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Optimum Reception in Non-Gaussian Electromagnetic Interference Environments: II. Optimum and Suboptimum Threshold Signal Detection in Class A and B Noise

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PREFACE

This is the second in a series of studies by the present authors which addresses the critical problem of signal detection in highly nongaussian electromagnetic interference (EMI) environments. (The first in this series is the Report-OT-75-67, "Optimum Reception in an Impulsive Interference Environment", June 1975, by A.D. Spaulding and D. Middleton, for the Office of Telecommunications - U.S. Dep't. of Commerce [Ref. [1a]], subsequently published in somewhat shorter form in the IEEE Transactions on Communications in 1977, [1b].

Because of the recent development (1974-) of effective, tractable statistical-physical models of typical EMI environment ([2]-[10a]), which provide at least the complete first-order statistics of the received interference (as it appears following the initial linear stages of narrow-band receivers), it has become possible to determine and compare the limiting threshold (i.e. weak-signal) performance of both optimum and conventional receivers in such disturbances. The latter are found to be heavily degraded vis-à-vis the former, because of the highly nongaussian character of these typical telecommunication environments, where both man-made and natural "noise" can and usually do predominate. Optimality is important, since from it one can establish the limiting behaviour of suitably designed receiving algorithms, as well as evaluate the performance of current suboptimum receivers. These results, in turn, are fundamental to the technical basis of effective spectrum use and management. Included here as well, is the aforementioned construction of adequate EMI models and the explicit identification of the pertinent data bases required for both empirical and analytic applications.

These studies accordingly focus on signal detection, with particular attention to the structure of the nongaussian EMI and its "scenario", i.e. propagation laws, source distributions, signal waveforms, etc., as well as the corresponding (desired) signal scenario. In this way observables of the EMI environment are directly incorporated into the results, e.g., optimum signal processing algorithms, suboptimum procedures, and performance measures.

Among the many topics under investigation in this series are: (1), the rôle of the interference class (Class A, B noise) on detection algorithms and performance; (2), the effects of the EMI scenario on performance; (3), the various matched filters appropriate to different propagation conditions for the desired signal; (4), the effects of approximate or inaccurate EMI parameter data on structure and performance (i.e. "robustness" questions); (5), receiver structure and performance for varieties of digital signal waveforms in common usage; and many related problems, which one hopes to examine as the work progresses.

Finally, it should be stressed that, although attention is directed here primarily to (EM) telecommunication environments, the concepts, methods, and results of this work are quite generally applicable to other communication fields and physical systems. This is a direct consequence of the canonical formulation of the detection problem itself, on the one hand, and of the canonical nature of the broad spectrum of interference scenarios encompassed by the recently-developed non-gaussian noise or interference models on the other. Consequently, it is expected that the approaches and results obtained here should have impact well beyond the particular applications to EMI telecommunication systems discussed herein.

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by

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ABSTRACT

In this second part of an ongoing study, the general problem of optimum and suboptimum detection of threshold (i.e. weak) signals in highly non-gaussian interference environments is further developed from earlier work ([1a],[1b];[34]). Both signal processing algorithms and performance measures are obtained canonically, and specifically when the electromagnetic interference environment (EMI) is either Class A or Class B noise. Two types of results are derived: (1), canonical analytic threshold algorithms and performance measures, chiefly error probabilities and probabilities of detection; and (2), various typical numerical results which illustrate the quantitative character of performance. Suboptimum systems are also treated, among them simple cross- and auto-correlators (which are optimum in gaussian interference), and clipper-correlators which employ hard limiters (and are consequently optimum in "Laplace noise"). The various modes of reception considered explicitly here include: (i), coherent and incoherent reception; (ii), "composite" or mixed reception (when there is a nonvanishing coherent component in the received signal; (iii), "on-off" and binary signals, as well as varieties of fading and doppler spread.

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Both local optimality (LO) and asymptotic optimality (AO) are demonstrated, along with the critical influence of the proper bias in the optimum algorithms, which maintain their LO and AO character as sample size is increased, without having to add additional terms in the original threshold expansion (and thus produce insurmountable system complexity for the very large samples required for effective detection of weak signals). It is shown that for AO, as well as LO, two conditions may be needed to establish the largest magnitude of the minimum detectable input signal which can be permitted and still maintain the optimal character of the algorithm. In addition to the more general Bayes risk and probabilistic measures of performance, Asymptotic Relative Efficiencies (ARE's) are also included and their limitations discussed. A number of numerical examples which illustrate the determination of performance and performance comparisons are provided, with an extensive set of Appendices containing many of the analytic details developed and presented here for future use, as well.

KEY WORDS AND PHRASES:

Threshold signal detection, optimum threshold detection algorithms, performance measures, performance comparisons, electromagnetic interference environments (EMI), suboptimum detectors, locally optimum and asymptotically optimum algorithms; Class A, B noise; correlation detectors; clipper-correlators; error probabilities; minimum detectable signals, processing gain, bias, EMI scenarios; composite threshold detection algorithms; on-off binary signal detection; non-gaussian noise and interference.

1. INTRODUCTION

Nongaussian noise and interference have been recognized for some time [10], [10a] as an increasingly significant factor in the degradation of the performance of most electronic systems and of telecommunication systems in particular [1a,b]. Both natural and man-made noise contribute noticeably here, with the latter becoming the dominant component in most instances, as time goes on. At the same time, most telecommunication systems - specifically receivers - have been designed to be (approximately) optimal against gaussian noise (both internal and external). This has been accomplished by means of "matched filters" ([11],[12]), whose particular structures depend on the mode of reception, i.e., on whether or not reception is "coherent" of "incoherent" [Sec. 19.4, [12]]. Now, because of the growing presence of nongaussian interference of all kinds, these conventional or "classical" (correlation) receivers are found to be badly degraded O(20-50db) typically, and new designs (or "algorithms") for optimality are accordingly required [1a,b], [13].

Analytically quantifiable procedures for optimal signal processing at <u>all</u> desired signal levels in arbitrary interference are not generally possible, however. Thus, to obtain a "general" solution either one must restrict the class of signals and interference, mode of observation, etc., or one must limit the approach to <u>threshold signals</u>, where now there is no restriction on signal type and interference class. Such an approach is accordingly <u>canonical</u>, [14], with several considerable advantages over more specific but less general methods. These advantages are: (i), an explicit operational development of the required optimum signal processing algorithms (i.e. detection or signal extraction); (ii), an explicit formalism for evaluating error-probability performance directly in terms of the various first and second moments of the processing algorithm (vis-à-vis the various hypothesis states involved, e.g. H_0: interference alone, H_1: desired signal plus interference, etc.); and (iii), a similar procedure for obtaining the performance of specified sub-optimum systems in the electromagnetic interference (EMI) environment.

Optimality here is expressed in the general sense of minimum average risk or cost (i.e. Bayes risk ([12], Chapters 18,19), and in the more special sense of minimum probability of error, or maximum probability of correct signal detection, etc., which is, of course, ultimately embedded in the more general Bayes formalism. Of course, as the signal level increases the signal threshold algorithm is no longer optimum, but it is still better on an absolute basis than it is for very small signals. Moreover, it remains better, in many instances, than the original suboptimum systems to which it is often vastly superior in the threshold régime (as noted above).

For these threshold signals optimality is achieved under the strictly mathematical condition of vanishingly small input signals. In the practical cases, however, as we show here, effective optimality is maintained as long as the small desired input signal does not exceed some upper bound (itself small). [The desired signal is, of course, nonvanishing in all practical applications.] These optimum threshold algorithms can be shown to be optimum in two senses: (i), locally optimum (LO), i.e. essentially yielding the smallest error probabilities for small signals θ, $(0<\theta<\epsilon<<1)$, with finite sample sizes $(n<\infty)$; and (ii), asymptotically optimum (AO), where for these same LO algorithms, the error probabilities (or average risk, more generally) remains minimal (and can approach zero) as sample-size increases indefinitely $(n\to\infty)$. For the latter we emphasize that the structure of these threshold optimum (LO) algorithms remains unchanged as $n\to\infty$, provided the correct bias, $B_n^*(\theta)$, is employed. Without the proper bias term in the threshold algorithm, the processing is suboptimum, and moreover, is not only not LO but is also not AO. [These questions are discussed in detail in Secs. 2.4, 6.1, 6.4, and particularly in Appendix A3 ff.]

The concept of optimum threshold reception is comparatively venerable. Perhaps the first exposition of the concept was presented for detection by Middleton in 1953, 1954, [15] and [16], where the approach was to demonstrate a series development the generalized likelihood function in various orders of cross- and autocorrelation components, mostly non-linear in the received waveform data. Among the important subsequent works are those of Rudnick in 1961 [17], who expressed the threshold detector in an alternative closed form, more useful in applications, and that of Capon [18], also in 1961, who introduced the notion of asymptotic relative efficiences (ARE's) for performance measures.

A further important step, including these earlier advances and embedding the overall approach fully in the Bayes formalism of statistical communication theory ([10]; Section 19.4, Chapter 20, of [12]), was presented by Middleton in 1966 [14]; (see also [21]). Thomas and coworkers ([21]-[24]) have applied these methods, particularly to non-parametric reception, since about 1965; at about the same time Antonov [25], 1967, and a little later Levin and his colleagues ([26]-[28], approx. 1969 and subsequently, used these concepts for signal detection and estimation. More recently (1978), Sheehy for example, has applied these ideas to acoustic signals. [See also [48] for some recent observations on the current status of work in this area.] In this present study we shall use Middleton's 1966 paper [14] as a starting point for the derivation of specific detection algorithms and performance measures, alone the lines, to some extent, of [1a,b], and particularly, [34].

Although the general threshold detection formalism has been available since 1966, cf. [14], its practical applicability has been limited until recently because of the lack of physically realistic and tractable nongaussian noise models. Most of the interference models suggested have been ad hoc attempts to represent such phenomena, without sufficient physical basis and analytic structure to apply generally. This difficulty was largely removed in the mid-70's and subsequently, by the development of statistical-physical models of interference, which are both analytically tractable and well-verified by experiment, [2]-[9]. Specifically, first-order probability distributions and densities have been obtained, with the model parameters themselves determined analytically from the physical EMI scenario involved [8],[9], or empirically [6],[7], when such information is unavailable. These models are canonical also, in the sense that the <u>form</u> of the results is independent of the particular physical mechanism involved, the principal conditions being; (i), that the potential number of possible sources producing the resultant interference be large, and (ii), that each source emits independently of the others [cf. Sec. 3 below].

Two main classes of interference are distinguished: Class A noise, which is "coherent" in the receiver in that it produces negligible transients therein; and Class B noise, which is alternatively "incoherent", producing essentially nothing but transient responses. The former is non-impulsive, while the latter is usually highly impulsive. Typical examples of Class A interference are other, man-made telecommunications for the same channel or spectral region. Similarly, automobile ignition noise and atmospherics are common types of Class B interference, cf. [6]. We stress the fact that these interference models, and their classification, are not limited to EMI, but apply equally well (with different numerical values, of course) in other physical areas where the same basic source conditions noted above apply.

In the fullest formal sense these general signal processing algorithms (e.g. for detection and extraction) usually require nth-order statistical descriptions of the interference. Fortunately, we can greatly simplify the analysis, without serious loss in either methodology or performance, by using independent (noise) samples. Such procedures are conservative, in that they provide upper bounds on performance, in the sense of larger error probabilities for given input signal levels and sample sizes, or greater signal levels or sample sizes, for the same error probabilities, etc. At the same time we can now use the new canonical statistical-physical interference models noted above, to provide a truly realistic account of the EMI environment in which our signal processing tasks are to be carried out.

Because the parameters of these Class A and B models are themselves derivable from the underlying EMI scenario (i.e. source distribution, propagation law and fading effects, signal structure, etc., (cf. Sec. 3 ff.), we can gain further insight into the rôle of the EMI scenario on system performance, and from this predict how changes in source distributions, propagation conditions, etc., may affect receiver operation. In effect, what we have done by introducing these physically-derived interference models is to show explicitly how the underlying physical mechanisms and conditions can influence system design and behaviour.

In our present study we shall confine our attention to threshold signal detection in canonical Class A or Class B interference, reserving the extension of the analysis to general signal levels along the lines indicated in [1a]) for a subsequent study. Our specific goals are to obtain

(i). the optimum threshold signal detection algorithms for both the coherent and incoherent modes of reception,

(ii). the associated optimum performance for these algorithms, and (iii). comparisons with selected suboptimum receivers, namely, receivers conventionally optimized against gaussian noise, viz. cross- and auto-correlation detectors, and against impulsive noise, e.g., clipper-correlators.

(iv). An important fourth goal is to study the effects of "mismatch", i.e., when approximate or incorrect parameter values and/or noise distributions are employed in system design and operation.

Accompanying this is the concept of "robustness": how little (or how much) is performance degraded by these various types of "mismatch".

Most of the results to be achieved under the above are new, although a few special cases have been obtained earlier [13]; also [1a,b]. In addition to the analysis, selected numerical results illustrate typical performance situations in typical Class A and B EMI environments. Algorithm structure is shown in a number of "flow diagrams", which indicate the organization of the various operational elements.

Specifically, among the principal new results achieved here are the demonstration of asymptotic optimality (AO) of the (optimum) threshold algorithms, when the correct bias is used, various explicit results for coherent and incoherent detection, including composite detectors when there is a nonvanishing coherent signal component, and upper bounds on the minimum detectable signal, required to preserve optimality of the threshold algorithm. Parallel results for binary signals are similarly obtained.

This Report is organized as follows: Section 2 presents a concise overview of the general threshold theory needed for both matched and mismatched, optimum and suboptimum systems, developed mainly from [14]. Section 3 summarizes the pertinent statistics and EMI scenario and parameter structures needed for the Class A and B interference treated here, based mostly on [6], [9], [13]. Section 4 considers threshold detection algorithms themselves, in detail. Section 5 treats "matched filters" and the operational interpretations of these algorithms, while Section 6 examines the performance of these various optimum and suboptimum detectors in analytic detail. In Section 7 selected numerical results are obtained and discussed, for typical classes of (desired) signal waveforms. Section 8 completes the work with a short discussion of both the principal general and specific results, as well as suggested next steps in the analysis. The Appendices provide most of the technical details, and the computer software, needed in the main text.

We remark, finally, that the calculated great improvement of systems optimized properly to these highly nongaussian interference environments vis-à-vis conventionally optimized receivers (i.e. against gauss noise) stems fundamentally from the following conditions:

(1), the fact that the former are adaptive systems, which sense the (parameters of the) EMI environment currently with the the detection process, and

(2), the fact that the entire density function (pdf) is then suitably employed to give the correct threshold algorithm, while the latter remain sensitive only to second-moment statistics (which, of course, are sufficient when the noise is gaussian).

The degree of improvement over conventional detectors depends, as expected, on how nongaussian (in intensity and statistical structure) the interference is. When the interference reduces to gauss, so also does the (optimum) detector algorithm, again as we would expect. It should be noted, however, that the degradation of conventional (simple-correlation) receivers is greatly reduced vis-à-vis the optimum algorithm when (sub-optimum) clipper-correlators are employed. Nevertheless, optimum threshold algorithms may still provide a worthwhile improvement, O(3-10db), over the clipper-correlators, particularly when "composite" or mixed coherent and incoherent processing can be employed. In any case, the results of an optimality study are always needed in any effort to assess ultimate performance and practical departures from it. Finally, recent additional studies [49-54] are to be noted for possible extension of present work.

2. GENERAL THRESHOLD DETECTION THEORY:

Threshold detection theory, as is well-known [14], is a general subelement of the Bayes, or (minimum) average risk theory of signal reception ([19],[12], Chap. 18, et seq.), and as such carries with it all the same general statistical structure and concepts of the latter, more comprehensive formulation. Moreover, the general Bayesian detection theory naturally provides the starting point from which the former is developed. We begin, accordingly, with a very brief summary of the general formalism for both optimum and sub-optimum detection.

2.1 Remarks on General Detection Theory:

Optimum reception, and, in particular optimum detection, is well-known to require the minimization of the probabilities of decision errors. This is achieved (in the usual context of minimizing the average risk, or cost, of decisions) by constructing the "test statistic", or reception algorithm, $\Lambda_n(\underset{\sim}{X}|S)$. Here Λ_n is the (generalized) likelihood ratio, defined in the standard way [Ref. 12, Chapter 18] by

$$\Lambda_n^{(1)} \equiv \frac{p\langle F_n(\underset{\sim}{X}|\underset{\sim}{S})\rangle_S}{q\, F_n(\underset{\sim}{X}|0)} \quad , \tag{2.1}$$

where $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is the set of n samples of received data; $\underset{\sim}{S}$ represents the desired signal; $\langle\,\rangle_S$, the average over the signal or its (possibly) random parameters, while p,q (=1-p) are respectively the a priori probabilities that a received data set $\underset{\sim}{X}$ does or does not contain the desired signal. The quantity $F_n(\underset{\sim}{X}|\underset{\sim}{S})$ is the probability density function for the set $\underset{\sim}{X}$, under the condition of the presence of a signal $(\underset{\sim}{S})$ in the usual fashion. The optimum detection process, then, consists of comparing $\Lambda_n^{(1)}$ (or any monatonic function of $\Lambda_n^{(1)}$ (say, the logarithm, $\log \Lambda_n^{(1)}$)) with a suitably chosen threshold, $\mathcal{K}$, e.g.

$$\left.\begin{array}{ll} \underline{\text{decide } H_0}: & \text{"no signal present", if } \log \Lambda_n^{(1)} < \log \mathcal{K} \\ \underline{\text{decide } H_1}: & \text{"signal, as well as interference} \\ & \text{is present", if } \quad \log \Lambda_n^{(1)} \geq \log \mathcal{K} \end{array}\right\} . \tag{2.2}$$

Similarly, for non-optimum systems, the reception algorithm, or processing of the data, is some (pre-determined) function, $g(\underset{\sim}{X})$, and the decision process has, like (2.2), the form

$$\left.\begin{array}{ll} \underline{\text{decide } H_0}: & \text{if } g(\underset{\sim}{X}) < \log K \text{ , e.g. noise alone} \\ \underline{\text{decide } H_1}: & \text{if } g(\underset{\sim}{X}) \geq \log K\text{, e.g. signal as well as noise,} \end{array}\right\} \tag{2.3}$$

where now the threshold K is $\mathcal{K}(K)$, and usually $K = a\mathcal{K}$, with a some (positive) constant.

<u>Performance</u> is generally expressed as some linear function of the Type I and Type II error probabilities, (α,β), e.g.

$$\alpha = \alpha(S|N) = \int_{\log \mathcal{K}}^{\infty} w_1(g|0)dg \; ; \; \beta = \beta(N|S) = \int_{-\infty}^{\log \mathcal{K}} w_1(g|S)dg, \tag{2.4a}$$

which for optimal systems, (minimizing average risk), becomes

$$\alpha^* = \int_{\log \mathcal{K}}^{\infty} w_1(g^*|0)dg^* \; ; \; \beta^* = \int_{-\infty}^{\log \mathcal{K}} w_1(g^*|S)dg^* . \tag{2.4b}$$

The $w_1(g^*|0)$ etc. are the (1st-order) pdf's with respect to H_0, H_1 of the optimum or suboptimum test statistic or "detection algorithm", $g^* = \log \Lambda_n^{(1)}$ or $g(\underset{\sim}{X})$. The associated average costs or risks are (cf. Secs. (2.3, 2.4, Ref. 20)

$$R^* = \mathcal{L}(\alpha^*,\beta^*) = \mathcal{R}_0 + p(C_0^{(1)}-C_1^{(1)})(\tfrac{\mathcal{K}}{\mu}\alpha^*+\beta^*) = A_0+B_0(\tfrac{\mathcal{K}}{\mu}\alpha^*+\beta^*) \tag{2.5a}$$

$$R = \mathcal{L}(\alpha,\beta) = \mathcal{R}_0 + p(C_0^{(1)}-C_1^{(1)})(\tfrac{\mathcal{K}}{\mu}\alpha+\beta) = A_0+B_0(\tfrac{\mathcal{K}}{\mu}\alpha+\beta) \; , \tag{2.5b}$$

$$\mathcal{K} \equiv [C_0^{(1)}-C_0^{(0)}]\big/[C_0^{(1)}-C_1^{(1)}] \; (\equiv \mathcal{K}_{01}) \; (>0), \tag{2.5c}$$

so that <u>system comparisons</u> are then logically made on a comparison of R, R^* for the same thresholds $K = \mathcal{K}$, where now $\mu \equiv p/q$. The convention here is that $C^{(j)} = C^{(H_j)}_{(\text{decision})}$: the superscripts refer to the hypothesis state (H_j), and the subscripts to the decisions actually made, and errors naturally "cost" more than correct decisions. [For a detailed development see Ref. 12, Chapter 19, Ref. 20, Chapter 2.]

The formalism above is adapted to the common situation where the alternative reception situation (Hypothesis H_1) is a "signal and noise" as opposed to H_o: "noise alone". In many telecommunication applications the choice is between two types of signals in noise (or interference), and the test statistic (2.1) becomes now for these binary signal cases.

$$\Lambda_n^{(21)} = \frac{p_2\langle F_n(\underset{\sim}{X}|\underset{\sim}{S}_2)\rangle_2}{p_1\langle F_n(\underset{\sim}{X}|\underset{\sim}{S}_1)\rangle_1} = \Lambda_n^{(2)}/\Lambda_n^{(1)} \quad \text{with } \Lambda^{(i)} = \text{Eq. (2.1)}; \quad i = 1,2;(S_i) \, . \tag{2.6}$$

The decision process (2.2) is, correspondingly,

$$\left.\begin{array}{l} \underline{\text{decide } H_1}\text{: "a signal } (S_1) \text{ present in noise", if } \operatorname{Log} \Lambda_n^{(21)} < \log \mathcal{K}_{12} \\ \underline{\text{decide } H_2}\text{: "a signal } (S_2) \text{present in noise", if } \log \Lambda_n^{(21)} \geq \log \mathcal{K}_{12} \end{array}\right\}, \tag{2.7}$$

with

$$\mathcal{K}_{12} \equiv (C_2^{(1)}-C_1^{(1)})/(C_1^{(2)}-C_2^{(2)}) \quad (>0). \tag{2.7a}$$

(It is assumed that all signals $\{S_1\}$ are distinct ("disjoint") from all signals $\{S_2\}$, so that there is no ambiguity in establishing correct and incorrect decisions. When the signal classes overlap, however, modifications in the cost assignments, i.e. the selection of the $C_i^{(j)}$ above, must be made: see Sec. 2.2, [20].)

Performance in the case of alternative signal classes is obtained as above [(2.4), (2.5)], now with the obvious notational modifications:

$$\alpha^{(*)} \to \beta_2^{(1)(*)} = \beta^{(*)}(S_2|S_1) = \int_{-\infty}^{\log \mathcal{K}_{12}} w_1(g^{(*)}|H_1)dg^{(*)} \; ;$$

$$\beta^{(*)} \to \beta_1^{(2)*} = \beta^{(*)}(S_1|S_2) = \int_{\log \mathcal{K}_{12}}^{\infty} w_1(g^{(*)}|H_2)dg^{(*)} \; , \tag{2.8}$$

where $g^{(*)}$, etc. = g^* (=log $\Lambda_n^{(21)}$) or g , etc., and the various w_1 refer to the optimum and suboptimum detection algorithms and their associated error probabilities.

2.2 Threshold Detection

Thus, in the detection phase of reception - which is always the initial, or acquisition phase at least - and usually subsequently - each signal unit is to be detected, i.e., a decision made as to the presence (or absence) of the signal symbol, to form a stream of decisions, generating the signal sequence, which is then ultimately decoded into the desired message (possibly corrupted by interference, etc.). However, in the majority of practical situations, the explicit development of the optimum algorithm $\Lambda_n^{(1)}$, or log $\Lambda_n^{(1)}$, cannot be achieved, only approximated. Moreover, the evaluation of performance, via the error probabilities (α^*,β^*), cf. (2.4b), is even more difficult. Ingenious approximations are required, and even these are not sufficient. Only by a literal (i.e. purely computational) realization of Λ_n can we expect to obtain the optimum processor (as is sometimes done.)

In any case, for the important purposes of <u>predicting</u> performance, analytical methods, for all signal levels, are not generally realizable, and we must (apart from brute-force simulation) seek other approaches. Fortunately, as we have remarked in Sec. 1 above, it is possible to obtain canonical results analytically, in the critical limiting case of <u>weak signals</u>, which, also fortunately, is of very considerable interest, as it is the situation which establishes the limiting performance, i.e., the best that can be done either for optimum processors $g(X)^*$, or for specified systems, $g(X)$, which are suboptimal. In general, the limiting, optimal algorithm for any interference has been shown [14] to be (for additive signal and noise processes) the expansions of the (log) likelihood ratio about zero signal $(\theta=0)$:

$$\log \Lambda_n^{(1)} \doteq g(\underset{\sim}{x})^* \equiv \log \mu + \theta \underset{\sim}{\tilde{y}} \underset{\sim}{\tilde{\bar{s}}}' + \frac{\theta^2}{2!} [\underset{\sim}{\hat{y}}(\underset{\sim}{\rho}_s - \underset{\sim}{\bar{s}}' \underset{\sim}{\hat{\bar{s}}}')\underset{\sim}{y} + \mathrm{trace}(\underset{\sim}{\rho}_s \underset{\sim}{\hat{z}})] + \hat{B}_n(\underset{\sim}{\theta})^*, \tag{2.9}$$

where (cf. Sec. III, Ref. [13]):

$$\left.\begin{aligned} &\theta = \sqrt{\overline{a_o^2}}\ ;\ \underset{\sim}{S} = [a_{oj}s_j\sqrt{\psi}]\ ;\ \psi = \langle N^2\rangle,\ \langle N\rangle = 0 \\ &\underset{\sim}{s}' = [a_{oj}s_j/\sqrt{\overline{a_o^2}}\ ];\ \theta_j \equiv a_{oj}s_j\ ; \\ &\overline{a_o^2} = \langle S^2\rangle/\psi\ ;\ \underset{\sim}{s} = [s(t_j-\epsilon)]\ ; \end{aligned}\right\}\ ;\ \mu \equiv p/q \qquad (2.9a)$$

and $\underset{\sim}{s}$ is a normalized signal wave form, such that $\langle s^2\rangle = 1$; $\overline{a_o^2}$ = input signal-to-noise power ratio; $\psi = \langle N^2\rangle$ = (total) mean square noise (or interference) power. Here, $\underset{\sim}{y}$ and $\underset{\sim}{z}$ are the column and square matrices

$$\underset{\sim}{y} = [y_i] = [-\frac{\partial}{\partial x_i}\log W_o]\ ;\ \rho_s \equiv [\overline{s_i's_j'}] = [\frac{\overline{a_{oi}a_{oj}\ s_is_j}}{\overline{a_o^2}}];\ \underset{\sim}{x} = \underset{\sim}{X}/\sqrt{\psi}\ ;$$

$$\underset{\sim}{z} = [z_{ij}] = [\frac{\partial^2}{\partial x_i\partial x_j}\log W_o] = \tilde{\underset{\sim}{z}}\ , \qquad (2.10)$$

with

$$\langle F_n(\underset{\sim}{X}|\underset{\sim}{S})\rangle_S = \langle W_n(\underset{\sim}{x}\sqrt{\psi} - \theta\underset{\sim}{s}')_N\rangle_S;\quad W_o = W_n(\underset{\sim}{X})_N\ ,$$

this last for the postulated additive signal and noise, so that W_o is the joint pdf of $\underset{\sim}{X}$ (=$\underset{\sim}{V}$) when there is only noise.

Here $\hat{B}_n(\theta)^*$ [$=O(\theta^3$ or $\theta^4)$] is a bias, which is determined from the higher order terms in the expansion (2.9), averaged with respect to the null-hypothesis, e.g. H_o: no signal. The (correct) bias is critical for optimum performance in these threshold cases, where $n \gg 1$ necessarily. [See Appendix A3.] The resulting bias is also required to insure the consistency of the test (H_1 vs. H_o) as sample size (n) becomes infinite (as $\theta \to 0$), or for $n < \infty$, as $\theta = \epsilon \doteq 0$. The quantity $g(\underset{\sim}{X})^*$ we call the Locally Optimum Bayes Detector (or LOBD), as it gives a Bayes or minimum average risk, cf. (2.5a) and Appendix A3.

The general result (2.9) for the LOBD includes correlated samples, and both incoherent and coherent reception. For the latter, strictly, we have

$\bar{s}' \neq 0$, e.g. $\langle s(t-\epsilon)\rangle_\epsilon \neq 0$, where ϵ is the signal epoch vis-à-vis the observer (receiver), which by definition of coherence, is now assumed to be strictly given. At the other extreme, we have so-called incoherent reception, where $\bar{s}' = 0$, e.g., $\langle s(t-\epsilon)\rangle_\epsilon = 0$. In between these extremes, it is possible to have what we call quasi-coherent reception, where $w_1(\epsilon)$ is non-uniform, such that $\langle s\rangle_\epsilon \neq 0$, and may be small but not ignorable compared to the terms containing $\langle s_i s_j\rangle_\epsilon$, i.e. $O(\theta^2)$, in (2.9). These distinctions are particularly pertinent when dealing with narrow-band signals, where now $w_1(\epsilon)$ is defined over an RF carrier cycle, not over the whole duration of the signal. [In such cases, feedback loops are often used to "lock-on" from the initial instance of purely incoherent reception, to the eventual stage of more or less exact phase tracking, which permits strict synchronization of the local oscillator of the receiver, with the RF phase of the desired input signal. The result is then, of course, coherent reception, vs. the incoherent reception that occurs when this "phase-learning" process is not employed.]

The critical feature of coherent vs. incoherent detection is, of course, the fact that the LOBD for the former is $O(\theta)$, while the latter is $O(\theta^2)$, $\theta \ll 1$. The structures of the optimum threshold detector, or LOBD, are then, respectively, [cf. Appendix A-I, also]:

I. Coherent Reception: $(H_1$ vs. $H_0)$:

$$\log \Lambda_n^{(1)}\Big|_{coh} \doteq g(\underset{\sim}{x})^*_c = [\log \mu + \hat{B}_n(\theta)^*_{coh}] + \theta \tilde{\underset{\sim}{y}} \underset{\sim}{\bar{s}}' \ , \tag{2.11}$$

while for the latter we have

II. Incoherent Reception $(H_1$ vs. $H_0)$:

$$\log \Lambda^{(1)}_{n-inc} \doteq g(x)^*_{inc} = [\log \mu + \hat{B}_n(\theta)^*_{inc}] + \frac{\theta^2}{2!} [\tilde{\underset{\sim}{y}}(\underset{\sim}{\rho}_s)\underset{\sim}{y} + \text{trace}\, \underset{\sim}{\rho}_s \tilde{\underset{\sim}{z}}], \ (\underset{\sim}{\bar{s}}=0), \tag{2.12}$$

in which $\hat{B}^*_{n-inc} \neq \hat{B}^*_{n-coh}$, generally. For mixed modes of reception (i.e. "quasi-coherent" cases), we must use a suitably modified form of (2.9), cf. Appendix A3-6.

When there are two classes of signal to be distinguished, generally according to (2.6), (2.7), the general optimum threshold algorithm (2.9) is

$$\log \Lambda_n^{(21)} = \hat{B}_n^{(21)*} + \tilde{\underset{\sim}{y}}\, \underset{\sim}{\overline{\Delta\theta}}^{(21)} + \frac{1}{2!}[\tilde{\underset{\sim}{y}}(\underset{\sim}{\Delta\rho}_\theta^{(21)} - [\underset{\sim}{\bar{\theta}}^{(2)} \cdot \tilde{\underset{\sim}{\bar{\theta}}}^{(2)} - \underset{\sim}{\bar{\theta}}^{(1)} \cdot \tilde{\underset{\sim}{\bar{\theta}}}^{(1)}])\underset{\sim}{y}$$

$$+ \text{trace}\,(\underset{\sim}{\Delta\rho}_\theta^{(21)} \tilde{\underset{\sim}{z}})] \equiv g^{(21)*} , \tag{2.13}$$

where now

$$\left.\begin{aligned} \underset{\sim}{\overline{\Delta\theta}}^{(21)} &\equiv \underset{\sim}{\bar{\theta}}^{(2)} - \underset{\sim}{\bar{\theta}}^{(1)} = [\bar{a}_{oj}^{(2)}\, \bar{s}_j^{(2)} - \bar{a}_{oj}^{(1)}\, \bar{s}_j^{(1)}] = [\bar{\theta}_j^{(2)} - \bar{\theta}_j^{(1)}] \\ \underset{\sim}{\Delta\rho}_\theta^{(21)} &\equiv \overline{\underset{\sim}{\theta}^{(2)} \underset{\sim}{\theta}^{(2)}} - \overline{\underset{\sim}{\theta}^{(1)} \underset{\sim}{\theta}^{(1)}} = [\langle a_{oj}^{(2)} a_{oj}^{(2)} s_j^{(2)} s_i^{(2)} \rangle - \langle a_{oi}^{(1)} a_{oj}^{(1)} s_i^{(1)} s_j^{(1)} \rangle] \\ &\equiv \underset{\sim}{\rho}_\theta^{(2)} - \underset{\sim}{\rho}_\theta^{(1)} , \end{aligned}\right\} \tag{2.13a}$$

and $\hat{B}_n^{(21)*}$ is once more a suitable bias to insure optimality and consistency of the test H_2 vs. H_1 here. This bias is obtained, as before [cf. (2.10) et seq. and Appendix A-I] by averaging the next (non-vanishing) terms in the expansion of $\log \Lambda^{(21)}$ <u>again with respect to H_o</u>, since $\log \Lambda_n^{(21)} = \log \Lambda_n^{(2)} - \log \Lambda_n^{(1)}$ is the difference of two "on-off" detectors, viz.

$$\left.\begin{aligned} \log \mu_{21} + \hat{B}_n^{(21)*} &= \log \mu_{21} + \langle O(\theta^{(2)^3}) \rangle_{H_o} - \langle O(\theta^{(1)^3}) \rangle_{H_o} , \text{ or} \\ &= \log \mu_{21} + \langle O(\theta^{(2)^4}) \rangle_{H_o} - \langle \Theta(\theta^{(1)^4}) \rangle_{H_o} \end{aligned}\right\} ;$$

$$\mu_{21} = \frac{p_2/q}{p_1/q} = \frac{p_2}{p_1} . \tag{2.14}$$

Thus, (2.11) and (2.12) now become, for S_2 vs. S_1 in the same interference

I. Coherent Reception $(\bar{s}^{(1,2)}\neq 0):(H_2 \text{ vs. } H_1)$:

$$g^{(21)}(\underset{\sim}{x})^*_c = [\log \mu_{21}+\hat{B}_n^{(21)}(\theta)^*_c] + \tilde{\underset{\sim}{y}}(\overline{\underset{\sim}{\theta}^{(2)}}-\overline{\underset{\sim}{\theta}^{(1)}}) \,, \tag{2.15}$$

and

II. Incoherent Reception $(\bar{s}^{(1,2)}=0)$: H_2 vs. H_1:

$$g^{(21)}(\underset{\sim}{x})^*_{inc} = \log \mu_{21}+\hat{B}_n^{(21)}(\underset{\sim}{\theta})^*_{inc}+ \frac{1}{2!}\{\tilde{\underset{\sim}{y}}\, \underset{\sim}{\Delta\rho}_\theta^{(21)}\underset{\sim}{y}+\text{trace}(\underset{\sim}{\Delta\rho}_\theta^{(21)}\tilde{\underset{\sim}{z}})\}. \tag{2.16}$$

The decision process is given by (2.7), with (2.13), generally, and with (2.15), (2.16), respectively for the coherent and incoherent modes of reception. [Equations (2.11) and (2.13) apply in the "composite" or "quasi-coherent" cases, when there is enough coherence (via phased-locked loops, for example) to justify using both processing modes simultaneously, cf. II-C (Part II), (1a): These variants are reserved to a subsequent study, cf. Sec. 8.]

Finally, for suboptimum detectors we have,

$$\left.\begin{aligned} g^{(21)}(\underset{\sim}{x})^*_c &\to g^{(21)}(\underset{\sim}{x})_c = g^{(2)}(\underset{\sim}{x})_c-g^{(1)}(\underset{\sim}{x})_c \\ g^{(21)}(\underset{\sim}{x})^*_{inc} &\to g^{(21)}(\underset{\sim}{x})_{inc} = g^{(2)}(\underset{\sim}{x})_{inc}-g^{(1)}(\underset{\sim}{x})_{inc} \end{aligned}\right\}, \tag{2.7}$$

with decision rules (2.7) on replacing $\log \Lambda_n^{(21)} \to g^{(21)}(\underset{\sim}{x})^*$ by $g^{(21)}(\underset{\sim}{x})$, etc.

The decision process is, of course, carried out according to (2.3), (2.7), with $\log \Lambda_n$ replaced by g^*, cf. (2.9), (2.11)-(2.13), (2.15),(2.16).

2.3 Gaussian Interference

The threshold canonical forms of Sec. 2.2 readily reduce to the known structures when the noise or interference is gaussian. This is easily seen from (2.10) and the pdf

$$W_o(\underset{\sim}{x}) = \{(2\pi)^{n/2}(\det \underset{\sim}{k}_N)^{1/2}\}^{-1} e^{-\frac{1}{2}\tilde{\underset{\sim}{x}}\underset{\sim}{k}_N^{-1}\underset{\sim}{x}} , \tag{2.18}$$

where one has directly

$$\underset{\sim}{y} = [(\underset{\sim}{k}_N^{-1}\underset{\sim}{x})_i] \; ; \; \underset{\sim}{z} = [-\frac{\partial y_i}{\partial x_j}] = -\underset{\sim}{k}_N^{-1} . \tag{2.18a}$$

Thus, the threshold algorithms (2.10), (2.12) in the "on-off" cases become

I. Coherent Reception (H_1 vs. H_o):

$$g(\underset{\sim}{x})_c^*\Big|_{gauss} = [\log \mu + \hat{B}^*_{n-coh}]_{gauss} + \tilde{\underset{\sim}{x}}\underset{\sim}{k}_N^{-1}\underset{\sim}{\theta} \; ; \tag{2.19}$$

II. Incoherent Reception (H_1 vs. H_o):

$$g(\underset{\sim}{x})^*_{inc}\Big|_{gauss} = [\log \mu + \hat{B}^*_{n-inc} - \frac{1}{2}\langle\tilde{\underset{\sim}{\theta}}\underset{\sim}{k}_N^{-1}\underset{\sim}{\theta}\rangle]_{gauss} + \frac{1}{2!}\tilde{\underset{\sim}{x}}\underset{\sim}{k}_N^{-1}\underset{\sim}{\rho}_\theta\underset{\sim}{k}_N^{-1}\underset{\sim}{x} , \tag{2.20}$$

where

$$\underset{\sim}{\theta} = [a_{oi}s_i], \text{ cf. (2.9a)} \; ; \; \underset{\sim}{\rho}_\theta \equiv \langle[\theta_i\theta_j]\rangle = [\langle a_{oi}a_{oj}s_is_j\rangle]. \tag{2.20a}$$

These results are just those (Eq. 20.7, Eq. 20.11a, [12]) obtained many years ago for these gaussian situations.

Similarly, we find for the two-signal cases (2.15), (2.16), that the threshold algorithms reduce respectively to

I. Coherent Reception (H_2 vs. H_1):

$$g^{(21)}(\underset{\sim}{x})^*_c\Big|_{gauss} = [\log \mu_{21}+\hat{B}^{(21)*}_{n-c}]_{gauss} + \tilde{\underset{\sim}{x}}\underset{\sim}{k}_N^{-1}[\bar{\underset{\sim}{\theta}}^{(2)}-\bar{\underset{\sim}{\theta}}^{(1)}] , \qquad (2.21)$$

and

II. Incoherent Reception (H_2 vs. H_1):

$$g^{(21)}(\underset{\sim}{x})^*_{inc}\Big|_{gauss} = [\log \mu_{21}+\hat{B}^{(21)*}_{n-inc} - \tfrac{1}{2}\{\langle\tilde{\underset{\sim}{\theta}}^{(2)}\underset{\sim}{k}_N^{-1}\underset{\sim}{\theta}^{(2)}\rangle-\langle\tilde{\underset{\sim}{\theta}}^{(1)}\underset{\sim}{k}_N^{-1}\underset{\sim}{\theta}^{(1)}\rangle\}]_{gauss}$$

$$+ \frac{1}{2!}\tilde{\underset{\sim}{x}}\underset{\sim}{k}_N^{-1}(\underset{\sim}{\rho}_\theta^{(2)}-\underset{\sim}{\rho}^{(1)})\underset{\sim}{k}_N^{-1}\underset{\sim}{x} , \qquad (2.22)$$

with $\underset{\sim}{\rho}_\theta^{(2)} = [\langle a_{oi}^{(2)}a_{oj}^{(2)}s_i^{(2)}s_j^{(2)}\rangle]$, etc. [Equations (2.21), (2.22) agree, as expected, with the earlier results, Problem 20.12, p. 935, [12], and Section 20.4-5, [12], respectively, when the accompanying interference is gaussian noise.]

Thus, when the noise is gaussian, the resulting algorithms remain optimum (LOBD's) with a generalized cross- or auto-correlation structure for the processors, cf. (2.19)-(2.22). With independent noise sampling ($k_N^{(-1)}=(\delta_{ij})$), these algorithms reduce to the simpler specific LOBD structures A.1-24,25) with the biases now obtained from (4.9), (4.12).

2.4 Canonical Evaluation of Threshold Detection Performance:

By threshold detection we mean not only appropriately small input signals vis-à-vis the accompanying interference, but also appropriately large observation periods, expressed as a suitably large number $n' \leq n$) of effectively independent noise samples. Thus, for the LOBD, or g*, cf. (2.9) et seq., we consider the quasi-limiting cases of "small signals" ($\theta^2 \ll 1$) and large samples ($n \geq n' \gg 1$), or equivalently, large time-bandwidth products $n \doteq B_e T \gg 1$. Performance, in terms of the error probabilities (2.4b), is then found by direct application of the Central Limit Theorem (cf. Sec. 7.7-3, [12]) to the detection algorithm, or test statistic g^*. Accordingly g^* is asymptotically normally distributed, in the "on-off" cases (H_1 vs. H_o), with the first and second moments†

$$\left\langle g^{(*)}\right\rangle_{H_oH_1}\ ,\left\langle (g^{(*)})^2\right\rangle_{H_o,H_1} \rightarrow \text{var}\ g^*\Big|_{H_o,H_1} \equiv \sigma^{*2}_{o,1} \tag{2.23}$$

e.g.

$$w_1(g^*|H_o) \cong \frac{e^{-(g^*-\langle g^*\rangle_{H_o})^2/2\sigma_o^{*2}}}{\sqrt{2\pi\sigma_o^{*2}}}\ ;\ w_1(g^*|H_1) \cong \frac{e^{-(g^*-\langle g^*\rangle_{H_1})^2/2\sigma_1^{*2}}}{\sqrt{2\pi\sigma_1^{*2}}}\ . \tag{2.24}$$

In fact, applying (2.23), (2.24) to (2.4b) for "on-off" detection (H_1 vs. H_o), where the (conditional) false-alarm probability, α_F^* (or threshold $\mathcal{K}$), is preset, [the so-called Neyman-Pearson Observer, (Sec. 19.2-1, [12])], we have

$$\boxed{\alpha_F^* \simeq \frac{1}{2}\{1+\Theta[\frac{\langle g^*\rangle_o-\log\mathcal{K}}{\sigma_o^*\sqrt{2}}]\};\ \beta^* \simeq \frac{1}{2}\{1-\Theta[\frac{\langle g^*\rangle_1-\log\mathcal{K}}{\sigma_1^*\sqrt{2}}]\},} \tag{2.25}$$

so that the probability, P_D^*, of correctly detecting the presence of a signal is maximized to become

$$\boxed{P_D^* = p(1-\beta^*) \simeq \frac{p}{2}\{1+\Theta[\frac{\langle g^*\rangle_1-\langle g^*\rangle_o}{\sqrt{2}\ \sigma_1^*} - \frac{\sigma_o^*}{\sigma_1^*}\ \Theta^{-1}(1-2\alpha_F^*)]\},} \tag{2.26}$$

on eliminating threshold $\mathcal{K}$. Here

$$y = \Theta(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \text{erf}\ x;\quad x = \Theta^{-1}(y) \tag{2.26a}$$

are the well-known error function and its inverse. [Equation (2.16) is, of course, equivalent to minimizing the error probability ($\beta\rightarrow\beta^*$), with $\alpha=\alpha_F^*$ fixed.

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† But, see the ultimate condition (2.29) ff, when for optimality $\sigma_1^* \rightarrow \sigma_o^*$, etc.

Similarly, when the threshold is set to $\mathcal{K} = 1$, i.e. when $(\alpha\to\alpha^*, \beta\to\beta^*)$ are jointly minimized, we have the so-called Ideal Observer [cf. Sec. 19.2-2, [12]], so that the total probability of decision error is

$$P_e^* = p\beta^* + q\alpha^* \simeq \frac{1}{2}\{1 - p\Theta[\frac{\langle g\rangle_1^*}{\sqrt{2}\,\sigma_1^*}] + q\Theta[\frac{\langle g^*\rangle_o}{\sqrt{2}\,\sigma_o^*}]\}, \quad = 1, \tag{2.27a}$$

which for symmetrical channels (i.e. p=q=1/2) reduces further to

$$\boxed{P_e^*\Big|_{sym} \simeq \frac{1}{2}\{1 - \frac{1}{2}\Theta[\frac{\langle g^*\rangle_1}{\sqrt{2}\,\sigma_1^*}] + \frac{1}{2}\Theta[\frac{\langle g^*\rangle_{H_o}}{\sqrt{2}\,\sigma_o^*}]\},\ \mathcal{K} = 1;\ p=q=1/2\ .} \tag{2.27b}$$

The Neyman-Pearson, or fixed false alarm observer is appropriate to the initial stages of detecting the presence of a desired signal, while the Ideal Observer ($\mathcal{K} = 1$) is the more suitable criterion (i.e. total decision error probability) when particular elements of a signal are to be detected, i.e. "marks" or "spaces" (in these "on-off" cases), in the course of message transmission, where now P_e^* is directly proportional to the bit-error rate.

Equations (2.23)-(2.27b) apply equally well, formally, for suboptimum detectors, $g(\underset{\sim}{x})$: we simply replace g^* by g, $\sigma_{1,o}^*$ by $\sigma_{1,o}$, P_D^*, P_e^* by P_D, P_e in the above. Furthermore, we have explicitly for the averages (2.18)

$$\langle h^k\rangle_{o,1} \equiv \int_{-\infty}^{\infty} w_n(\underset{\sim}{x}|H_{o,1})_N h(\underset{\sim}{x})^k d\underset{\sim}{x}\ ,\ (h=g,g^*)\ , \tag{2.28a}$$

with

$$w_n(\underset{\sim}{x}|H_o)_N = W_n(\underset{\sim}{x})_N\ ;\ w_n(\underset{\sim}{x}|H_1) = W(\underset{\sim}{x}-\underset{\sim}{s})_N\ , \tag{2.28b}$$

cf. (2.9), for the postulated additive signal and noise cases here.

The relations P_D^*, P_D, P_e^*, P_e, etc., (2.25) et seq., hold asymptotically for all input signal levels (as long as the number (n'<n) of effectively

independent noise samples remains large). However, the LOBD's, g^*, [(2.9), (2.11), (2.12) etc.] are then no longer optimum, in the locally optimum sense ($\theta^2 \ll 1$, $n' \gg 1$), but can become drastically suboptimum as the input signal level ($\sim\theta$) becomes larger. In keeping with the concept of the LOBD, which is a truncated series development in θ, cf. (2.9), which depends on the mode of observation (or reception) i.e. coherent or incoherent [cf. (2.10) et seq.], we must be similarly consistent with respect to the appropriate power of θ in determining the above probability measures of performance. Because of the asymptotically optimum (AO) condition, cf. Appendix A3, which determines the bias $B_n^*(\theta)$ as the average of the next highest non-vanishing (H_0-) average in the series development $\log \Lambda_n = g^* + \ldots$, cf. (2.9), we must likewise require that $\sigma_1^{*2} = \sigma_0^{*2} + F_n(\theta \text{ or } \theta^2)$, where $F_n^* \ll 1$. This (AO) condition,

$$\boxed{|F_n^*(\theta \text{ or } \theta^2)| \ll \sigma_0^{*2}\ , \ \therefore\ \sigma_1^{*2} \doteq \sigma_0^{*2}\ ,\ n \gg 1,} \tag{2.29}$$

in turn, requires that the <u>input</u> signal level remains appropriately small, to insure that g^* (=LOBD) is indeed "locally optimum" and asymptotically optimum.

We can make the condition (2.29) somewhat more explicit by considering for these additive signal and noise cases (2.28b) the expansions

$$\langle g^{*k} \rangle_{H_1} = \int (g^*)\ w_n(\underset{\sim}{x}|H_0)d\underset{\sim}{x} - \bar{a}_0 \langle (g^*)^k \underset{\sim}{s}\underset{\sim}{w}_n'/w_n \rangle_{o,s} + \frac{\overline{a_o^2}}{2} \langle (g^*)^k \underset{\sim}{s}\underset{\sim}{w}_n''\underset{\sim}{s}/w_n \rangle_{o,s} + \ldots\ ,\ k=1,2, \tag{2.30a}$$

so that

$$\sigma_1^{*2} = \sigma_0^{*2} + F_n^* \doteq \sigma_0^{*2}\ , \tag{2.30b}$$

and

$$\therefore \quad F_n^* = \bar{a}_o[2\langle g^*\rangle_o\langle g^*\tilde{\bar{\underset{\sim}{s}}}\underset{\sim}{w}_n'/w_n\rangle_o - \langle (g^*)^2\tilde{\bar{\underset{\sim}{s}}}\underset{\sim}{w}_n'/w_n\rangle_o]$$

$$+ \frac{\overline{a_o^2}}{2}[\langle (g^*)^2\tilde{\underset{\sim}{s}}\underset{\sim}{w}_n''\underset{\sim}{s}/w_n\rangle_{o,s} - 2\langle g^*\rangle_o\langle g^*\tilde{\bar{\underset{\sim}{s}}}\underset{\sim}{w}_n'/w_n\rangle_o] + 0(a_o^3) << \sigma_o^{*2} \;, \quad (2.30c)$$

with

$$\underset{\sim}{w}_n' \equiv [\frac{\partial}{\partial x_i} w_n(x_1,\ldots,x_n)] \;;\; \underset{\sim}{w}_n'' = [\frac{\partial^2}{\partial x_i \partial x_j} w_n] \;, \text{ etc.} \quad (2.30d)$$

Thus, for <u>coherent reception</u> the first term of (2.30c) determines the required smallness of ($\bar{a}_o$), while the second term supplies the needed condition on (a_o^2) in the <u>incoherent cases</u> (since,(2.30d),$\bar{s}$=0 then, etc.). Suboptimum algorithms, g, are handled similarly, with $g^* \to g$ in the above. We shall encounter explicit examples of $F_n^* << 1$, (2.30c), later, in Section 6 ff. In any case, (2.26) and (2.27b) now reduce to

$$P_D^{(*)} \doteq \frac{p}{2}\left\{1+\Theta\left[\frac{\langle g^{(*)}\rangle_1 - \langle g^{(*)}\rangle_o}{\sqrt{2}\,\sigma_o^{(*)}} - \Theta^{-1}(1-2\alpha_F^{(*)})\right]\right\} \;, \quad (2.31)$$

$$P_e^{(*)} \doteq \frac{1}{2}\left\{1 - \frac{1}{2}\Theta\left[\frac{\langle g^{(*)}\rangle_1}{\sqrt{2}\,\sigma_o^{(*)}}\right] + \frac{1}{2}\Theta\left[\frac{\langle g^{(*)}\rangle_o}{\sqrt{2}\,\sigma_o^{(*)}}\right]\right\}_{\mu=1} \;, \quad (2.32)$$

$$F_n^{(*)} << \sigma_o^{(*)2} \;,$$

Here, super (*) denotes optimum by super * alone and suboptimum otherwise, i.e. a blank superscript.

For the common telecommunication situations involving the "symmetrical" 2-signal situations H_2: S_2+N vs. H_1: S_1+N, cf. (2.13)-(2.17), performance is calculated as above with the help of (2.8). Now, however, we have $\alpha^* \to \beta_2^{(1)*}$, $\beta^* \to \beta_2^{(2)*}$, $\mathcal{K} \to \mathcal{K}_{12}$, cf. (2.7), (2.7a), and (2.24) is appropriately modified $g^* \to g^{(2)*}$, (2.13) et seq., $H_o \to H_1$; $H_1 \to H_2$, n >> 1. Thus, for example, (2.32) is extended to

$$P_e^{(*)(21)} \doteq \frac{1}{2}\left\{1-p_2\Theta\left[\frac{\langle g^{(2)(*)}\rangle_2}{\sqrt{2}\,\sigma_1^{(*)}}\right]+p_1\Theta\left[\frac{\langle g^{(2)(*)}\rangle_1}{\sqrt{2}\,\sigma_1^{(*)}}\right]\right\}, \quad \dot{\mu}_{21} \equiv p_2/p_1 \; ; \; p_1+p_2 = 1 \tag{2.33}$$

where $\sigma_2^{(*)^2} \doteq \sigma_1^{(*)^2}$, and the higher order terms in θ (or θ^2) are dropped in the means and variances, consistent with the order of development of $g^{(21)(*)}$, as explained above in the case of the "on-off" detection algorithms, cf. (2.29). We shall see some examples of this in Sec. 6 ff., as well.

Finally, the explicit evaluation and comparison of threshold performance, by LOBD's (g^*), or specified sub-optimum systems (g), may be effected by comparing P_D^* vs. P_D, or P_e^* vs. P_e, for the same perameters: observation time ($\equiv$ sample size n), input signal-to-noise ratio $\theta(=\sqrt{\overline{a_0^2}}$, or $\overline{a_0^2}$) and input signal and noise levels, etc. Comparisons may also be made using the associated error probabilities (α^*,β^*), or (α,β), in the Bayes and average risks (2.5a,b). Other useful ways of comparison include calculations of the various Asymptotic Relative Efficiencies (ARE's), and Efficacies, cf. Appendix, [14].(See also, p. 921, of [1a] and our remarks in Sec. 8.) [In addition to the results of Secs. 6,7 here, examples of comparisons based on the error probabilities are also given further in [1a], [13], [14].]

3. A SUMMARY OF CLASS A AND B INTERFERENCE MODELS: 1st-ORDER STATISTICS:

In this section we provide appropriate first-order statistics of Class A and B interference. This includes the general EMI scenario, from which the principal parameters of Class A and B models may be calculated, as well as a rather general desired signal scenario, which encompasses most practical applications.

We shall henceforth approximate the general threshold theory [Sec. 2] by restricting the analysis to independent noise or interference samples (n). As explained in Section 1 above (and as we shall see in Secs. 4-7 subsequently), this greatly simplifies the analysis, without significantly affecting the results. Moreover, it permits us to use the recently developed (and experimentally verified, [5],[6]) first-order probability models of Class A and B interference, which canonically describe most classes of noise and interference.

3.1 Desired Signal Scenarios:

The desired signals are here narrow-band input waveforms,* which appear likewise as narrow band signals at the output of the front-end stages of the receiver, i.e. before any subsequent linear or nonlinear processing. These desired signals often have the same generic form as those producing the interference (in Class A cases). One has explicitly (in sampled form)

$$\underset{\sim}{S}(t,\underset{\sim}{\theta}') = \left[\frac{a_j \hat{I}_{oS}^{1/2}(t,\phi)}{r_d^{\gamma}} \cos[\omega_o(t_j-\epsilon)-\phi_s(t_j)-\phi_o]\right] = [a_{oj} s_j \sqrt{\psi}] \,, \quad \text{cf. (2.9a)}, \quad \psi = \bar{I}_N \,, \tag{3.1}$$

where $\psi(\equiv \bar{I}_N)$ is the mean total noise intensity (measured at the same point in the receiver as the desired signal). Here $r_d \equiv r_D/\hat{r}_o = c_o\lambda/\hat{r}_o$ is the normalized distance of the source to the receiver, $\hat{r}_o$ is the normalizing distance, c_o = speed of propagation, so that λ is a distance measured in units of time (secs.). The quantity $\underline{a_j}$ is a dimensionless scale factor embodying the effects of fading.

In an alternative form we may write (3.1) as

$$\underset{\sim}{S} = \left[\frac{a_j G_o(t_j,\phi)}{\lambda^{\gamma}} \cos[\omega_o(t_j-\epsilon)-\phi_j-\phi_o]\right] = [a_{oj}\, s_j \sqrt{\psi}] = [\frac{A_{oj}}{\sqrt{2}}\, s(t_j-\epsilon)] \tag{3.1a}$$

where now

$$G_o(t,\phi) \equiv \hat{I}_{os}(t,\phi)^{1/2} \hat{r}_o^{\gamma}/c_o^{\gamma} \,, \tag{3.1b}$$

and the "mean amplitude", A_o, over the sampling period t $,T_o+t_o)$ is obtained from

$$A_o^2/2 = \frac{1}{T}\int_{t_o}^{t_o+T} S(t)^2 dt \,. \tag{3.1c}$$

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* The canonical theory is in no way limited by this practical condition, cf. (2.9) et seq.

The normalized signal waveform (s_j) is likewise defined by (3.1a) with the help of (3.1c), cf. Sec. 19.4, [12], Eq. (19.49a).

In many applications digital signals may be used, with no significant amplitude modulation, so that G_o and $\hat{I}_{os}$ are no longer time-dependent. Thus, we can write (3.1a) as

$$\underset{\sim}{S} = \left[\left[\frac{a_j G_o(\phi)/\sqrt{2}}{\lambda^\gamma}\right] \sqrt{2}\cos[\omega_o(t_j-\epsilon)-\phi_j-\phi_o]\right] \equiv \left[\frac{A_{oj}}{\sqrt{2}}\, s(t_j-\epsilon)\right] = [a_{oj} s_j \sqrt{\psi}], \tag{3.2}$$

which defines the normalized signal s_j now by

$$s_j \equiv \sqrt{2}\cos[\omega_o(t_j-\epsilon)-\phi_j-\phi_o]; \therefore A_{oj} = [a_j G_o(\phi)/\lambda^\gamma], \tag{3.2a}$$

so that $\langle s_j^2 \rangle_\epsilon = 1$, as required.

Since the location of the desired signal source is not necessarily known at the receiver, λ is a random variable, as is the fading parameter $\underline{a}$, and the beam-pattern function, $G_o(\phi)$, as well. For most observation periods Rayleigh fading is the expected mechanism, e.g., $\underline{a}$ obeys the pdf

$$w_1(a) = \frac{2a\, e^{-a^2/\overline{a^2}}}{\overline{a^2}}, \quad a \geq 0 . \tag{3.3}$$

The average effects of the (resolvable) multipath are determined by the value of the propagation exponent (γ), which, for example, is usually larger than unity for rough terrain, e.g. $\gamma = 2$ is an often-used empirical value; (γ need not be an integer, however). Moreover, the desired source may be moving (comparatively slowly), so that its location vis-à-vis the receiver is described by a random walk pdf of the form [30], [31]