}$, since because of coherence $\overline{s_i} = s_{max} = \sqrt{2}$, etc.] to

$\underline{\sigma^{*2}_{1c} \doteq \sigma^{*2}_{oc} :}$

$$\left| \{\overline{a_o^2}\, L^{(2,2)}/2L^{(2)} - \bar{a}_o^2 L^{(2)}\} + \frac{L^{(2)}}{n} \sum_{ij}{}' (\overline{a_{oi}a_{oj}} - \bar{a}_o^2) \right| << 1 \qquad \text{(A.2-15b)}$$

and clearly there is a dependence on sample size (n). For slow and rapid fading (A.2-15b) reduces further to

(i). <u>slow fading and no fading:</u>

$$\left| \overline{a_o^2}\, L^{(2,2)}/2L^{(2)} - \overline{a_o^2} L^{(2)} \right| << 1 ; \qquad \text{(A.2-15c)}$$

(ii). <u>rapid fading:</u>

$$\left| \{\overline{a_o^2}\, L^{(2,2)}/2L^{(2)} - \bar{a}_o^2 L^{(2)}\} \right| << 1 \quad ; \qquad \text{(A.2-15d)}$$

(iii). <u>no fading:</u>

$$\overline{a_0^2}\,|(L^{(2,2)}/2-L^{(2)^2})/L^{(2)}| \ll 1, \tag{A.2-15e}$$

cf. (A.2-17a) ff.

In the strictly coherent régimes (no fading), we have $\langle\theta_i\theta_j\rangle = \langle\theta_i\rangle\langle\theta_j\rangle$ here. Moreover

$$L^{(2,2)} = \int_{-\infty}^{\infty} \left(\frac{w_1'}{w_1}\right)^2 w_1'' dx = w_1'\left(\frac{w_1'}{w_1}\right)^2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{2w_1''}{w_1^2} - 2\,\frac{w_1'^2\cdot w_1'}{w_1^3}\right) w_1' dx$$

$$= 2\int_{-\infty}^{\infty}\left(\frac{w_1'}{w_1}\right)^4 w_1\,dx = 2\langle\ell^4\rangle_0\,, \tag{A.2-16a}$$

$$\therefore\ \frac{L^{(2,2)}}{2} - L^{(2)^2} = \frac{2\langle\ell^4\rangle_0}{2} - \langle\ell^2\rangle_0 = \mathrm{var}_0\,\ell^2\ ;\ \ L^{(2)} = \langle\ell^2\rangle_0 = \mathrm{var}_0\,\ell. \tag{A.2-16b}$$

Accordingly, the condition on $\langle\theta_i\rangle$, (A.2-15a), becomes

$$\boxed{\sum_i^n \langle\theta_i\rangle^4 \mathrm{var}_0\,\ell_i^2 \Big/ \sum_i^n \langle\theta_i\rangle^2 \mathrm{var}_0\,\ell_i \ll 1.}\quad , \tag{A.2-17}$$

for $\sigma_1^{*2} \doteq \sigma_0^{*2}$.

<u>When stationarity obtains</u>, in addition, $L_i^{(2)} = L^{(2)}$, etc., $\langle\theta_i\rangle = \bar{a}_0\bar{s}$, all i, so that (A.2-17) reduces further to

$$\boxed{\bar{a}_0^2\,\bar{s}^2\left(\frac{\mathrm{var}_0\,\ell^2}{\mathrm{var}_0\,\ell}\right) \ll 1}\quad ,\ \ \overline{a_0}\,,\ \bar{s} > 0\,, \tag{A.2-17a}$$

which is independent of sample size (n), as is (A.2-17) essentially, if ℓ_i does not vary too much (i=1,...n).

A.2-2: Incoherent Detection:

Here we seek the mean and variance of g^*_{inc}, (A.1-21), when (A.1-20a) is the general bias in the non-stationary cases. We proceed as in Sec. A.2-1 and consider first the H_1-average of g^*_{inc}:

$$\langle g^*_{inc}\rangle_{1,\theta} = B^*_{n-inc} + \frac{1}{2}\sum_{ij}^{n} \langle\theta_i\theta_j\rangle\langle\ell_i\ell_j+\ell_i'\delta_{ij}\rangle_{1,\theta} \qquad \text{(A.2-18)}$$

Specifically, we have (cf. A.1-11):

(i=j):

$$\langle\ell_i^2+\ell_i'\rangle_{1,\theta} = \left\langle\int_{-\infty}^{\infty}\left(\frac{w_1''}{w_1}\right)_i w_1(x_i-\theta_i)dx_i\right\rangle_{\theta_i} \qquad \text{(A.2-19a)}$$

$$= \left\langle\int_{-\infty}^{\infty}\frac{w_1''}{w_1}\left[w_1-\theta w_1'+\frac{\theta^2}{2!}w_1''+\frac{-\theta^3}{3!}w_1^{(''')}+\dots\right]dx\right\rangle_{\theta,i}$$

$$= 0 - 0 + \frac{\langle\theta_i^2\rangle}{2!}\left(\int_{-\infty}^{\infty}\left(\frac{w_1''}{w_1}\right)^2 w_1\,dx\right)_i - 0 + \frac{\langle\theta_i^4\rangle}{4!}\left(\int_{-\infty}^{\infty}\left(\frac{w_1^{('')}}{w_1}\right)\left(\frac{w_1^{(4)}}{w_1}\right)w_1\,dx\right)_i \dots$$

$$= \frac{\langle\theta_i^2\rangle}{2}L_i^{(4)} + \frac{\langle\theta_i^4\rangle}{4!}L_i^{(2,4)} + O(\overline{\theta^6}) \ , \qquad \text{(A.2-19b)}$$

where

$$L_i^{(2,4)} \equiv \left(\int_{-\infty}^{\infty}\left(\frac{w_1''}{w_1}\right)\left(\frac{w_1^{(4)}}{w_1}\right)w_1\,dx\right)_i \ , \qquad \text{(A.2-19c)}$$

and we have used the symmetry property of w_1, $w_1^{('')}$, etc., and the antisymmetry of w_1', $w_1^{(3)}$, etc.

Similarly, we have

<u>(i≠j):</u>

$$\langle \ell_i \ell_j + \ell_i' \delta_{ij} \rangle_{1,\theta} = \langle \langle \ell_i \rangle_1 \langle \ell_j \rangle_1 \rangle_\theta = \langle \theta_i \theta_j \rangle L_i^{(2)} L_j^{(2)} + O(\bar{\theta}^4), \quad \text{Eq. (A.2-7b)}, \tag{A.2-20}$$

so that combining (A.2-19b) and (A.2-20) in (A.2-18) yields specifically

$$\langle g_{inc}^* \rangle_{1,\theta} = B_{n-inc}^* + \frac{1}{2} \left\{ \sum_{ij}{}' \langle \theta_i \theta_j \rangle^2 L_i^{(2)} L_j^{(2)} + \sum_i \frac{\langle \theta_i^2 \rangle^2}{2} L^{(4)} + O(\overline{\theta_i^6}) \right\} \tag{A.2-21a}$$

$$= B_{n-inc}^* + \frac{1}{4} \left\{ \sum_{ij} \langle \theta_i \theta_j \rangle^2 [(L_i^{(4)} - 2L_i^{(2)^2}) \delta_{ij} + 2L_i^{(2)} L_j^{(2)}] \right\}, \tag{A.2-21b}$$

which now combined with (A.1-20a) for the bias B_{n-inc}^* gives directly

$$\langle g_{inc}^* \rangle_{1,\theta} = \log \mu + \frac{1}{8} \sum_{ij} \langle \theta_i \theta_j \rangle^2 [L_i^{(4)} - 2L_i^{(2)^2}) \delta_{ij} + 2L_i^{(2)} L_j^{(2)}] \tag{A.2-22a}$$

$$= \log \mu - \hat{B}_{n-inc}^* , \tag{A.2-22b}$$

cf. (A.1-20a).

The H_o-moment of g_{inc}^* is found at once to be

$$\langle g_{inc}^* \rangle_{o,o} = B_{n-inc}^* + \frac{1}{2} \sum_{ij} \langle \ell_i \ell_j + \ell_i' \delta_{ij} \rangle_{o,o} \langle \theta_i \theta_j \rangle, \tag{A.2-23}$$

where

$$\underline{(i=j):} \qquad \left\langle \ell_i^2+\ell_i' \right\rangle_{o,o} = \int_{-\infty}^{\infty} \frac{w_{1i}''}{w_{1i}} \; w_1(x_i)dx_i = w_{1i}' \Big|_{-\infty}^{\infty} = 0 \ , \qquad \text{(A.2-24a)}$$

$$\underline{(i\neq j):} \qquad \left\langle \ell_i \right\rangle_o \left\langle \ell_j \right\rangle_o = 0 \ , \ \text{(A.1-13b)}, \qquad \text{(A.2-24b)}$$

so that

$$\boxed{\left\langle g_{inc}^* \right\rangle_{o,o} = B_{n-inc}^* = \text{Eq. (A.1-20a)} = \log \mu + \hat{B}_{inc}^* .} \qquad \text{(A.2-25)}$$

We proceed similarly for $\text{var}_{1,\theta} g_{inc}^*$, cf. (A.2-9). From (A.1-21) specifically we write

$$\text{var}_{1,\theta} g_{inc}^* = \frac{1}{4} \sum_{ijk\ell}^{n} \{ \left\langle F(x_i,x_j|\theta_i,\theta_j) F(x_k,x_\ell|\theta_k,\theta_\ell) \right\rangle_{1,\theta}$$

$$- \left\langle F(x_i,x_j|\theta_i,\theta_j) \right\rangle_{1,\theta} \left\langle F(x_k,x_\ell|\theta_k,\theta_\ell) \right\rangle_{1,\theta} \ , \qquad \text{(A.2-26)}$$

where

$$F(x_i,x_j|\theta_i,\theta_j) \equiv (\ell_i \ell_j + \ell_i' \delta_{ij}) \left\langle \theta_i \theta_j \right\rangle \ . \qquad \text{(A.2-26a)}$$

Let us consider the first average in (A.2-26). We have

$$\sum_{ijk\ell} \left\langle F_{ij} F_{k\ell} \right\rangle_{1,\theta} = \left\langle [\sum_i \frac{w_{1i}^{('')}}{w_{1i}} \left\langle \theta_i^2 \right\rangle + \sum_{ij}^{(i\neq j)} \ell_i \ell_j \left\langle \theta_i \theta_j \right\rangle] \right.$$

$$\left. \cdot [\sum_k \frac{w_{1k}^{('')}}{w_{1k}} \left\langle \theta_k^2 \right\rangle + \sum_{k\ell}^{(k\neq \ell)} \ell_k \ell_\ell \left\langle \theta_k \theta_\ell \right\rangle] \right\rangle_{1,\theta} \ , \qquad \text{(A.2-27)}$$

cf. (A.1-19). We proceed as for (A.1-19) et seq. and distinguish the following terms [through $O(\theta^6)$ in (A.2-27), or equivalently, through $O(\overline{\theta^2})$ in the coefficients of $\langle\theta_i^2\rangle$, etc.]:

(1). (i≠k):

$$\left\langle\left\langle\frac{w_{1i}^{(")}}{w_{1i}}\right\rangle_1\left\langle\frac{w_{1k}^{(")}}{w_{1k}}\right\rangle_1\right\rangle_\theta = \left\langle\int_{-\infty}^{\infty}\frac{w_{1i}^{(")}}{w_{1i}}\,w_1(x_i-\theta_i)dx_i\int_{-\infty}^{\infty}\frac{w_{1k}^{(")}}{w_{1k}}\,w_1(x_k-\theta_k)dx_k\right\rangle_\theta \quad \text{(A.2-28a)}$$

$$= \left\langle\int_{-\infty}^{\infty}\frac{w_{1i}^{(")}}{w_{1i}}[w_{1i}-\theta_i w_{1i}'+\frac{\theta_i^2}{2}w_{1i}''\cdots]dx_i\right.$$

$$\left.\cdot\int_{-\infty}^{\infty}\frac{w_{1k}^{(")}}{w_{1k}}[w_{1k}-\theta_k w_{1k}'+\frac{\theta_k^2}{2}w_{1k}''\cdots]dx_k\right\rangle_\theta$$

$$= \left\langle[0-0+\frac{\theta_i^2}{2}\left\langle(\frac{w_{1i}''}{w_{1i}})^2\right\rangle_o+\ldots][0-0+\frac{\theta_k^2}{2}\left\langle(\frac{w_{1k}''}{w_{1k}})^2\right\rangle_o+\ldots]\right\rangle_\theta$$

$$= \frac{\langle\theta_i^2\theta_k^2\rangle}{4}L_i^{(4)}L_k^{(4)}+0(\overline{\theta^6}) \;. \quad \text{(A.2-28b)}$$

(2). (i=k):

$$\left\langle(\frac{w_{1i}^{(")}}{w_{1i}})^2\right\rangle_{1,\theta} = L_i^{(4)}+\frac{\langle\theta_i^2\rangle}{2}\left\langle(\frac{w_{1i}^{(")}}{w_{1i}})^3\right\rangle_o+\ldots = L_i^{(4)}+\frac{\langle\theta_i^2\rangle}{2}L_i^{(6)}+\ldots, \quad \text{(A.2-29a)}$$

where

$$\boxed{L_i^{(6)} \equiv \int_{-\infty}^{\infty}(\frac{w_{1i}^{(")}}{w_{1i}})^3\,w_{1i}dx_i \;.} \quad \text{(A.2-29b)}$$

Next, let us consider the product terms $J_4 \equiv \langle\sum_{ij}{}' a_{ij}\sum_{k\ell}{}'a_{k\ell}\rangle_{1,\theta}$ (where the prime, as before, indicates that terms j=i, etc., are omitted in the summations). Let us rewrite J_4 as

$$J_4 \equiv \sum_{ijk\ell}\left\langle(1-\delta_{ij})(1-\delta_{k\ell})a_{ij}a_{k\ell}\right\rangle_{1,\theta} =$$

$$= \sum_{ijk\ell}\left\langle a_{ij}a_{k\ell}\right\rangle_{1-\theta} - 2\sum_{ij\ell}\left\langle a_{ii}a_{j\ell}\right\rangle_{1-o} + \sum_{ij}\left\langle a_{ii}a_{jj}\right\rangle_{1-o}\ , \qquad \text{(A.2-30a)}$$

where $a_{ij} = \langle\theta_i\theta_j\rangle\ell_i\ell_j$, etc. Of fourth-order **product** averages $J_{4/4}$, we have from the leading term of (A.2-30a):

(3) <u>$(i\neq j)\neq(k\neq\ell)$:</u>

$$\left\langle\ell_i\ell_j\ell_k\ell_\ell\right\rangle_{1,\theta} = \left\langle\int_{-\infty}^{\infty}\cdots\int\left(\frac{w'_{1i}}{w_{1i}}\right)\left(\frac{w'_{1j}}{w_{ij}}\right)\left(\frac{w'_{ik}}{w_{1k}}\right)\left(\frac{w'_{1\ell}}{w_{1\ell}}\right)[w_{1i}-\theta_i w'_{1i}+\frac{\theta_i^2}{2}w''_{1i}+\ldots]\right.$$

$$\left.\cdot[w_{ij}-\theta_j w'_{ij}+\ldots][w_{1k}-\theta_k w'_{1k}+\ldots][w_{1\ell}-\theta_\ell w'_{1\ell}+\ldots]dx_i\ldots dx_\ell\right\rangle_\theta$$

$$= 0+\langle\theta_i\theta_j\theta_k\theta_\ell\rangle L_i^{(2)}L_j^{(2)}L_k^{(2)}L_\ell^{(2)}+O(\overline{\theta^8})\ , \qquad \text{(A.2-30b)}$$

which accordingly do not contribute $O(\overline{\theta^2})$, i.e. $O(\overline{\theta^6})$ in J_4 when we include the $\langle\theta_i\theta_j\rangle\langle\theta_k\theta_\ell\rangle$ factors in a_{ij}, $a_{k\ell}$, etc. Of third-order products, $J_{4/3}$, we need to consider the first two terms of (A.2-30a), where now

(4). <u>$(i\neq j);(k\neq\ell)$</u>: (a). and k=i, or ℓ=i, or k=j, or ℓ=j, ∴ 4x(k=i) contributions

$$\therefore\ J_{4/3} = 4\sum_{ij\ell}^{i\neq j\neq\ell}\left\langle a_{ij}a_{i\ell}\right\rangle_{1,\theta} - 2\sum_{ij\ell}^{i\neq j\neq\ell}\left\langle a_{ii}a_{j\ell}\right\rangle_{1,\theta} \qquad \text{(A.2-30c)}$$

$$= \sum_{ij\ell}^{i\neq j\neq\ell}\left\langle\langle\ell_i^2\rangle_1\langle\ell_j\rangle_1\langle\ell_\ell\rangle_1\right\rangle_\theta[4\langle\theta_i\theta_j\rangle\langle\theta_\ell\theta_i\rangle-2\langle\theta_i^2\rangle\langle\theta_\ell\theta_j\rangle]. \qquad \text{(A.2-30d)}$$

Now

$$\left\langle\langle \ell_i^2\rangle_1 \langle \ell_j\rangle_1 \langle \ell_\ell\rangle_1\right\rangle_\theta = \left\langle \int_{-\infty}^{\infty}\cdots\int \left(\frac{w_{1i}'}{w_{1i}}\right)^2 \left(\frac{w_{ij}'}{w_{ij}}\right)\left(\frac{w_{1\ell}'}{w_{1\ell}}\right)[w_{1i}-\theta_i w_{1i}+\ldots]\right.$$

$$\left.\cdot[w_{1j}-\theta_j w_{1j}'+\ldots][w_{1\ell}-\theta_\ell w_{1\ell}'+\ldots]dx_i\,dx_j\,dx_\ell\right\rangle_\theta$$

$$= 0+L_i^{(2)}\langle\theta_j\theta_\ell\rangle L_j^{(2)}L_\ell^{(2)}+0(\overline{\theta^4}) \ , \qquad \text{(A.2-31)}$$

so that $J_{4/3}$ becomes

$$J_{4/3} = \sum_{ij\ell}''' L_i^{(2)}L_j^{(2)}L_\ell^{(2)}[4\langle\theta_i\theta_j\rangle\langle\theta_j\theta_\ell\rangle\langle\theta_\ell\theta_i\rangle-2\langle\theta_i^2\rangle\langle\theta_j\theta_\ell\rangle^2]. \qquad \text{(A.2-31a)}$$

For second-order products $J_{4/2}$ we have directly from J_4 as a whole:

$$\left.\begin{array}{l}\text{(b)}\quad i=k;j=\ell: \\ \qquad i=\ell;j=k:\ (i\neq j;k\neq\ell)\end{array}\right\}\ \langle\ell_i^2\ell_j^2\rangle_{1,\theta}\times 2\ :$$

$$\langle\ell_i^2\ell_j^2\rangle\times 2 = 2\left\langle\int_{-\infty}^{\infty}\cdots\int\left(\frac{w_{1i}'}{w_{1i}}\right)^2\left(\frac{w_{ij}'}{w_{ij}}\right)^2[w_{1i}-\theta_i w_{1i}'+\frac{\theta_i^2}{2}w_{1i}''+\ldots]\right.$$

$$\left.\cdot[w_{1j}-\theta_j w_{1j}'+\frac{\theta_j^2}{2}w_{1j}''+\ldots]dx_i\,dx_j\right\rangle_\theta$$

$$= 2L_i^{(2)}L_j^{(2)}+L_i^{(2)}L_j^{(2,2)}\langle\theta_j^2\rangle+L_j^{(2)}L_i^{(2,2)}\langle\theta_i^2\rangle+0(\overline{\theta^4}) \qquad \text{(A.2-32a)}$$

$$\therefore J_{4/2} = 2\sum_{ij} L_i^{(2)} L_j^{(2)} \langle \theta_i \theta_j \rangle^2 + 2\sum_{ij} \langle \theta_i \theta_j \rangle^2 \langle \theta_i^2 \rangle L_i^{(2,2)} L_j^{(2)} + O(\overline{\theta^8}) \ . \qquad \text{(A.2-32b)}$$

In addition to the sets of terms (1)-(4) in the product (A.2-27) there are also the following:

(5). $\left\langle \frac{w''_{1i}}{w_{1i}} \ell_k \ell_\ell \right\rangle_{1,\theta}$: $\underline{k \neq \ell}$: i=k, or i=ℓ; or i≠k≠ℓ:

[(x2): for (i≠j): k=i, or k=j, or k≠i≠j:] (A.2-33)

We have

(5a) $\underline{k \neq \ell:\ i=k:}$

$$\left\langle \frac{w''_{1i}}{w_{1i}} \ell_i \ell_\ell \right\rangle_{1,\theta} = \left\langle \int_{-\infty}^{\infty} \cdots \int \frac{w''_{1i}}{w_{1i}} \frac{w'_{1i}}{w_{1i}} \frac{w'_{1\ell}}{w_{1\ell}} [w_{1i} - \theta_i w'_{1i} + ..] \right.$$

$$\left. \cdot [w_{1\ell} - \theta_\ell w'_{1\ell} + ..] dx_1 .. dx_\ell \right\rangle_\theta \qquad \text{(A.2-34a)}$$

$$= 0 + \langle \theta_i \theta_\ell \rangle \int_{-\infty}^{\infty} \cdot\cdot \int \frac{w''_{1i}}{w_{1i}} \left(\frac{w'_{1i}}{w_{1i}}\right)^2 \left(\frac{w'_{1\ell}}{w_{1\ell}}\right)^2 w_{1i}\, w_{1\ell}\, dx_i\, dx_\ell + O(\overline{\theta^4})$$

$$= 0 + L_i^{(2,2)} L_\ell^{(2)} \langle \theta_i \theta_\ell \rangle + O(\overline{\theta^4}) \ . \qquad \text{(A.2-34b)}$$

(5b). $\underline{k \neq \ell:\ i=\ell:}$

$$\left\langle \frac{w''_{1i}}{w_{1i}} \ell_i \ell_k \right\rangle_{1,\theta} = 0 + L_i^{(2,2)} L_k^{(2)} \langle \theta_i \theta_k \rangle + O(\overline{\theta^4}) \text{ , similarly.} \qquad \text{(A.2-35)}$$

(5c). $k\neq\ell$: $i\neq k(\neq\ell)$:

$$\left\langle \frac{w''_{1i}}{w_{1i}} \ell_k \ell_\ell \right\rangle_{1,\theta} = \left\langle \int_{-\infty}^{\infty} \cdots \int \frac{w''_{1i}}{w_{1i}} \frac{w'_{1\ell}}{w_{1\ell}} \frac{w'_{1k}}{w_{1k}} [w_{1i} - \theta_i w'_{1i} + \frac{\theta_i^2}{2} w''_{1i} + \ldots] \right.$$

$$\left. \cdot [w_{1k} - \theta_k w'_{1k} + \ldots][w_{1\ell} - \theta_\ell w'_{1\ell} + \ldots] dx_i dx_k dx_\ell \right\rangle_\theta$$

$$= 0 + \frac{L_i^{(4)}}{2} \langle \theta_i^2 \theta_k \theta_\ell \rangle L_k^{(2)} L_\ell^{(2)} = 0 + 0(\overline{\theta^4}) . \qquad \text{(A.2-36)}$$

From (A.2-33) we repeat the above, equivalent to multiplying by a factor 2 in the relevant summations.

Combining the results of (1)-(5) for the average (A.2-27) then yields:

$$\sum_{ijk\ell} \langle F_{ij} F_{k\ell} \rangle_{1,\theta} = \sum_{ij} \langle \theta_i \theta_j \rangle^2 [(L_i^{(4)} - 2L_i^{(2)^2}) \delta_{ij} + 2L_i^{(2)} L_j^{(2)}]$$

$$+ \sum_i \frac{\langle \theta_i^2 \rangle^3}{2} L_i^{(6)} + 6 \sum_{ij} \langle \theta_i^2 \rangle \langle \theta_i \theta_j \rangle^2 L_j^{(2)} L_i^{(2,2)}$$

$$+ \sum_{ijk}^{(i\neq j\neq k)} L_i^{(2)} L_j^{(2)} L_k^{(2)} [4 \langle \theta_i \theta_j \rangle \langle \theta_j \theta_k \rangle \langle \theta_k \theta_i \rangle$$

$$- 2 \langle \theta_i^2 \rangle \langle \theta_j \theta_k \rangle^2] + 0(\overline{\theta^8}) . \qquad \text{(A.2-37)}$$

From the above it is seen directly that

$$\sum_{ijk\ell} \langle F_{ij} F_{k\ell} \rangle_{o,o} = \sum_{ij} \langle \theta_i \theta_j \rangle^2 [(L_i^{(4)} - 2L_i^{(2)^2}) \delta_{ij} + 2L_i^{(2)} L_j^{(2)}] = -8\hat{B}^*_{n-inc},$$

cf. (A.1-20a). (A.2-38)

From (A.2-19b), (A.2-20) we obtain

$$\sum_{ijk\ell}\left\langle F_{ij}\right\rangle_{1,\theta}\left\langle F_{k\ell}\right\rangle_{1,\theta}$$

$$= \sum_{ij}\left\{\begin{matrix} 0+\langle\theta_i\theta_j\rangle^2 L_i^{(2)}L_j^{(2)}(1-\delta_{ij}) \\ \\ 0+\dfrac{\langle\theta_i^2\rangle^2}{2}L_i^{(4)}\delta_{ij}\end{matrix}\right\}\cdot\sum_{k\ell}\left\{\begin{matrix} 0+\langle\theta_k\theta_\ell\rangle^2 L_k^{(2)}L_\ell^{(2)}(1-\delta_{k\ell}) \\ \\ 0+\dfrac{\langle\theta_k^2\rangle^2}{2}L_k^{(4)}\delta_{k\ell}\end{matrix}\right\} = 0+0(\overline{\theta^8}), \tag{A.2-39}$$

so that this average is always ignorable $[0(\overline{\theta^6})]$ in (A.2-26).

Accordingly, applying (A.2-37) - (A.2-39) to (A.2-26) gives us

$$\sigma^{*2}_{1\text{-}inc} \equiv \mathrm{var}_{1,\theta}g^*_{inc} \doteq -2\hat{B}^*_{n\text{-}inc}$$

$$= \frac{1}{4}\sum_{ij}^{n}\langle\theta_i\theta_j\rangle^2[(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}] \quad (>0)$$

$$\therefore\ \sigma^{*2}_{1\text{-}inc} \doteq \sigma^{*2}_{o\text{-}inc}\ [\equiv \mathrm{var}_{o,o}g^*_{inc}\ ,\ \text{cf. (A.2-38)}]\ . \tag{A.2-40}$$

This last relation, viz. $\sigma^{*2}_{1\text{-}inc} \doteq \sigma^{*2}_{o\text{-}inc}$, as required by the nature of the LOBD expansion [cf. Sec. 2.4], puts the necessary condition on the smallness of the input signal by demanding that terms $0(\overline{\theta^6})$ in $\mathrm{var}_{1,\theta}g^*_{inc}$, viz., in (A.2-37) be small vis-à-vis $\sigma^{*2}_{o\text{-}inc}$. Specifically, this condition is

$[\sigma^{*2}_{1\text{-}inc} \doteq \sigma^{*2}_{o\text{-}inc}]$:

$$\Big| \sum_{ij} \langle\theta_i^2\rangle\langle\theta_i\theta_j\rangle^2 \Big[\Big(\frac{L_i^{(6)}}{2}\,\delta_{ij} + 6L_i^{(2)}L_j^{(2,2)}\Big]$$
$$+ \sum_{ijk}^{(i\neq j\neq k)} L_i^{(2)}L_j^{(2)}L_k^{(2)}\big[4\langle\theta_i\theta_j\rangle\langle\theta_j\theta_k\rangle\langle\theta_k\theta_i\rangle - 2\langle\theta_i^2\rangle\langle\theta_j\theta_k\rangle^2\big]$$
$$<< \sum_{ij} \langle\theta_i\theta_j\rangle^2\big[(L_i^{(4)} - 2L_i^{(2)^2})\delta_{ij} + 2L_i^{(2)}L_j^{(2)}\big] \,. \qquad \text{(A.2-41)}$$

The condition (A.2-41) is considerably more complex than (A.2-17) for the coherent cases, as we might expect from the generally complex nature of the correlated signal samples $\langle\theta_i\theta_j\rangle$, etc. Writing

$$Q_n \equiv \frac{1}{n} \sum_{ij} m_{ij}^2 \rho_{ij}^2 \ (\geq 1) \ ;$$
$$R_n \equiv \frac{1}{n} \sum_{ijk}''' \{4m_{ij}m_{jk}m_{ki}\rho_{ij}\rho_{jk}\rho_{ki} - 2m_{jk}^2\rho_{jk}^2\}, \qquad \text{(A.2-41a)}$$

with

$$m_{ij} \equiv \overline{a_{oi}a_{oj}}/\overline{a_o^2} \ ; \ \rho_{ij} \equiv \langle s_i s_j\rangle \qquad \text{(A.2-41b)}$$

as before, cf. (6.25), (in the stationary noise cases, e.g. $L_i^{(2)} = L^{(2)}$, etc.), we get directly for (A.2-41)

$$\frac{\overline{a_o^2}\left|\frac{L^{(6)}}{2}+6L^{(2)}L^{(2,2)}Q_n+L^{(2)^3}R_n\right|}{L^{(4)}+2L^{(2)^2}(Q_n-1)} << 1 \ , \quad (L^{(6)} \neq 0) \ , \qquad \text{(A.2-42)}$$

Which is the (essentially) general condition on input signal $(\overline{a_o^2})$ that $\sigma_1^{*2} \doteq \sigma_o^{*2}$. Here we have, from A.1, A.2 above, in summary

$$\left.\begin{aligned}
L^{(2)} &\equiv \left\langle \left(\frac{w_1'}{w_1}\right)^2 \right\rangle_o = \left\langle \ell^2 \right\rangle_o \ (>0), && \text{Eq. (A.1-15);} \\
L^{(2,2)} &\equiv 2\left\langle \left(\frac{w_1'}{w_1}\right)^4 \right\rangle_o = 2\left\langle \ell^4 \right\rangle_o \ (>0) \ , && \text{Eq. (A.2-16);} \\
L^{(4)} &\equiv \left\langle \left(\frac{w_1''}{w_1}\right)^2 \right\rangle_o = \left\langle (\ell'+\ell^2)^2 \right\rangle_o \ (>0), && \text{Eq. (A.1-19b);} \\
L^{(6)} &\equiv \left\langle \left(\frac{w_1''}{w_1}\right)^3 \right\rangle_o = \left\langle (\ell'+\ell^2)^3 \right\rangle_o \ (\lesseqgtr 0), && \text{Eq. (A.2-29b)} \ .
\end{aligned}\right\} \qquad \text{(A.2-42a)}$$

We remark that whereas $L^{(2)}$, $L^{(2,2)}$, $L^{(4)}$ are always positive, $L^{(6)}$ can be negative (and zero). [In this last instance, we may have to include an additional term B $\overline{a_o^2}$ in the numerator of (A.2-42), e.g. $(L^{(6)}/2) \to (L^{(6)}/2)+B\,\overline{a_o^2}$, when Q_n, R_n vanish.]

For purely incoherent signals, we have $\rho_{ij} = \delta_{ij}$; such signals can result from scatter mechanisms, heavy doppler "smear", and/or rapid fading, or combinations of all these mechanisms. Then, $Q_n = 1$, and $R_n = 0$, cf. (A.2-41a), so that (A.2-42) becomes

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$$a_o^2 \, F(L^{(2)},\ldots|Q_n,R_n)^{-1} = \overline{a_o^2} \cdot \frac{\left|\frac{L^{(6)}}{2} + 6L^{(2)}L^{(2,2)}\right|}{L^{(4)}} \ll 1. \tag{A.2-42b}$$

At the other extreme of purely coherent signals, (e.g. sinusoidal wave trains), we have $\rho_{ij} = \cos \omega_o(t_i - t_j)$ etc., with $m_{ij}=1$, etc. Letting $T = n\Delta t$, $t_i = i\Delta t = x$, etc., we have (n large)

$$\left.\begin{aligned} I_1 &\equiv 4 \sum_{ijk}''' \rho_{ij}\rho_{jk}\rho_{ki} \doteq \left(\frac{n}{T}\right)^3 4 \iiint_0^T \cos \omega_o(x-y)\cos \omega_o(y-z)\cos \omega_o(z-x)\,dxdydz \\ I_2 &\equiv 2 \sum_{ijk}''' \rho_{jk}^2 \doteq 2n \left(\frac{n}{T}\right)^2 \iint_0^T \cos^2 \omega_o(y-z)\,dydz \end{aligned}\right\}. \tag{A.2-42c}$$

Expanding and integrating gives, after some algebra:

$$\begin{cases} R_n = \frac{1}{n}(I_1 - I_2) = \frac{1}{n}\left(\left[n^3 + 2n^3\left\{\frac{1-\cos 2n\omega_o\Delta t}{(2n\omega_o\Delta t)^2} + \frac{\sin 2n\omega_o\Delta t}{2n\omega_o\Delta t}\right\}\right]_{I_1} \right. \\ \qquad\qquad \left. - \left[n^3 + 2n^3\left\{\frac{1-\cos 2n\omega_o\Delta t}{(2n\omega_o\Delta t)^2}\right\}\right]_{I_2}\right) \\ \qquad = 2n \cdot \left(\frac{\sin 2n\omega_o\Delta t}{2\omega_o\Delta t}\right) \; ; \qquad\qquad (A.2\text{-}42d) \\ Q_n = \frac{n}{2}\left\{1 + 2\left(\frac{1-\cos 2n\omega_o\Delta t}{(2n\omega_o\Delta t)^2}\right)\right\}. \qquad\qquad (A.2\text{-}42e) \end{cases}$$

Consequently, the condition (A.4-42) becomes here (n>>1):

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$$\overline{a_o^2}\, F_n^{-1} = \overline{a_o^2}\left|\frac{3L^{(2,2)}}{L^{(2)}} + 2L^{(2)}\,\frac{\sin 2n\omega_o \Delta t}{2\omega_o \Delta t}\right| <<1,\ n >> 1, \tag{A.2-42f}$$

and we can drop the | |, since $L^{(2,2)}-2L^{(2)^2} = 2\,\text{var}_o \ell^2 (>0)$. As required (and expected), $F_{n\to\infty}$ is effectively independent of sample-size (n). Finally, both Q_n, R_n are $O(n^\lambda,\ 0 \leq \lambda < 1)$ when the input signal structure is partially incoherent; $\lambda=0$ usually.

A.2-3: Binary Signal Detection: Optimum Coherent Detection:

Here we extend the analysis above for "on-off" operation [Secs. A.2-1,2] to the important cases of (optimum) binary signal detection, where the optimum algorithm is given generally by (2.15) and (A.1-7), viz:

$$g^{(21)}(\underset{\sim}{x})^*_c = \{\log \mu_{21} + \hat{B}_{nc}^{(21)*}\} + \underset{\sim}{\tilde{y}}\{\langle\underset{\sim}{\theta}^{(2)}\rangle - \langle\underset{\sim}{\theta}^{(1)}\rangle\}, \tag{A.2-43}$$

where

$$\underset{\sim}{\tilde{y}}\{\overline{\underset{\sim}{\theta}^{(2)}} - \overline{\underset{\sim}{\theta}^{(1)}}\} = \sum_{i=1}^{n} \ell_i \{\langle a_{oi}^{(2)} s_i^{(2)}\rangle - \langle a_{oi}^{(1)} s_i^{(1)}\rangle\}\ ;\quad \underset{\sim}{\theta}^{(1,2)} = \{a_{oi} s_i\}^{(1,2)} \tag{A.2-43a}$$

$$\hat{B}_{nc}^{(21)*} = \frac{1}{2!}\Big\langle \underset{\sim}{\tilde{y}}\{\underset{\sim}{\rho}_\theta^{(2)} - \underset{\sim}{\rho}_\theta^{(1)} - (\langle\underset{\sim}{\tilde{\theta}}\rangle^{(2)}\langle\underset{\sim}{\theta}^{(2)}\rangle - \langle\underset{\sim}{\tilde{\theta}}^{(1)}\rangle\langle\underset{\sim}{\theta}^{(1)}\rangle)\}\underset{\sim}{y}$$

$$+ \langle\underset{\sim}{\theta}^{(2)} z \underset{\sim}{\theta}^{(2)}\rangle - \langle\underset{\sim}{\theta}^{(1)} z \underset{\sim}{\theta}^{(1)}\rangle\Big\rangle_{H_o}, \tag{A.2-43b}$$

cf. (2.14), with $\underset{\sim}{\rho}_\theta^{(1,2)} \equiv \langle a_{oi}a_{oj}s_is_j\rangle^{(1,2)}$, etc., and $\underset{\sim}{y}$, $\underset{\approx}{z}$ given by A.1-7, A.1-9 etc. Note that the bias is obtained once more by taking the average $\langle\ \rangle_{H_o}$ with respect to H_o: N alone, since these binary detectors are the difference of a pair of "on-off" detectors, cf. (2.14).

Specializing again to independent (noise) sampling, according to (A.1-10) et seq., we get directly [cf. (4.3)]

$$g_c^{(21)*} = \log \mu_{21} + \hat{B}_{nc}^{(21)*} - \sum_i^n \ell_i[\langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle]; \qquad \overline{\Delta\theta}_i \equiv \langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle. \tag{A.2-44}$$

Our main problem now is to obtain the bias $\hat{B}_{nc}^{(21)*}$. Having already obtained the bias in the "on-off" cases, cf. (A.1-16) above, we invoke the fact that these binary algorithms are the difference of two "on-off" algorithms, cf. (A.2-43,44), and (2.13)-(2.17) to get directly

$$\hat{B}_{nc}^{(21)*} = -\frac{1}{2}\{\sum_{i=1}^n L_i^{(2)}[\langle\theta_i^{(2)}\rangle^2 - \langle\theta_i^{(1)}\rangle^2]\}. \tag{A.2-45}$$

(This may also be obtained using (A.1-13)-(A.1-15) directly on (A.2-43b).) Thus, the LOBD for coherent reception in the binary cases (independent noise sampling) can be written explicitly

$$g_c^{(21)*} = \log \mu_{21} - \frac{1}{2}\{\sum_i^n L_i^{(2)}[\langle a_{oi}^{(2)}s_i^{(2)}\rangle^2 - \langle a_{oi}^{(1)}s_i^{(1)}\rangle^2]\} - \sum_i^n \ell_i[\langle a_{oi}^{(2)}s_i^{(2)}\rangle - \langle a_{oi}^{(1)}s_i^{(1)}\rangle]\,. \tag{A.2-45a}$$

Note that with gauss noise (A.2-45a) reduces to, cf. (A.1-22):

$$g_c^{(21)*}\Big|_{gauss} = \{\log \mu_{21} - \frac{1}{2}\sum_i^n [\langle\theta_i^{(2)}\rangle^2 - \langle\theta_i^{(1)}\rangle^2]\} - \sum_i^n [\langle\theta_i^{(2)}\rangle - \langle\theta_i^{(1)}\rangle]x_i, \tag{A.2-46}$$

which shows, as expected, that the (cross-) correlator is now the LOBD once more, with a weighting and bias appropriately structured for these binary cases, cf. (A.1-24).

Our next problem is to obtain the means and variances of $g_c^{(21)*}$, now under (H_2, H_1) respectively, in place of (H_1, H_0) for the "on-off" situations. We take direct advantage of our preceding results in Sec. A.2-1 for the average H_1 and appropriately apply it $g_c^{(21)*}$, (A.2-45a), changing H_1 to H_2 as demanded. First, we see that (A.2-1) becomes now

$$\begin{aligned}\langle g_c^{(21)*}\rangle_{2,1:\theta} &= \Big\langle \int_{-\infty}^{\infty} \prod_i^n w_1(x_i - \theta_i^{(2),(1)})_N g_c^{(21)*}(\underset{\sim}{x})d\underset{\sim}{x}\Big\rangle_{H_2,H_1:\theta} \\ &= B_c^{(21)*} - \sum_i^n \langle\Delta\theta_i\rangle\Big\langle\int_{-\infty}^{\infty} \ell(x_i)w_1(x_i-\theta_i^{(2),(1)})_N dx_i\Big\rangle_{2,1:\theta}\end{aligned} \tag{A.2-47}$$

for H_2, or H_1 averages (over $\theta^{(2)}$, $\theta^{(1)}$ respectively). Comparing (A.2-47), (A.2-1), and (A.2-3), we have at once

$$\begin{aligned}\langle g_c^{(21)*}\rangle_{2,1:\theta} &\doteq B_c^{(21)*} + \sum_i^n \langle\Delta\theta_i\rangle\langle\theta_i^{(2),(1)}\rangle L_i^{(2)} + O(\overline{\theta^4}) \\ &= \log \mu_{21} \pm \frac{1}{2}\sum_{i=1}^n \langle\Delta\theta_i\rangle^2 L_i^{(2)} + O(\overline{\theta^4})\ ;\end{aligned} \tag{A.2-48}$$

where (+,-) refer respectively to the (H_2, H_1) averages, and we have used (A.2-45).

The second moment is obtained in the same way. We have

$$\left\langle (g_c^{(21)*})^2 \right\rangle_{2,1:\theta} = \left\langle (B_c^{(21)*})^2 - 2B_c^{(21)*} \sum_i^n \langle \Delta\theta_i \rangle \ell_i \right.$$

$$\left. + \sum_{ij}^n \langle \Delta\theta_i \rangle \langle \Delta\theta \rangle_j \ \ell_i \ell_j \right\rangle_{2,1:\theta} \qquad \text{(A.2-49)}$$

Here, as before, $\langle \ell_i \rangle_{2,1:\theta} = -\langle \theta_i^{(2),(1)} \rangle L_i^{(2)}$. For the moments $\langle \ell_i \ell_j \rangle_{2,1:\theta}$ we simply parallel the analysis of (A.2-6)-(A.2-11) to get finally

$$(\sigma_{oc-2,1}^{(21)*})^2 \equiv \mathrm{var}_{2,1:\theta}\ g_c^{(21)*}$$

$$= \sum_{ij}^n \langle \Delta\theta_i \rangle \langle \Delta\theta \rangle_j \ \{ \langle \ell_i \ell_j \rangle_{2,1:\theta} - \langle \ell_i \rangle_{2,1:\theta} \langle \ell_j \rangle_{2,1:\theta} \} \qquad \text{(A.2-50)}$$

$$= \sum_i^n \langle \Delta\theta_i \rangle^2 \{ L_i^{(2)} + \langle (\theta_i^{(2),(1)})^2 \rangle \tfrac{1}{2} \cdot L_i^{(2,2)} + \ldots - \langle \theta_i^{(2),(1)} \rangle^2 L_i^{(2)^2} \}$$

$$+ \sum_{ij}{}' \langle \Delta\theta_i \rangle \langle \Delta\theta_j \rangle \{ \langle \theta_i^{(2),(1)} \theta_j^{(2),(1)} \rangle L_i^{(2)} L_j^{(2)} + \ldots$$

$$- \langle \theta_i^{(2),(1)} \rangle \langle \theta_j^{(2),(1)} \rangle L_i^{(2)} L_j^{(2)} \} \ . \qquad \text{(A.2-50a)}$$

Note that the leading term of $(\sigma_{c-2,1}^{(21)*})^2$ is independent of the particular hypothesis state H_2, or H_1. More important, we see that

$$\boxed{\sigma^{(21)*2}_{oc-2,1} \doteq -2B^{(21)*}_{ncoh}} \qquad \text{(A.2-50b)}$$

as is evident from (A.2-45) and the leading term $(O(\theta^2)$ in (A.2-50).

In the stationary cases where $a_o^{(1)} = a_o^{(2)} = a_o$, and $\bar{s}_i^{(1)} \neq \bar{s}_i^{(2)}$ because, of course, we must have <u>different</u> signals in order to convey information), we see that (A.2-50a) reduces to the conditions

<u>$(\sigma^{(21)*}_{oc-2})^2 \doteq (\sigma^{(21)*}_{oc-1})^2$:</u>

$$\Big| \{\overline{a_o^2}\, L^{(2,2)}/2L^{(2)} - \bar{a}_o^2 L^{(2)}\} \sum_i^n (\bar{s}_i^{(2)} - \bar{s}_i^{(1)})^2 \bar{s}_i^{(2),(1)}$$

$$+ L^{(2)} \sum_{ij}^{n}{}' (\bar{s}_i^{(2)} - \bar{s}_i^{(1)})(\bar{s}_j^{(2)} - \bar{s}_j^{(1)}) \bar{s}_i^{(2),(1)} \bar{s}_j^{(2),(1)} (\overline{a_{oi}a_{oj}} - \bar{a}_o^2) \Big|$$

$$<< \sum_i^n (\bar{s}_i^{(2)} - \bar{s}_i^{(1)})^2 , \qquad \text{(A.2-50b)}$$

for $s^{(2)}, s^{(1)}$, respectively, which are to be compared with (A.2-15b) earlier. Clearly, there is dependence on sample size (n) and on the statistics of the signal amplitude (a_o). Thus, for slow, rapid, and no fading we get directly the following simplified conditions (for each $s^{(2)}$, $s^{(1)}$):

(i). <u>slow-fading:</u>

Equations (A.2-50b), with $\overline{a_{oi}a_{oj}} - \bar{a}_o^2 \to \overline{a_o^2} - \bar{a}_o^2 = \text{var } a_o$; (A.2-50c) (ii). rapid fading: $\overline{a_{oi}a_{oj}} = \bar{a}_o^2(1-\delta_{ij})$:

$$\frac{\left|\{\overline{a_o^2}L^{(2,2)}/2L^{(2)}-\bar{a}_o^2L^{(2)}\}\sum_i^n(\bar{s}_i^{(2)}-\bar{s}_i^{(1)})^2\bar{s}_i^{(2),(1)}\right|}{\sum_i^n(\bar{s}_i^{(2)}-\bar{s}_i^{(1)})^2} \ll 1; \tag{A.2-50d}$$

(iii). no fading: $\overline{a^2}_o = \bar{a}_o^2 = a_o^2$:

$$\frac{\left|a_o^2\{L^{(2,2)}/2L^{(2)}-L^{(2)}\}\sum_i^n(\bar{s}_i^{(2)}-\bar{s}_i^{(1)})^2\bar{s}_i^{(2),(1)}\right|}{\sum_i^n(\bar{s}_i^{(2)}-\bar{s}_i^{(1)})^2} \ll 1\ , \tag{A.2-50e}$$

which are to be compared with (A.2-15c-e).

The extension of the consistency condition here [cf. Sec. (2.4) and (A.2-15a) et seq.] to the bias associated with these threshold binary cases, which puts one condition on how large the input signals (θ_2,θ_1) can be [cf. Sec. 6.4 also], requires that $(\sigma_{c-2,1}^{(21)*})^2$ be invariant of the hypothesis states H_1, H_2. Accordingly, the higher-order terms in (A.2-50a) must be suitably small vis-à-vis the leading term. This gives a pair of joint conditions on $\langle\theta_i^{(2),(1)}\rangle$ now, viz.

$$\left|\sum_i^n\langle\Delta\theta_i\rangle^2\{\langle(\theta_i^{(2),(1)})^2\rangle\frac{L_i^{(2,2)}}{2}-\langle\theta_i^{(2),(1)}\rangle^2L_i^{(2)^2}\}\right| \ll (\sigma_{oc}^*)^2$$

$$\equiv \sum_i^n\langle\Delta\theta_i\rangle^2L_i^{(2)}, \tag{A.2-51}$$

where we have used the strict (no fading) coherence condition of reception $\langle\theta_i\theta_j\rangle = \langle\theta_i\rangle\langle\theta_j\rangle$, which eliminates the $\sum_{ij}'$ terms in (A.2-50a). Similarly, $\langle\theta_i^2\rangle = \langle\theta_i\rangle^2$, and so (A.2-51) is modified with the help of (A.2-16) to the condition

$(\sigma_{c-2,1}^{(21)*})^2 \doteq (\sigma_{oc}^{*})^2$:

$$\boxed{\sum_i^n \langle\Delta\theta_i\rangle^2 \langle\theta_i^{(2),(1)}\rangle^2 \mathrm{var}_o \ell_i^2 / \sum_i^n \langle\Delta\theta_i\rangle^2 \mathrm{var}_o \ell_i << 1} \quad (1,2). \tag{A.2-51a}$$

With stationary régimes, $L_i^{(2)} = L^{(2)}$, etc., $\langle\theta_i\rangle^{(1,2)} = (\bar{a}_o\ \bar{s})^{(1,2)}$ all i, so that (A.2-51a) reduces further to

$$(\bar{a}_o^{(2),(1)} \bar{s}^{(2),(1)})^2 [\mathrm{var}_o \ell^2 / \mathrm{var}_o \ell] << 1, \quad \bar{a}_o, \bar{s} > 0, \tag{A.2-51b}$$

which not too surprisingly is just our earlier condition (A.2-17a), now for each input signal separately. Equation (A.2-51b) is independent of sample size (n).

A.2-4: Binary Signal Detection: Optimum Incoherent Detection:

We may proceed as above, now for optimum incoherent threshold detection of binary signals. The optimum algorithm is given by (4.5), where now the bias is found most simply by again observing that detector structure here is the difference of two "on-off" types of incoherent algorithm. Accordingly, the binary LOBD is now (for independent noise samples)

$$\boxed{g_{inc}^{(21)*} = \log \mu_{21} + \hat{B}_{inc}^{(21)*} + \frac{1}{2!} \sum_{ij}^n \Delta\rho_{ij}^{(21)} (\ell_i \ell_j + \ell_i' \delta_{ij})} \ , \tag{A.2-52}$$

[cf. (2.16) for dependent samples], where specifically

$$\Delta\rho_{ij}^{(21)} \equiv \left\langle a_{oi}^{(2)} a_{oj}^{(2)} s_i^{(2)} s_j^{(2)} \right\rangle - \left\langle a_{oi}^{(1)} a_{oj}^{(1)} s_i^{(1)} s_j^{(1)} \right\rangle$$

$$= \left\langle a_o^{(2)^2} \right\rangle m_{ij}^{(2)} \rho_{ij}^{(2)} - \left\langle a_o^{(1)^2} \right\rangle m_{ij}^{(1)} \rho_{ij}^{(1)} \quad ; \qquad \text{(A.2-52a)}$$

$$\therefore \hat{B}_{inc}^{(21)*} = -\frac{1}{8} \{ \sum_{ij}^{n} [(L_i^{(4)} - 2L_i^{(2)^2}) \delta_{ij} + 2L_i^{(2)} L_j^{(2)}] \}$$

$$\cdot [\left\langle a_o^{(2)^2} \right\rangle m_{ij}^{(2)} \rho_{ij}^{(2)} - \left\langle a_o^{(1)^2} \right\rangle m_{ij}^{(1)} \rho_{ij}^{(1)}] \quad . \qquad \text{(A.2-52b)}$$

The reduction of the LOBD (A.2-52) when the noise is gaussian is immediate: $\ell_i = -x_i$, $\ell_i' = -1$ and $L_i^{(2)} = 1$; $L_i^{(4)} = 2$, cf. Sec. A.1-3. We get

$$g_{inc}^{(21)*}\Big|_{gauss} = [\log \mu_{21} - \frac{1}{2} \sum_i^n \{ \left\langle \theta_i^{(2)^2} \right\rangle - \left\langle \theta_i^{(1)^2} \right\rangle \} - \frac{1}{4} \sum_{ij} \{ \left\langle \theta_i^{(2)} \theta_j^{(2)} \right\rangle^2$$

$$- \left\langle \theta_i^{(1)} \theta_j^{(1)} \right\rangle^2 \}] + \frac{1}{2!} \sum_{ij}^{n} x_i x_j \, \Delta\theta_{ij}^{(21)} \quad , \qquad \text{(A.2-53)}$$

cf. (4.11), (4.12), as required.

We make the same kind of modifications of the results of Sec. A.2-2 here, for the incoherent binary cases, as we did above in the coherent cases. We find directly that the means (under H_2, H_1) become

$$\left\langle g_{inc}^{(21)*}\right\rangle_{2,1:\theta} = \log \mu_{21}+\hat{B}_{inc}^{(21)*} + \frac{1}{4}\sum_{ij}^{n} \Delta\rho_{ij}^{(21)}\left\langle\theta_i^{(2),(1)}\theta_j^{(2),(1)}\right\rangle$$

$$\cdot[(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}]+O(\overline{\theta^6}) \ , \qquad \text{(A.2-54a)}$$

i.e.

$$\left\langle g_{inc}^{(21)*}\right\rangle_{2,1:\theta} = \log \mu_{21}\pm \frac{1}{8}\sum_{ij}^{n}(\Delta\rho_{ij}^{(21)})^2$$

$$\cdot[L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}] + O(\overline{\theta^6}) \ , \qquad \text{(A.2-54b)}$$

where (+) refers to the H_2 and (-) to the H_1 averages.

We proceed similarly for $\text{var}_{2,1:\theta}g_{inc}^{(21)*}$. Using (A.2-52) we see that Equation (A.2-26) is now modified to

$$\text{var}_{2,1:\theta}g_{inc}^{(21)*} = \frac{1}{4}\sum_{ijk\ell}^{n}\{\left\langle F_{ij}^{(21)}F_{k\ell}^{(21)}\right\rangle_{2,1:\theta}-\left\langle F_{ij}^{(21)}\right\rangle_{2,1:\theta}\left\langle F_{k\ell}^{(21)}\right\rangle_{2,1:\theta}\} \ , \qquad \text{(A.2-55)}$$

where

$$F_{ij}^{(21)} \equiv F(x_i,x_j|\Delta\rho_{ij}^{(21)}) \equiv (\ell_i\ell_j+\ell_i'\delta_{ij})\Delta\rho_{ij}^{(21)} \ . \qquad \text{(A.2-55a)}$$

By inspection, from (A.2-27)-(A.2-29) we get

$$(\sigma_{inc-2,1}^{(21)*})^2 \equiv \text{var}_{2,1:\theta}g_{inc}^{(21)*} \doteq \frac{1}{4}\sum_{ij}^{n}(\Delta\rho_{ij}^{(21)})^2$$

$$\cdot[(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}] \equiv \sigma_{o-inc}^{(21)*} = -2\hat{B}_{inc}^{(21)*} . \qquad \text{(A.2-56)}$$

The condition on the smallness of $\theta^{(2)}$, $\theta^{(1)}$, i.e., the "consistency condition" on the bias, here becomes from the appropriate extension of (A.2-41), (A.2-42):

$\underline{\sigma_{inc-2}^{(21)*^2} \doteq \sigma_{inc-1}^{(21)*^2}}$:

$$\Big|\sum_{ij}^{n} \Delta\rho_{ij}^{(21)^2}\left\langle\theta_i^{(2),(1)}\theta_j^{(2),(1)}\right\rangle\{(\frac{L_i^{(6)}}{2})\delta_{ij}+6L_i^{(2)}L_j^{(2,2)}\}$$

$$+\sum_{ijk}^{(i\neq j\neq k)}[4\Delta\rho_{ij}^{(21)}\Delta\rho_{jk}^{(21)}\left\langle\theta_k^{(2),(1)}\theta_i^{(2),(1)}\right\rangle$$

$$-2\,\Delta\rho_{ii}^{(21)}\Delta\rho_{jk}^{(21)}\left\langle\theta_k^{(2),(1)}\theta_i^{(2),(1)}\right\rangle]L_i^{(2)}L_j^{(2)}L_k^{(2)}$$

$$<<\sum_{ij}^{n}\Delta\rho_{ij}^{(21)^2}[(L_i^{(4)}-2L_i^{(2)^2})\delta_{ij}+2L_i^{(2)}L_j^{(2)}]\,. \qquad \text{(A.2-57)}$$

This is to be compared with (A.2-50), (A.2-51) for the coherent cases; it is considerably more complex, which is not unexpected in view of the considerably greater complexity of the incoherent detection algorithm (A.2-50) vis-à-vis the coherent algorithm (A.2-45a).

In the case of narrowband signals, with slow fading (i.e. $m_{ij}=1$, etc.) and stationary noise, cf. (A.2-41a), we find that the condition (A.2-57) now reduces to

$$\left\{\begin{matrix}\left\langle a_o^{(2)^2}\right\rangle \\ \left\langle a_o^{(1)^2}\right\rangle\end{matrix}\right\} F_n^{(21)}(L^{(2)},\ldots | Q_n^{(21)}, R_n^{(21)})^{-1} \ll 1 \quad : \qquad \text{(A.2-58)}$$

$$\sigma_{inc-2}^{(21)*^2} \doteq \sigma_{inc-1}^{(21)*^2} \equiv (\sigma_{o-inc}^{(21)*})^2 ,$$

where specifically

$$F_n^{(21)} \equiv \frac{L^{(4)}+2L^{(2)^2}(Q_n^{(21)}-1)}{\left|\frac{L^{(6)}}{2} +6L^{(2)}L^{(2,2)}Q_n^{(21)}+L^{(2)^3}R_n^{(21)}\right|} , \qquad \text{(A.2-59)}$$

in which

$$Q_n^{(21)}-1 \equiv \frac{1}{n}\sum_{ij}^{n'} \frac{\{\langle a_o^{(2)^2}\rangle\rho_{ij}^{(2)}-\langle a_o^{(1)^2}\rangle\rho_{ij}^{(1)}\}^2}{\{\langle a_o^{(2)^2}\rangle-\langle a_o^{(1)^2}\rangle\}^2} \quad (\geq 0), \qquad \text{(A.2-60a)}$$

$$R_n^{(21)} \equiv \frac{1}{n}\sum_{ijk}^{'''} \frac{\{4\Delta\rho_{ij}^{(21)}\Delta\rho_{jk}^{(21)}\rho_{ki}^{(2)\text{ or }(1)}-2\Delta\rho_{ii}^{(21)}\Delta\rho_{jk}^{(21)}\rho_{ki}^{(2)\text{or}(1)}\}}{\{\langle a_o^{(2)^2}\rangle-\langle a_o^{(1)^2}\rangle\}^2} , \qquad \text{(A.2-60b)}$$

where

$$\Delta\rho_{ij}^{(21)} \equiv \langle a_o^{(2)^2}\rangle\rho_{ij}^{(2)} - \langle a_o^{(1)^2}\rangle\rho_{ij}^{(1)} , \text{ cf. (A.2-52a)}. \qquad \text{(A.2-60)}$$

[In the most general cases, $m_{ij}\neq 1$, $L^{(2)}\to L_i^{(2)}$,etc., we use (A.2-57) directly, remembering that $\Delta\rho_{ij}^{(21)}$ is given by (A.2-52a).]

In the important special cases of symmetric channels, where $\mu=1$ and where $a_o^{(2)} = a_o^{(1)} = a_o$, (A.2-58)-(A.2-60) are modified to

$$\boxed{\langle a_o^2\rangle F_n^{(21)^{-1}} \ll 1} : \begin{cases} Q_n^{(21)} - 1 = \frac{1}{n}\sum_{ij}^{n'} \{\rho_{ij}^{(2)} - \rho_{ij}^{(1)}\}^2 \cong n(\gg 1) ; & \text{(A.2-61a)} \\ \qquad\qquad \text{cf. (A.6-5c).} & \\ R_n^{(21)} = \frac{1}{n}\sum_{ijk}^{'''} [4(\rho_{ij}^{(2)} - \rho_{ij}^{(1)})(\rho_{jk}^{(2)} - \rho_{jk}^{(1)})\rho_{ki}^{(2)\text{or}(1)} \doteq 0, & \text{(A.2-61b)} \end{cases}$$

since $\rho_{ij}^{(2)} \doteq \rho_{ij}^{(1)}$, over the sums with proper choice of Δt, viz. $\Delta\pi k\Delta t' = \omega_{o2} - \omega_{o1}$. For example, for signals with an entirely coherent structure, e.g. $\rho_{ij}^{(2)} = \cos\omega_{o2}(t_i - t_j)$, etc., and the proper choice of Δt, in $[t_i = i\Delta t]$, etc., $Q_n^{(21)} \doteq 1$, $R_n^{(2)} = 0$, and $F_n^{(21)}$, (A.2-59), becomes

$$\left.\begin{array}{l}\underline{\text{coh. signals:}}\\ \underline{\text{incoh.signals:}}\end{array}\right\} F_n^{(21)} \doteq \frac{L^4}{\left|\frac{L^{(6)}}{2} + 6L^{(2)}L^{(2,2)}\right|} , \qquad \text{(A.2-62)}$$

cf. (A.2-42b). For other choices of Δt (vis-à-vis $\omega_{o2} - \omega_{o1}$) we have $Q_n^{(21)} = O(n^o)$, $R_n^{(21)} = O(n^o)$, so that the complete relation (A.2-59) is required for $F_n^{(21)}$. Equation (A.2-62) also applies for signals with an entirely incoherent structure, e.g. $\rho_{ij}^{(1),(2)} = \delta_{ij}$, regardless of the symmetry of the channel, as we can see directly from (A.2-60a,b) in (A.2-59).

APPENDIX A-3

The Optimal Character of the LOBD:

In this Appendix we demonstrate that the canonical LOBD's derived in this study [cf. Sec. 2.2] and by Middleton earlier, in 1966 [14], and recently [34], cf. also [1], [1a], are indeed optimum for small (but nonzero input signals) and all sample sizes (particularly for large samples, $n \gg 1$, and in the limit $n \to \infty$). This is in contrast to the conventionally defined locally optimum detectors, whose optimal character is limited to small-sample conditions. The practical as well as analytic superiority of these LOBD algorithms stems from the addition of a suitable "bias" term and the associated condition, consistent with the way the bias term is derived, that the variances of the test statistic (g^*) under (H_0,H_1) be the same (and similarly under (H_1,H_2) for binary signal reception). This equality of variances, in turn, insures that the input signal be suitably small but nonvanishing, essentially independent of sample-size as $n \to \infty$, under conditions readily achieved in practice.

The LOBD is not unique: there may be other algorithms which give the same optimal performance [cf. Sec. A.3-4], but most such are structurally (i.e. operationally) more complex, or converge more slowly to the limiting "global" optimum, or both. The LOBD is canonical (i.e. exhibits an invariant form) vis-à-vis both signal and noise statistics and structures. In fact, the LOBD is determined by the appropriate pdf of the interference and by the lower-order moments of the input signal, and in this fashion is different in some important respects from the Asymptotically Optimum Detectors (AOD's) developed recently (1976) by Levin [39] and his colleagues (1967-), [25]-[28], as we shall see below.

A.3-1. Introductory Remarks:

Conventional locally optimum detectors (LOD's) are defined by the term linear in the signal parameter (θ), in the expansion of the [globally optimum] likelihood ratio $\Lambda_n(\underset{\sim}{x};\theta)$ ($\equiv \mu\langle F_n(\underset{\sim}{x}|\theta)\rangle_\theta / F_n(\underset{\sim}{x}|0)$), or its logarithm $\log \Lambda_n(\underset{\sim}{x};\theta)$, about the null signal state $\theta = 0$, viz:

$$\log \Lambda_n^{(\ell_o)} \equiv \frac{\partial}{\partial\theta} \log \Lambda_n(\underset{\sim}{x};\theta)\Big|_{\theta=0} , \tag{A.3-1}$$

where the decision that a signal is present, (H_1) vs. noise alone (H_o), is made when

$$H_1 : \log \Lambda_n^{(\ell_o)} \geq K \quad ; \quad H_o : \log \Lambda_n^{(\ell_o)} < K , \tag{A.3-2}$$

with K some appropriately chosen threshold. This threshold is usually determined by the false alarm probability α_F, e.g.

$$P_1(\log \Lambda_n^{(\ell_o)} \geq K|H_o) = \alpha_F . \tag{A.3-3}$$

The detection algorithm based on (A.3-1) is called locally optimum (or "ϵ-optimum") if it gives the minimum missed-signal probability $\beta_n^{(\ell_o)}(\theta)$ for all values of θ in some finite range $(0 < \theta < \epsilon)$ for specified $\alpha = \alpha_F$. In the usual cases ϵ is taken to be small, so that local optimality applies to those cases where the input signal is small and sample-size (n) is finite. In this situation (i.e. local optimality) it is required that

$$\frac{\partial\beta_n^*(\delta_n^*)}{\partial\theta}\Big|_{\theta=0} = \frac{\partial\beta_n^{(\ell_o)}(\delta_n^{(\ell_o)})}{\partial\theta}\Big|_{\theta=0} \tag{A.3-4}$$

where δ_n^*, $\delta_n^{(\ell_o)}$ are respectively the decision rules for the strictly (or "globally" i.e. all signal levels) optimum and locally optimum algorithms.

Similarly, the locally optimum approach is extended to the more general Bayes decision formulation by replacing log $\Lambda_n^{(\ell_o)}$ by the conditional risk $r(\theta,\delta_n)$, so that if δ_n^* is a Bayes rule (i.e. one minimizing the conditional risk), then the locally optimum Bayes rule $(\sigma_n^{(\ell_o)})$ is determined from the conditions (obtained on expanding $r(\theta,\delta_n)$ about $\theta=0$):

$$r^{(\ell_o)}(0,\delta_n^{(\ell_o)}) = r^*(0,\delta_n^*) \; ; \quad \left.\frac{\partial r^{(\ell_o)}(\theta,\delta_n^{(\ell_o)})}{\partial\theta}\right|_{\theta=0} = \left.\frac{\partial r^*(\theta,\delta_n^*)}{\partial\theta}\right|_{\theta=0}, \tag{A.3-5}$$

where the decision that a signal is present is again made on the basis of the inequality (A.3-2), where now K depends on the various cost assignments. A more general Bayes formulation, based on the minimization of average risk, employs the same approach, with r^*, $r^{(\ell_o)}$ replaced by the average risks R^*, $R^{(\ell_o)}$ in (A.3-5), and K dependent not only on the cost assignments but also on the a priori probabilities associated with the signal and its presence or absence (p,q) in the data sample. See, for example, Sec. II of [14].

A critical problem with the conventional LOD's is that higher order terms in the expansion of log $\Lambda_n(\underset{\sim}{x};\theta)$ about $\theta=0$ can be discarded for weak input signals only if the sample size (n) is small. This is easily seen from the following argument: for the m^{th}-order term in the expansion, one has a contribution $(\theta^m/m!)O(n^m) = O([\theta n]^m/m!)$. Thus, for terms $m \geq 2$ to be discarded vis-à-vis m=1, for instance, one requires $\theta n \gg (\theta n)^2/2!$, or $\theta n \ll 1$ essentially. Even for small input signals [$\theta=O(10^{-3}$ or less], n must also be comparatively small, say n=20, to satisfy the inequality $\theta n \ll 1$. [Clearly, if the m^{th}-order inequality is satisfied, so also will all m+1, etc.] But for this situation the correct-signal detection probability, $p_D^{(\ell_o)}=1-\beta^{(\ell_o)}$, is the same order as $\alpha^{(\ell_o)} = \alpha_F$. Then, in order to achieve a correct detection probability $p_D^{(\ell_o)}$ which is close to unity for weak signals, it is necessary to increase the sample size (n) by a suitable amount. This inevitably brings in higher order terms (beyond the linear one in θ), which now cannot be ignored if optimal performance is to be maintained. These more complex algorithms are no longer close to the LOD's, either in structure or performance, nor can they be made so generally.

Accordingly, we must seek an appropriate modification, and extension, of the locally optimal (i.e., weak-signal) detection concept, which preserves the comparatively simple structure embodied in (A.3-1) and which at the same time permits the use of large (and ultimately very large ($n \to \infty$)) samples, which are required in practice for detecting weak signals. This must be done without destroying the optimal nature of the algorithm itself. As we shall see below subsequently, the canonical LOBD algorithms derived by Middleton [14] in 1966 for optimum threshold signal detection, under some simple conditions, do indeed provide such desired extensions and generalization. We emphasize that we are considering here fully canonical developments, whose general form [cf. (2.9),(2.11), (2.12)] is invariant of the particular waveform and statistical structures of both the signal and noise.

A.3-2 Asymptotically Optimum Signal Detection Algorithms (AODA's): General Remarks:

To develop the desired LOBD algorithms, which are to remain locally optimum for all sample sizes, with suitably small but nonzero input signals, we shall parallel the recent approach of Levin [39] and his colleagues [25]-[28] and employ the concept of an Asymptotically Optimum Detection Algorithm (AODA). This, however, unlike the AODA's used by Levin [39], is modified to admit nonvanishing input signals (as $n \to \infty$) and hence to provide consistency, (i.e. $\beta_n^* \to 0, n \to \infty$) of the LOBD algorithm, as well.

One class of asymptotically optimum detection algorithm (AODA) for signals in a general noise background is one for which structure and performance approach that of the appropriate (strictly) optimum algorithm for fixed (non-zero) error probabilities $[\alpha^{(\ell_0)}, \beta^{(\ell_0)}]$, as sample size

(n) becomes infinitely large and the input signal-to-noise ratio approaches zero. This is the class considered by Levin [39]. Another, related class of AODA is that for which structure and performance again approach that of the corresponding (strictly) optimum algorithm, but now for error probabilities [β^*, or $q\alpha^*+p\beta^*$, etc.] which vanish as sample-size becomes infinitely great <u>and</u> at the same time the input signal-to-noise ratio remains <u>non-zero</u>, although necessarily small. It is this latter class of AODA which we consider here, and to which the LOBD belongs, as we shall demonstrate.

The principal idea on which the theory of asymptotically optimum detection algorithms is based is to find an <u>asymptotically sufficient statistic</u> in the sense that its distribution converges in probability to a normal distribution when the sample size (n) increases without limit and the input signal amplitude ($\sim a_o$) is suitably small. For the class of AODA's considered by Levin [39] the signal <u>amplitude</u> $\lambda_n a_o s(t) \to 0$, $\lambda_n \to 0$. For the class of AODA's examined here, $0 < a_o^2 << 1$: the input signal is small but never vanishingly so. In any case, the reasonable assumption is that if such asymptotically sufficient statistics are substituted for the known optimum decision rule, or are otherwise shown to be equivalent to it, for normal distributions, the result is an AODA which, as $n \to \infty$, becomes strictly optimum. The canonical character of the resulting AODA then stems from the generic form of the noise distribution alone, as expressed formally by an appropriate expansion of the (always optimum) likelihood ratio about the null-signal (H_o) condition. The explicit form of the expansion, however, is not unique, and therefore it is desireable to choose those expansions which: (i), converge rapidly to the (strict) optimum (as $n \to \infty$); and (ii), which are not excessively complex in structure.

In more precise fashion let us give a <u>definition</u> of the notion of "asymptotic optimality", for the class, $0 < a_o^2 << 1$. As an example, let us consider a detection algorithm, $\hat{\delta}_n = \delta_n^{(0)}$, to be the strictly optimum algorithm, which for fixed false-alarm probability α_n and fixed sample size (n) minimizes the missed-signal probability $\hat{\beta}_n(\delta_n; a_o(t))$, that signal $a_o s(t)$ will not be detected (the Neyman-Pearson Observer). Then, for some sequence of algorithms $\{\delta_n\}$ we denote the corresponding missed-signal probability by $\beta_n(\delta_n;a_o s)$. We next call the sequence of algorithms $\{\delta_n^{(a_o)}\}$ asymptotically optimal if for any other sequence of algorithms $\{\delta_n\}$ the relation

$$\lim_{n\to\infty} [\beta_n(\delta_n;a_o s)-\beta_n(\delta_n^{(ao)},a_o s)] \geq 0, \text{ when } \lim n\to\infty(\delta_n^{(ao)}\to\delta_n^{(0)}\to\delta_\infty^{(0)}), \tag{A.3-6}$$

is valid for fixed false alarm level $\alpha_\infty=\alpha$, where $\alpha_\infty = \lim_{n\to\infty} \alpha_n(\delta_n;0)\equiv\alpha$. In the case of the Ideal Observer, the corresponding relation is

$$\lim_{n\to\infty}\{q\alpha_n(\delta_n;a_o s)+p\beta_n(\delta_n;a_o s)-q\alpha_n(\delta_n^{(ao)},a_o s)-p\beta_n(\delta_o^{(ao)},a_o s)\} \geq 0. \tag{A.3-6a}$$

Of course, for our class of AODA's being examined here, $\beta_n(\delta_n^{(ao)},a_o s) \to 0$ as $n \to \infty$, $(a_o s>0)$, to insure the required consistency of the AODA: a necessary condition for a properly chosen sequence of algorithms $\{\delta_n^{(ao)}\}$ is that they provide a consistent test of the hypotheses states H_o (noise alone) and H_1 (signal and noise), with $\beta_n < 1 - \alpha_n$, all n.

A.3-3 The LOBD as an Asymptotically Optimum Detection Algorithm - AODA:

Here we shall show that the LOBD is an AODA, as well as being locally optimum for all sample sizes, n. [In fact, the latter follows at once from the former here, because of the convergence of the LOBD with finite n to the limiting AODA, as $n \to \infty$.]

Remembering that the (generalized) likelihood ratio $(\Lambda_n^{(1)})$, cf. (2.1), or any monotonic function of it, e.g. $\log \Lambda_n^{(1)}$ for instance, is always (strictly) optimum, for all n, including $n\to\infty$, we see that (cf. 2.2, [14]) it is entirely reasonable to seek acceptable candidates for an AODA by an approprate expansion of the (logarithm) of the likelihood ratio about the null signal (θ=0). The LOBD, g_n^*, (2.9), here for additive signal and noise,* and specifically for coherent and incoherent reception, as described by (2.11), (2.12), is one such class of expansions. Thus, we write

$$\log \ell_n^{(1)}(\underset{\sim}{x};\theta) = g_n^*(\underset{\sim}{x};\theta)+t_n(\underset{\sim}{x},\theta), \text{ with } \log \Lambda_n^{(1)}=\log \mu+\log \ell_n^{(1)}(\underset{\sim}{x};\theta), \tag{A.3-7}$$

cf. (2.1), where

$$\ell_n^{(1)}(\underset{\sim}{x};\theta) \equiv \langle F_n(\underset{\sim}{x}|\theta)\rangle_\theta / F_n(\underset{\sim}{x}|\theta) \tag{A.3-7a}$$

is also a generalized likelihood ratio, and $\hat{g}_n^*$ is the LOBD $\hat{g}_n^*=g_n^*-\log \mu$, without the a priori "bias term", log μ. Here the necessary and sufficient condition that the LOBD, $\hat{g}_n^*$ (and hence g_n^* $(=\hat{g}_n+\log \mu)$ itself) is an AODA (as $n\to\infty$), is that the "remainder", t_n, coverge to zero with respect to the sequence (n) of pdf's governing the null $(H_0|N)$ and alternative $(H_1|S+N)$ hypotheses, viz., with respect to H_1: $F_n(\underset{\sim}{x};\theta)\equiv\langle F_n(\underset{\sim}{x}|\theta)\rangle_\theta$, and H_0: $F_n(\underset{\sim}{x};0)$, as $n\to\infty$. The general problem is to find suitable expansions for which the above is true. Our particular problem here is to show that the LOBD, $\hat{g}_n^*$, is an AODA. It is clear that the LOBD $\hat{g}_n^*$ here is not a unique locally optimum or asymptotically optimum algorithm. Other, more complex structures can give equivalent results, but they can not be any better than the LOBD and its AODA form, and they suffer from the operational defect of complexity and possibly slower convergence (as $n\to\infty$) to the limiting AODA here.

Our next step is to establish specific conditions for which the "remainder" term t_n vanishes as $n\to\infty$, on H_0, H_1. For this we shall use (a limiting form of) Le Cam's theorem and his concept of the asymptotic equivalence of sequences of distributions [40],[40a]. Here, two sequences of pdf's

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* The general approach of Sec. A.3-3 is not necessarily limited to purely additive cases. The results for nonadditive cases are reserved to a later study.

$\{F_n(\underset{\sim}{x};\theta)\}$ and $\{F_n(\underset{\sim}{x};0)\}$ are termed asymptotically equivalent (AE) if the convergence in probability of any statistic (say, t_n above) to zero, i.e. $\lim_{n\to\infty} t_n \to 0$ (in prob.), for one sequence of pdf's, e.g. $\{F_n(\underset{\sim}{x};0), n\to\infty\}$ here, entails the convergence in probability of that statistic (t_n) to zero, for the other sequence, $\{F_n(\underset{\sim}{x};\theta), n\to\infty\}$. As Le Cam has shown [40a], the necessary and sufficient condition for the asymptotic equivalence (AE) of two sequences of distributions (pdf's) is

$$AE_0 \equiv \int_{-\infty}^{\infty} e^z\, w_1(z|H_0)_\infty dz = 1 \tag{A.3-8}$$

where $w_1(z|H_0)_\infty$ is thus the limiting pdf (as $n\to\infty$) for $z = \lim_{n\to\infty} \log \ell_n^{(1)}$ under hypothesis $H_0(N)$, e.g. $\theta=0$. [Note here that the sample values $\{x_i\}$ in ℓ_n need not be independent!]

For the two pdf's $F_n(\underset{\sim}{x};\theta)$ and $F_n(\underset{\sim}{x};0)$ to be asymptotically equivalent, it is sufficient that the logarithm of the likelihood ratio $\ell_n^{(1)}(\underset{\sim}{x};\theta)$, cf. (A.3-7a), be asymptotically normal (G) under hypothesis H_0, with the parameters $G_0((-\sigma_0^{*2}/2),\sigma_0^{*2})$ with $\sigma_0^{*2} = \lim_{n\to\infty} \sigma_{0n}^{*2}$, where $\sigma_{0n}^{*2}[=\mathrm{var}_0 \hat{g}_n^* = \langle \hat{g}_n^{*2}\rangle_0 - \langle \hat{g}_n^*\rangle_0^2]$ is the variance of $\hat{g}_n^*$ under H_0. Furthermore, from Le Cam's theorem it follows that if the pdf of $\log \ell_n^{(1)}$ under H_0 is asymptotically normal with $G_0(-\sigma_0^{*2}/2, \sigma_0^{*2})$, then the pdf of $\ell_n^{(1)}$ is also asymptotically normal, for the "close alternative", with the parameters $G_1(+\sigma_0^{*2}/2,\sigma_0^{*2})$. Then, if the above (sufficient) conditions on $\log \ell_n^{(1)} = \hat{g}_n^* + t_n$ are satisfied and $\hat{g}_n^*$ is the asymptotically normal form of $\log \ell_n^{(1)}$, it is at once evident that $t_n \to 0$ under H_0, H_1 and that $\hat{g}_n^*$ is an AODA. That the condition (A.3-8) is satisfied here is easily shown: if $\lim_{n\to\infty} \hat{g}_n^* \equiv z$ is $G_0(-\sigma_0^{*2}/2,\sigma_0^{*2})$, then (A.3-8) becomes

$$AE|H_0 = \int_{-\infty}^{\infty} e^z \cdot e^{-(z+\sigma_0^{*2}/2)^2/2\sigma_0^{*2}} \frac{dz}{\sqrt{2\pi\sigma_0^{*2}}} = \int_{-\infty}^{\infty} e^{-(z-\sigma_0^{*2}/2)^2/2\sigma_0^{*2}} \frac{dz}{\sqrt{2\pi\sigma_0^{*2}}}$$

$$= \int_{-\infty}^{\infty} e^{-(y-\sigma_0^*/2)^2/2} \frac{dy}{\sqrt{2\pi}} = 1\ . \tag{A.3-9}$$

[Similarly, for z to be $G_1(\sigma_o^{*2}/2, \sigma_o^{*2})$, (A.3-8) becomes under H_1, [40a]

$$AE|H_1 = \int_{-\infty}^{\infty} e^{-z} \cdot e^{-(z-\sigma_o^{*2}/2)^2/2\sigma_o^{*2}} \frac{dz}{\sqrt{2\pi\sigma_o^{*2}}} = \int_{-\infty}^{\infty} e^{-(z+\sigma_o^{*2}/2)^2/2\sigma_o^{*2}} \frac{dz}{\sqrt{2\pi\sigma_o^{*2}}}$$

$$= \int_{-\infty}^{\infty} e^{-(y+\sigma_o^{*}/2)^2/2} \frac{dy}{\sqrt{2\pi}} = 1 \text{ .]} \qquad \text{(A.3-9a)}$$

The "distance" between $\langle \hat{g}_n^* \rangle_{H_o}, \langle \hat{g}_n^* \rangle_{H_1}$ in the "close alternatives" (H_o, H_1) here is given asymptotically by

$$\lim_{n\to\infty}(\langle \hat{g}_n^* \rangle_1 - \langle \hat{g}_n^* \rangle_o) = \sigma_o^{*2}/2 - (-\sigma_o^{*2}/2) = \sigma_o^{*2} \text{ ,} \qquad \text{(A.3-10)}$$

with the normalized "distance"

$$\lim_{n\to\infty}\left\{ \frac{\langle \hat{g}_n^* \rangle_1 - \langle \hat{g}_n^* \rangle_o}{\sigma_o^*} \right\} = \sigma_o^* \text{ .} \qquad \text{(A.3-10a)}$$

We emphasize that the above results [(A.3-8) et seq.] apply for correlated samples, as well as for the independent samples $\{x_i\}$ of our detailed analysis here.

In the above we have assumed that $\sigma_o^{*2} < \infty$. The above conditions and results still apply when $\lim_{n\to\infty} \sigma_{on}^{*2} \equiv \sigma_o^{*2} \to \infty$, provided we replace the limits $(-\infty,\infty)$ in (A.3-9,9a) by $(-\infty-,\infty+)$ where $(-\infty-)$ and $(\infty+)$ are such that $\lim_{n\to\infty}(-\infty-) - a_n \leq 0-$ and $\lim_{n\to\infty}[(\infty+)-a_n] \geq 0+$, $\lim a_n \to \sigma_o^{*2}/2 (\to\infty)$. Thus, letting $z - \sigma_o^{*2}/2 = x$, we have (cf.(A.3-8) and (A.3-9))

$$AE|H_0 = \int_{(-\infty-)}^{\infty+} \lim_{n\to\infty} \frac{e^{-(z-\sigma^{*2}_{on}/2)^2/2\sigma^{*2}_{on}}}{(2\pi\sigma^{*2}_{on})^{1/2}}\, dz$$

$$= \int_{(-\infty-)}^{\geq 0+} \lim_{n\to\infty} \frac{e^{-x^2/2\sigma^{*2}_{on}}}{\sqrt{2\pi\sigma^{*2}_{on}}}\, dx = \int_{-\infty}^{\geq 0+} \delta(x-o)dx = 1, \qquad \text{(A.3-11)}$$

and similarly,

$$AE|H_1 = \int_{(-\infty-)}^{\infty+} \lim_{n\to\infty} \frac{e^{-(z+\sigma^{*2}_{on}/2)^2/2\sigma^{*2}_{on}}}{\sqrt{2\pi\sigma^{*2}_{on}}}\, dz$$

$$= \int_{\leq 0-}^{\infty+} \lim_{n\to\infty} \frac{e^{-x^2/2\sigma^{*2}_{on}}}{\sqrt{2\pi\sigma^{*2}_{on}}}\, dx = \int_{\leq 0-}^{\infty} \delta(x-o)dx = 1. \qquad \text{(A.3-11a)}$$

(For finite σ^{*2}_o these limits clearly reduce to those of (A.3-9,9a), as required.) This extension of the limits $(-\infty,\infty)$ insures that the integrand always remains within the suitably (infinite) domain of integration $(-\infty-,\infty+)$. Thus, a sufficient condition that the LOBD $\hat{g}^*_n$, when $\lim_{n\to\infty} \sigma^{*2}_{on} \to \sigma^{*2}_o \to \infty$, be an AODA is that $\hat{g}^*_n$ be asymptotically normal, with $\lim_{n\to\infty}$ (mean/variance = $-1/2{:}H_o$, $=+1/2{:}H_1$). The "distances" (A.3-10), (A.3-10a), of course, are infinite $O(\sigma^{*2}_o$ or $\sigma^*_o)$: (A.3-10a) with $\sigma^*_o \to \infty$ expresses the fact that the means under H_o, H_1 become infinitely separated, while the spread of each pdf increases less rapidly as $n\to\infty$, cf. Figure A.3-1. That the "distances" [σ^{*2}_o, or σ^*_o] are greater than zero (and greater than the spread ($\sim\sigma^*_o$) of each pdf) reflects the fact that $0 < \beta_h < 1-\alpha_n$, all $n\to\infty$, and when $\alpha^*_o \to \infty$, then $\beta_n \to \beta_\infty \to 0$: the test of H_1 is <u>consistent</u> as well as asymptotically optimum.

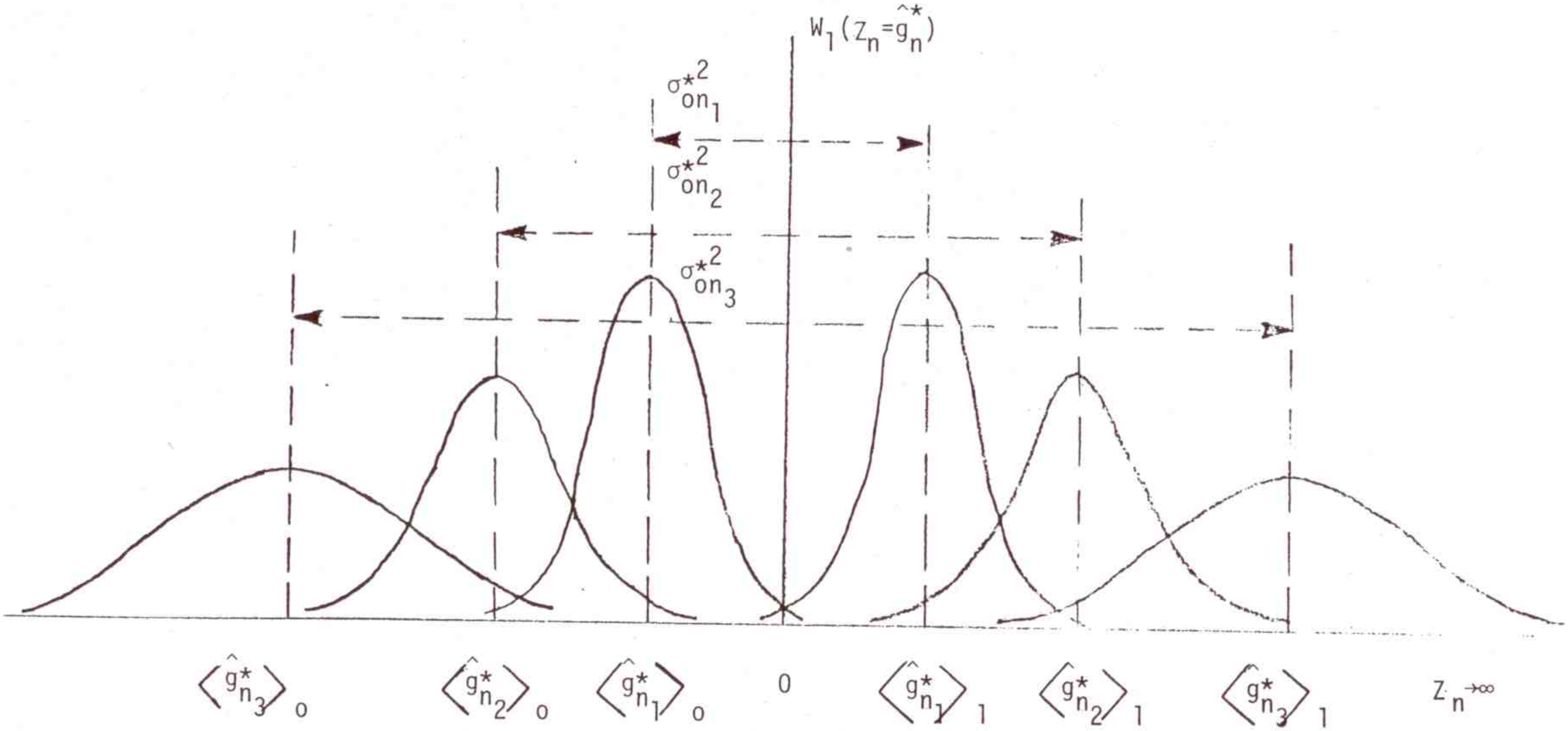

Figure A.3-1. The asymptotically normal pdf of the test statistic, $\hat{g}^*_n = Z_n$, as $n_1 < n_2 < n_3 \to \infty$.

The LOBD's here as derived by Middleton (for additive signal and noise) [14], cf. Sec. 2.2 and Appendix A.1, are clearly AODA's, as well as locally optimum (all n), when we note the results of Appendix 2 and Sec. 6. Specifically, we have $\hat{g}_n^*$ asymptotically normal $G_0(-\sigma_0^{*2}/2,\sigma_0^{*2})$, $G_1(+\sigma_0^{*2}/2,\sigma_0^{*2})$, for both coherent (2.11) and incoherent threshold reception (2.12), cf. Eqs. A.2-3,4 for the coherent means and Eqs. (A.2-14) for the coherent variances, and correspondingly, Eqs. (A.2-22b,25) for the incoherent means and Eq. (A.2-40) for the incoherent variances. [These results are summarized in Table 6.1, Sec. 6.] The full LOBD's, $g_n^* = \hat{g}_n^* + \log \mu$, are of course, AODA's also, with all the properties of $\hat{g}_n^*$: the various means under H_0, H_1 now have an added term, $\log \mu$, e.g., $G_0(-\sigma_0^{*2}/2,\sigma_0^{*2}) \to G_0(\log \mu - \sigma_0^{*}/2,\sigma_0^{*2})$, and $G_1(\sigma_0^{*2}/2,\sigma_0^{*2}) \to G_1(\log \mu + \sigma_0^{*2}/2,\sigma_0^{*2})$; the "distance" (A.3-10) etc. remains unchanged. The condition $\sigma_1^{*2} \doteq \sigma_0^{*2}$, cf. (2.29), (A.2-15), (A.2-41), etc. required for the AODA's here, in turn postulates a nonzero input signal-to-noise ratio, a_0^2 (>0), which is always suitably small, e.g. $a_0^2 \ll 1$. These LOBD's are not uniquely optimum, since it is possible that other expansions of $\log \ell_n^{(1)}$, cf. (A.1-7), may possess the desired properties, $G_{0,1}(\mp\sigma_0^{*2}/2,\sigma_0^{*2})$, in the limit. However, such other expansions usually include higher order terms (in θ) and are therefore much more complex in structure than the present LOBD's. In any case, there are no LO algorithms which are better than these LOBD's.

A.3-4 Remarks on A Comparison of Middleton's LOBD's [14] and Levin's AODA's, [39]:

There are certain distinct differences between Levin's approach [39] to the optimum threshold detection and that of Middleton [14]. The principal one is that the former is concerned with the asymptotic optimization of one type of expansion of the conditional likelihood ratio $\ell_n^{(1)}(\underset{\sim}{x}|\theta) \equiv F_n(\underset{\sim}{x}|\theta)/F_n(\underset{\sim}{x}|o)$, while the latter (Middleton) is concerned with the asymptotic optimization of the unconditional likelihood ratio $\ell_n^{(1)}(\underset{\sim}{x};\theta) \equiv \langle F_n(\underset{\sim}{x}|\theta)\rangle_\theta / F_n(\underset{\sim}{x}|o)$, cf. (A.3-7a), (2.9) etc. This may be summarized by

Levin (p. 128, [39]): log av. asymptot. expan. of conditional likelihood ratio

$$= \log\left\langle xp\ \ell^{(1)}_{asymp}(\underset{\sim}{x}|\theta)\right\rangle_\theta$$

Middleton (Sec. 2.2) : asymp expan. of log av. condit.
(Sec. A.3-3) likelihood ratio

$$= \text{asymp. xp} \log \left\langle \ell^{(1)}_n(\underset{\sim}{x}|\theta)\right\rangle_\theta . \qquad \text{(A.3-12)}$$

In the approach of Levin et al., [39], the input signal samples $a_{oi}s_i$ are replaced by a decreasing set of signal samples, $\gamma\, a_{oi}s_i/\sqrt{n}$, $(\gamma>0)$, so that the input signal vanishes in the asymptotic limit $(n\to\infty)$, and the error probabilities remain preset and nonvanishing, e.g. $0 < \beta_{n\to\infty} = \beta$; $0 < \alpha_n = \alpha$, etc., with $\beta < 1-\alpha$. This leads to finite values of σ_o^{*2} in the limit.

On the other hand, in Middleton's development (2.9) etc., which includes a proper selection of bias term, $\hat{B}_n^*(=B_n^*-\log\mu)$, the input signal-to-noise ratio always remains nonvanishing, so that $\lim_{n\to\infty} \beta_n^* \to 0$, for $\alpha^*>0$, etc. (and $\lim_{n\to\infty} q\alpha_n^* + p\beta_n^* \to 0$ in the communication examples where the Ideal Observer is appropriate). To assure the AO character of this LOBD it is required that $\sigma_1^{*2} \doteq \sigma_o^{*2}$, i.e. the variances of the LOBD under H_1 and H_o be essentially the same, which means, in turn, that $(a_o^2)_{in} << 1$, suitably, cf. (A.2-15), (A.2-41). In addition, the variance $\lim_{n\to\infty} \sigma_{on}^{*2} \to \sigma_o^{*2} \to \infty$, cf. A.3-3 above.

The two approaches above give equivalent results if we set $\beta_n^* = \beta^{(ao)}$, (>0) of Levin (in the Neyman Pearson cases, α fixed, for instance). This determines the unspecified constant, γ, and relates the various limiting parameters of Levin's approach to those in the LOBD's of Middleton. In particular, one can equate the missed-signal probabilities of detection of Sections 3.1.3-3.1.10, [39], to the corresponding results here (i.e., coherent, incoherent reception, post-detection optmization, "mismatch", etc.), and determine the corresponding values of γ.

It should be pointed out that Levin's approach is not restricted to additive signal and noise situations, only to those where the noise does not vanish when the signal does. Our present analysis can be extended to include such more general cases. Moreover, the present LOBD approach provides a natural distinction between various modes of reception (coherent, incoherent, mixed), and since $(a_0^2)_{in} > 0$ here, the useful notions of processing gain (Π^*), cf. Sec. 6. , and associated minimum detectable signal ($(\langle a_0^2\rangle^*_{min})$), likewise appear naturally. Both approaches provide a processing structure, but the LOBD structures of the present analysis appear to be the more appropriate in actual applications. [These points will be discussed in more detail in a later study.]

A.3-5. Extensions of the AODA to Binary Signals:

We may readily extend the earlier results of Sec. (A.3-3) on the AODA's for "on-off" cases to binary signal reception. Analogous to (A.3-7) we now write

$$\log \ell_n^{(21)}(\underset{\sim}{x};\theta) = \hat{g}_n^{(21)*}(\underset{\sim}{x};\theta)+t_n^{(21)}(\underset{\sim}{x},\theta), \text{ with } \log \Lambda_n^{(21)}=\log \mu_{21}+\log \ell_n^{(21)} , \tag{A.3-13}$$

where

$$\log \ell_n^{(21)} = \log\{\langle F_n(\underset{\sim}{x}|\theta_2)\rangle_2/\langle F_n(\underset{\sim}{x}|\theta_1)\rangle_1\} \equiv \log\{F_n(\underset{\sim}{x};\theta_2)/F_n(\underset{\sim}{x};\theta_1)\} \tag{A.3-13a}$$

and now $\hat{g}_n^{(21)*}$ is the LOBD for binary signal reception (coherent or incoherent) = $g_c^{(21)*}-\log \mu_{21}$.

The extension of Le Cam's theorem for asymptotic equivalence, [40a], now under H_1, H_2 (i.e. $AE_{1,2}$), of the two sequences of distributions $\{F_n(\underset{\sim}{x};\theta_2)$ $(\equiv\langle F_n(\underset{\sim}{x}|\theta_2)\rangle_2)\}$, $\{F_n(\underset{\sim}{x};\theta_1)(\equiv\langle F_n(\underset{\sim}{x}|\theta_1)\rangle_1)\}$ as $n\to\infty$, i.e. $\lim_{n\to\infty} t_n^{(21)} \to 0$ (in prob.) under H_1 and H_2 is immediate. The necessary and sufficient conditions for $AE_{1,2}$ here are

$$AE_1 \equiv \int_{-\infty}^{\infty} e^{z} w_1(z|H_1)_\infty dz = 1 \leftrightarrow AE_2 \equiv \int_{-\infty}^{\infty} e^{-z} w_1(z|H_2)dz \ , \qquad \text{(A.3-14)}$$

(i.e. AE_1 implies AE_2 and vice-versa), where $w_1(z|H_{1,2})_\infty$ are the limiting pdf's for $z(=\lim_{n\to\infty} \log \ell_n^{(21)})$ under H_1 and H_2.

For $F_n(\underset{\sim}{x};\theta_1)$, $F_n(\underset{\sim}{x};\theta_2)$ to be asymptotically equivalent, i.e. to have the "remainder term", $t_n^{(21)}$ vanish under H_1, H_2 as $n\to\infty$, it is again sufficient that: (i), $z = \lim_{n\to\infty} \ell_n^{(21)}(\underset{\sim}{x};\theta)$ be asymptotically normal under H_1, with parameters

$$G[-\frac{(\sigma_o^{(21)*})^2}{2} \ , \ (\sigma_o^{(21)*})^2] \ , \ (\sigma_o^{(21)*} = \lim_{n\to\infty} \sigma_{on}^{(21)*}) \ .$$

This also insures that (ii), $\ell_n^{(21)}(\underset{\sim}{x};\theta)$ is asymptotically normal under H_2, with parameters

$$G(+\frac{(\sigma_o^{(21)*})^2}{2} \ ,(\sigma_o^{(21)*})^2).$$

Now, from Section A.2-3,4 preceding we see that, indeed (for any sample size n)

$$\left\langle \hat{g}_{c,inc}^{(21)*} \right\rangle_{2:\theta_1} = \pm \frac{1}{2} \left(\sigma_o^{(21)*}\right)^2 \Big|_{C,inc} , \quad \left.\begin{array}{l}\text{Eqs. (A.2-48), with (A.2-50a)} \\ \text{Eqs. (A.2-54b), with (A.2-56)}\end{array}\right\} \qquad \text{(A.3-15)}$$

as required, where $(\sigma_o^{(21)*})^2$ is the appropriate variance (as $n \to \infty$) of $\hat{g}_{c,inc}^{(21)*}$. Applying the above to (A.3-14) along the lines of (A.3-11), (A.3-11a) at once shows the desired sufficiency. Thus, not unexpectedly, the LOBD's $g_{c,inc}^{(21)*}$ are AODA's here in the binary signal cases, as well as in the "on-off" detection situations examined initially. [Again, see the remarks following Eqs. (A.3-10,11).] The comments in Section A.3-4 also apply here, as well.

A.3-6 Rôle of the Bias in the AODA's: The Composite LOBD:

In the preceeding sections of Appendix A.1-A.3 we have seen that the bias, $\hat{B}_n^*$, must have the proper structure in order that the LOBD in question be an AODA as $n\to\infty$. In fact, from the sufficient conditions that the "on-off" LOBD, g_n^* ($\equiv \hat{g}_n^*+\log\mu$), be asymptotically gaussian, i.e. $\lim_{n\to\infty}$ $G(\log\mu\mp\sigma_{on}^{*2}/2,\sigma_{on}^{*2})$, where $\sigma_{on}^{*2}=\mathrm{var}_o g_n^*$, (($\mp$: H_o,H_1) in H_o, H_1 respectively, we can at once obtain the conditions on the bias that the resulting LOBD is an AODA.

Thus, from $G(\log\mu\mp\sigma_{on}^{*2}/2,\ \sigma_{on}^{*2})$ for g_n^* [or $G(\mp\sigma_{on}^{*2}/2,\sigma_{on}^{*2})$ for $\hat{g}_n^*$] we have directly

$$\underline{H_o:}\quad \langle\hat{g}^*\rangle_{o,o}=\hat{B}_n^*+\langle H_n(\underset{\sim}{x})^*\rangle_{o,o}=\frac{-\sigma_{on}^{*2}}{2}\quad;$$

$$\underline{H_1:}\quad \langle\hat{g}^*\rangle_{1,\theta}=\hat{B}_n^*+\langle H_n(\underset{\sim}{x})^*\rangle_{1,\theta}=\frac{\sigma_{on}^{*2}}{2}+O(\overline{\theta^4}\text{ or }\overline{\theta^6}),\tag{A.3-16}$$

where the terms $O(\overline{\theta^4}\text{ or }\overline{\theta^6})$ are negligible vis-à-vis $\sigma_{on}^{*2}/2$ (as a result of the "small-signal" condition that $\sigma_{on}^{*2}\doteq\sigma_{on}^{*2}$, cf. (2.29)). Here specifically

$$H_n(\underset{\sim}{x})^*=\sum_i^n h_i^{(1)}(x_i)+\frac{1}{2}\sum_{ij}^n h_{ij}^{(2)}(x_i,x_j)\tag{A.3-17a}$$

$$=\sum_i -\langle\theta_i\rangle\ell_i+\frac{1}{2}\sum_{ij}\{\langle\theta_i\theta_j\rangle(\ell_i\ell_j+\ell_i'\delta_{ij})-\langle\theta_i\rangle\langle\theta_j\rangle\ell_i\ell_j\}:$$

$$\frac{\text{"composite" LOBD}}{\text{Eq. (2.9)}},\ \langle\theta\rangle>0\quad;\tag{A.3-17b}$$

$$H_n(x)^* = \sum_i -\langle\theta_i\rangle \ell_i \quad : \qquad \frac{\text{coherent LOBD}}{\text{Eq. (2.11)}}, \ \langle\theta\rangle > 0; \tag{A.3-17c}$$

$$= \sum_{ij} \{\langle\theta_i\theta_j\rangle(\ell_i\ell_j + \ell_i'\delta_{ij})\}: \qquad \frac{\text{incoherent LOBD}}{\text{Eq. (2.12)}}, \ \langle\theta\rangle = 0. \tag{A.2-17d}$$

Applying (A.3-17) to (A.3-16), we see that <u>two conditions</u> jointly involving the bias and the AO character of $\hat{g}_n^*$ must be satisfied simultaneously. These are

$$\text{I.} \quad \langle\hat{g}^*\rangle_{1,\theta} - \langle\hat{g}^*\rangle_{o,o} \doteq \sigma_{on}^{*2} (= \text{var}_o \hat{g}_n^*) \quad ;$$

$$\text{II.} \quad \hat{B}_n^* = -\frac{1}{2}(\langle\hat{g}^*\rangle_{1,\theta} + \langle\hat{g}^*\rangle_{o,o}) \doteq -\sigma_{on}^{*2}/2 \ . \tag{A.3-18}$$

For both purely coherent and incoherent detection, cf. (A.3-17c,d), we have already shown that I and II, (A.3-18), are satisfied, subject to the "small-signal" condition $\sigma_{on}^{*2} >> |F_n^*(\langle\theta_o\rangle, \text{or} \langle\theta^2\rangle)|$, cf. (2.29), which insures that $\sigma_{1n}^{*2} \doteq \sigma_{on}^{*2}$. [See, specifically (A.2-14), (A.2-40), and (A.2-50b), (A.2-5b) in the binary signal cases.] However, as a preliminary to examining the composite LOBD, (A.3-17b), in regard to satisfying conditions I,II, (A.3-18), let us briefly outline the evaluations. We have

<u>Coherent Reception:</u>

$$\underline{\text{I.}} \quad -\sum_i \langle\theta_i\rangle\langle\ell_i\rangle_{1,\theta} + \sum_i \langle\theta_i\rangle\langle\ell_i\rangle_{o,o} = \sum_i \langle\theta_i\rangle^2 L_i^{(2)} + \sum_i \langle\theta_i^2\rangle\langle\theta_i\rangle^2 L_i^{(2,2)} + 0$$

$$\doteq \text{var}_o \hat{g}_{coh}^* = \sigma_{on\text{-}coh}^{*2} \quad ; \tag{A.3-19a}$$