## 3. RADIO WAVE PROPAGATION IN THE ANISOTROPIC MESOSPHERE

The previous section has shown how to describe the pertinent properties of the mesosphere. We now turn to a study of how these properties affect the propagation of radio waves. Some matrix theory is involved here, most of which may be found in texts such as that of Courant and Hilbert (1953).

### 3.1 Basic Equations for Plane-Wave Propagation

Because we seem required to treat an anisotropic medium, we shall proceed with some caution and begin with Maxwell's equations

$$\nabla \times \mathbf{H} = -i\omega\mathbf{D} \qquad \text{and} \qquad \nabla \times \mathbf{E} = i\omega\mathbf{B}, \tag{19}$$

where $\omega = 2\pi f$ is the radial frequency. As we have seen in the previous section, the constitutive equations will take the form

$$\mathbf{D} = \epsilon_o\mathbf{E} \qquad \text{and} \qquad \mathbf{B} = \mu_o(\mathbf{I} + 2\mathbf{N})\mathbf{H} \tag{20}$$

so that it is the introduction of the tensor **N** that is unfamiliar. If desired, one can include the nondispersive part of the refractivity given in (1) as either part of the electric permittivity or as a further part of the permeability.

We look for a plane wave solution to (19) and (20). We suppose a Cartesian coordinate system (not that of the previous section) oriented so that all the vector fields vary only in the z-coordinate, thus representing a wave traveling in the direction of the z-axis. Then (19) assumes the form of the six simultaneous equations

$$\begin{aligned} -dH_y/dz &= -i\omega\epsilon_o E_x, \qquad & -dE_y/dz &= i\omega B_x, \\ dH_x/dz &= -i\omega\epsilon_o E_y, \qquad & dE_x/dz &= i\omega B_y, \\ 0 &= -i\omega\epsilon_o E_z, \qquad & 0 &= i\omega B_z. \end{aligned} \tag{21}$$

On the left sides we have derivatives of **H** and **E** and on the right sides are linear combinations of the same vectors. Thus the standard methods for solving sets of linear differential equations should be applicable here.

The equations in the last row of (21) are only algebraic. They say that $E_z$ and $B_z$ vanish so that the vectors **E** and **B** lie wholly in the xy-plane perpendicular to the direction of propagation. The same is not true, however, of the vector **H**. Using (20), the equation $B_z = 0$ can be written

$$2N_{zx}H_x + 2N_{zy}H_y + (1 + 2N_{zz})H_z = 0, \tag{22}$$

where $N_{zx}, \ldots$ are elements of the matrix representing **N**. Then (22) can be solved for $H_z$ and is generally not zero.

Differentiating the first column of (21) and using the second column, we find

$$d^2H_x/dz^2 = -(k^2/\mu_o)B_x \quad \text{and} \quad d^2H_y/dz^2 = -(k^2/\mu_o)B_y, \tag{23}$$

since $k^2 = \omega^2\epsilon_o\mu_o$. Because of (20) the functions $B_x$, $B_y$ may be expressed as linear combinations of the three components of **H**. We solve (22) for $H_z$ and replace its appearance in (23) by this solution. In this way $B_x$, $B_y$ will become linear combinations of the two unknowns $H_x$, $H_y$.

Actually, because **N** is small we may shorten this suggested process. The solution to (22) has the variables $H_x$, $H_y$ multiplied by coefficients of the order of N, and the appearances of $H_z$ in (23) also have coefficients of this order. Thus $H_z$, while not zero, is small, and if the above process is carried out the coefficients of $H_x$, $H_y$ in (23) are changed by something on the order of $N^2$. We therefore ignore the terms involving $H_z$ in (23) and so write

$$d^2\mathbf{H}/dz^2 = -k^2(I + 2\mathbf{N}^m)\mathbf{H}, \tag{24}$$

where **H** is now a two-dimensional vector in the xy-plane, and $\mathbf{N}^m$ is the 2x2 submatrix of **N** obtained by discarding the last column and the last row.

A trial solution to (24) might take the form

$$\mathbf{H}(z) = \exp(ikz\mathbf{G})\mathbf{H}_o, \tag{25}$$

where $\mathbf{H}_o$ would be an "initial value" (when z = 0) of **H**, **G** is a 2x2 matrix whose value we seek, and where we do indeed mean to take the exponential of that matrix. This exponential is to be defined in the standard way using either the infinite power series or the properties of differentiation. It will follow that (25) satisfies (24) provided

$$G^2 = I + 2N^m \qquad (26)$$

or, in other words, provided **G** is a square root of the right-hand side. The most obvious square root has the simple approximation

$$G = I + N^m \qquad (27)$$

with an error again on the order of $N^2$. Substituting (27) in (25) we have the final result

$$H(z) = \exp[ikz(I + N^m)]H_o. \qquad (28)$$

There are, of course, other solutions to (24), for to identify a unique solution to a second-order differential equation it is necessary also to specify an initial first derivative. Such additional solutions are derived from other square roots in (26) and have to do with waves traveling in the negative z-direction.

One final detail here concerns how to calculate the electric field vector **E**. From the first column of (21) and from (28) we find

$$E = (i/\omega\epsilon_o)\ e_z \times dH/dz = -Z_o\ e_z \times (I + N^m)H, \qquad (29)$$

where $Z_o$ is the intrinsic impedance of free space and $e_z$ is the unit vector in the direction of propagation. Note that **E** is not orthogonal to **H** but that the discrepancy is only of order N.

In (28), the appearance of **N** is in the exponent and is multiplied by the very large number kz. It can, therefore, have a strong effect on radio propagation. In (29), however, its appearance has a constant relation to the rest of the expression. It can be ignored, leaving us with the usual formula

$$E = -\ Z_o\ e_z \times H. \qquad (30)$$

### 3.2 The Refractivity Matrix

In Section 2.2, we saw how the refractivity tensor **N** could be represented as a 3x3 matrix. We now need a representation for $N^m$. Another way of writing (18) is to separate out the component parts so that

$$\mathbf{N} = N_o\mathbf{P}_o + 2N_+\mathbf{P}_+ + 2N_-\mathbf{P}_- \tag{31}$$

where, using the previously defined basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_o$,

$$\mathbf{P}_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_\pm = (1/2)\begin{bmatrix} 1 & \mp i & 0 \\ \pm i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

These $\mathbf{P}_\alpha$ are orthogonal projections onto the three eigenspaces that correspond to linear polarization in the direction of the geomagnetic field and the two orthogonal circular polarizations. The expression in (31) is called the "spectral decomposition" of the tensor operator **N**. Note, however, that despite its appearance, **N** is not Hermitean symmetric. The eigenvalues $N_o$, $2N_\pm$ are complex-valued and this destroys many of the properties one usually associates with similar entities.

The tensor **N** depends for its definition on the special vector $\mathbf{e}_o$, the unit vector in the direction of the geomagnetic field and the analysis in Section 3.1 introduced a second special vector $\mathbf{e}_z$, the unit vector in the direction of propagation. In both cases the x- and y-coordinates were perpendicular to the respective special directions but were otherwise arbitrary. To fix them, we now suppose that $\mathbf{e}_x$ is the unit vector in the direction of $\mathbf{e}_z \times \mathbf{e}_o$. Thus it is perpendicular to both longitudinal vectors. There will be two different y-coordinates that complete the respective right handed Cartesian coordinate systems.

In this way, we have constructed an "old" coordinate system with basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_o$) in which **N** is represented as in (18) or (31); and we have a "new" system with basis ($\mathbf{e}_x$, $\mathbf{e}_y'$, $\mathbf{e}_z$) in which we want to represent **N**. Let $\phi$ be the angle between the geomagnetic field and the direction of propagation--between $\mathbf{e}_o$ and $\mathbf{e}_z$. Then the rotation matrix, which gives the new coordinates in terms of the old, is

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{33}$$

and similarity transformations provide new representations of the projections $P_\alpha$:

$$P_o' = R\ P_o\ R^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin^2\phi & \sin\phi\ \cos\phi \\ 0 & \sin\phi\ \cos\phi & \cos^2\phi \end{bmatrix}$$

$$P_\pm' = R\ P_\pm\ R^{-1} = 1/2 \begin{bmatrix} 1 & \mp i\ \cos\phi & \pm i\ \sin\phi \\ \pm i\ \cos\phi & \cos^2\phi & -\sin\phi\ \cos\phi \\ \mp i\ \sin\phi & -\sin\phi\ \cos\phi & \sin^2\phi \end{bmatrix} \tag{34}$$

A corresponding representation for **N** follows from (31).

To complete the process of Section 3.1, we simply discard the third rows and third columns in the matrices of (34) to obtain the 2x2 matrices

$$Q_o^m = \begin{bmatrix} 0 & 0 \\ 0 & \sin^2\phi \end{bmatrix}, \qquad Q_\pm^m = 1/2 \begin{bmatrix} 1 & \mp i\cos\phi \\ \pm i\cos\phi & \cos^2\phi \end{bmatrix} \tag{35}$$

whence

$$\mathbf{N}^m = N_o Q_o^m + 2N_+ Q_+^m + 2N_- Q_-^m = \begin{bmatrix} N_+ + N_- & -i(N_+ - N_-)\cos\phi \\ i(N_+ - N_-)\cos\phi & N_o \sin^2\phi + (N_+ + N_-)\cos^2\phi \end{bmatrix} \tag{36}$$

Until now, we have treated the refractivity and its effects as associated with the magnetic H-vector. This is the physically natural approach, but it is probably not entirely satisfying to the engineer. To change to a direct analysis of the electric E-vector, we note that (30) refers to two-dimensional vectors in the xy-plane perpendicular to the direction of propagation and can be rewritten as

$$E = -Z_o\ \mathbf{K}\ \mathbf{H} \tag{37}$$

where **K** is the 2x2 matrix

$$\mathbf{K} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{38}$$

We apply a similarity transformation to obtain

$$\mathbf{N}^e = \mathbf{K}\,\mathbf{N}^m\,\mathbf{K}^{-1} = \begin{bmatrix} N_o \sin^2\phi + (N_+ + N_-)\cos^2\phi & -i(N_+ - N_-)\cos\phi \\ i(N_+ + N_-)\cos\phi & N_+ + N_- \end{bmatrix} \qquad (39)$$

and then the plane wave E-vector is given by

$$E(z) = \exp[ikz(I + \mathbf{N}^e)]\ E_o, \qquad (40)$$

where $E_o$ is the initial value.

### 3.3 Characteristic Waves

The computation of the exponential in (28) or (40) may be carried out using any of several techniques (see, e.g., Moler and Van Loan, 1978). For example, we could use the standard series expression to write

$$\exp(s\mathbf{A}) = I + (s/1!)\mathbf{A} + (s^2/2!)\mathbf{A}^2 + \ldots \qquad (41)$$

for any complex number s and any square matrix **A**. Although this series always converges, the calculations are tedious and subject to round-off error.

Another technique involves the spectral decomposition of the matrix. It provides a physical insight that other methods lack, and it involves fairly easy and usually robust computations. We first look for complex numbers $\rho$ (the "eigenvalues") and vectors **v** (the "corresponding eigenvectors"), which satisfy

$$\mathbf{A}\ \mathbf{v} = \rho\mathbf{v}\ . \qquad (42)$$

It will follow that

$$\mathbf{A}^n\mathbf{v} = \rho^n\mathbf{v} \qquad (43)$$

and

$$\exp(s\mathbf{A})\mathbf{v} = \exp(s\rho)\mathbf{v}\ . \qquad (44)$$

Thus, it is easy to compute how the exponential acts on these special vectors. To find how it acts on some other vector, one expands it into a linear combination of the eigenvectors and then applies (44) to each term.

In particular, consider the matrix $\mathbf{N}^e$. To solve the equivalent of (42) we first treat the scalar equation (the "characteristic equation")

$$\det(\rho I - \mathbf{N}^e) = 0 \quad . \tag{45}$$

Since these are 2x2 matrices, this equation is quadratic in $\rho$ and there should be two solutions $\rho_1$ and $\rho_2$. Given these numbers, it is fairly easy to find the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and then (40) becomes

$$\mathbf{v}_j(z) = \exp[ikz(1 + \rho_j)]\ \mathbf{v}_j \quad , \qquad j = 1, 2 \tag{46}$$

whenever the initial field equals an eigenvector. The vector functions $\mathbf{v}_j(z)$ are plane wave solutions to the original Maxwell's equations. They are called *characteristic waves* and they have the property that, while they may change in size and phase, they always retain their original appearance and orientation.

The two eigenvectors are linearly independent and for any initial field we may find complex numbers $E_1$ and $E_2$, so that

$$E_o = E_1\ \mathbf{v}_1 + E_2\ \mathbf{v}_2 \quad . \tag{47}$$

Then the exponential in (40) quickly becomes

$$E(z) = e^{ikz}[E_1 \exp(ikz\rho_1)\ \mathbf{v}_1 + E_2 \exp(ikz\rho_2)\ \mathbf{v}_2] \ , \tag{48}$$

so that the propagating vector field is now represented as a linear combination of the two characteristic waves.

As a general rule the eigenvalues $\rho_j$ have the same order of magnitude as N and have positive imaginary parts so that as z increases, E(z) decreases exponentially. Generally these imaginary parts differ, so that one of the two components in (48) decreases faster than the other. After some distance, it becomes relatively small and E(z) approaches the appearance of the other, more dominant characteristic wave.

It will probably also be true that the real parts of the eigenvalues differ. The two characteristic waves travel at different speeds and the phase relation between the two components in (48) varies continuously. What this usually means is that the ellipse of polarization appears to rotate in space as the wave progresses, thus exhibiting a "Faraday rotation."

There are several aids that may be used to compute eigenvalues and eigenvectors. For example, we have

$$\rho_1\rho_2 = \det(\mathbf{N}^e) = 4N_+N_-\cos^2\phi + N_o(N_++N_-)\sin^2\phi$$

and (49)

$$\rho_1 + \rho_2 = \mathrm{trace}(\mathbf{N}^e) = 2(N_+ + N_-) + (N_o - N_+ - N_-)\sin^2\phi \ ,$$

from which the two $\rho_j$ may be found. Let us suppose that $\rho$ is one of these two and that we seek the corresponding eigenvector $\mathbf{v}$. We suppose its components have the values $v_x$, $v_y$, so that the equation $\mathbf{N}^e\mathbf{v} = \rho\mathbf{v}$ becomes a set of two equations in these two unknowns. The second of these equations is

$$i(N_+ - N_-)\cos\phi \ v_x + (N_+ + N_-)v_y = \rho v_y \tag{50}$$

and one solution is

$$v_x = \rho - N_+ - N_- \qquad \text{and} \qquad v_y = i(N_+ - N_-)\cos\phi \ . \tag{51}$$

Since $\rho$ is an eigenvalue, it is guaranteed that the first equation is also satisfied. Of course, any scalar multiple of (51) will also be an eigenvector and the usual practice is to normalize so it has unit size.

There remains the problem of finding the numbers $E_1$, $E_2$ of (47). For this purpose, it should be pointed out that the two eigenvectors are usually not orthogonal. Because the $N_\alpha$ are complex-valued, the matrix $\mathbf{N}^e$ is not Hermitean symmetric and the usual theorems do not apply. It is best to simply treat (47) as two equations in the two unknowns and to employ a straightforward approach for the solutions.

A case of special interest occurs when $\phi = 0$. The solutions to (49) are $2N_+$ and $2N_-$, and when these are inserted into (51) we find the corresponding eigenvectors are, respectively, right circularly polarized and left circularly polarized. This agrees with (15) and this is because the direction of propagation is along the geomagnetic field.

When $\phi = \pi/2$, the eigenvalues are $N_o$ and $N_+ + N_-$, and the corresponding eigenvectors are linearly polarized with the E-vector pointing respectively along the x-axis and along the y-axis. For the first of these, the H-vector points along the y-axis which, we note, is now the direction of the geomagnetic field.

### 3.4 Polarization and Stokes Parameters

We have seen how the polarization of a field vector may change as it propagates through this medium. To describe this change and its engineering consequences, we need to be able to describe the polarization in a quantitative way.

When our analyses concern a complex field such as E, we are, of course, using a shorthand notation for something like $Re[\exp(i2\pi ft)E]$. As time increases through one cycle, this latter real vector describes an ellipse, which is then the "ellipse of polarization." An obvious way to describe that ellipse is to measure both its ellipticity and the angle its major axis makes with some reference direction and, perhaps, an indication of the sense of rotation around the ellipse. In degenerate cases, one speaks qualitatively of linear polarization and of right and left circular polarization.

One standard and universally applicable way to measure polarization is through the use of what are called the *Stokes parameters*. These are discussed in many texts (see e. g., Born and Wolf, 1959, especially Section 1.4); here we shall try only to summarize some of their attributes.

As in Section 3.2, let E lie in the x,y-plane and let $E_x$, $E_y$ be the complex-valued components. Then the four Stokes parameters, $g_o$, $g_1$, $g_2$, $g_3$, are real numbers given by

$$\begin{aligned} g_o &= |E_x|^2 + |E_y|^2 \\ g_1 &= |E_x|^2 - |E_y|^2 \\ g_2 &= 2\,Re[E_x^*\,E_y] \\ g_3 &= 2\,Im[E_x^*\,E_y] \end{aligned} \qquad (52)$$

or, in more compact form,

$$g_2 + ig_3 = 2\,E_x^*\,E_y,$$

where the star has been used to indicate the complex conjugate.

We first note that $g_o$ is positive and equals the total field strength. Then also we quickly find

$$g_o^2 = g_1^2 + g_2^2 + g_3^2 \quad , \tag{53}$$

so that in a three-dimensional space with $g_1$, $g_2$, $g_3$ axes, the Stokes parameters of a field vector lie on the surface of a sphere of radius $g_o$. This is the *Poincaré sphere* and provides an attractive geometric picture of the situation.

Given the Stokes parameters, we can write

$$\begin{aligned} E_x &= [(g_o + g_1)/2]^{\frac{1}{2}} \exp(i\psi) \ , \\ E_y &= [2(g_o + g_1)]^{-\frac{1}{2}}(g_2 + ig_3) \exp(i\psi) \ , \end{aligned} \tag{54}$$

where $\psi$ is an arbitrary phase angle. Thus not only does the vector E determine the Stokes parameters but also they in turn determine the vector--up to within that arbitrary phase angle. Since the absolute phase of the field is probably not measurable, the Stokes parameters seem to represent all the useful information for the field.

What relates the parameters directly to the ellipse of polarization is the representation of the Poincaré sphere in spherical coordinates

$$\begin{aligned} g_1 &= g_o \cos2\tau \cos2\delta \ , \\ g_2 &= g_o \cos2\tau \sin2\delta \ , \\ g_3 &= g_o \sin2\tau \ . \end{aligned} \tag{55}$$

It turns out that $\delta$ $(0 \leq \delta < \pi)$ is the angle between the major axis of the ellipse and the x-axis, while $\tan \tau = \pm b/a$ $(-\pi/4 \leq \tau \leq \pi/4)$, where a and b are the major and minor semiaxes and the sign is chosen according to the sense of rotation. On the sphere, then, the azimuth measures the tilt of the field while the declination measures the ellipticity.

Thus the four Stokes parameters provide the total field strength and a complete description of the polarization. Often, however, one wants to describe only the polarization, and for this one can use the "normalized Stokes parameters." These are obtained by normalizing the vector E so it has unit size or more directly by dividing all parameters by $g_o$. For the normalized Stokes parameters, $g_o$ is always 1 and the Poincaré sphere has unit radius. Treating this sphere as a globe, one sees immediately that the northern hemisphere and the North Pole correspond to right-hand polarization and right circular polarization, while the southern hemisphere and the South Pole correspond to left-hand polarization and to left circular polarization. The Equator corresponds to linear polarization with the "East Pole" at $(g_1,g_2,g_3) = (1,0,0)$ corresponding to polarization along the x-axis and $(-1,0,0)$ to polarization along the y-axis.

If **E** represents the electric field vector at the aperture of a receiving antenna, then we expect the voltage V at the antenna terminals to be a linear function of **E**. We may write

$$V = G\ \mathbf{E}\cdot\mathbf{P}^* \tag{56}$$

where G is the (voltage) gain of the antenna and **P** is a complex vector of unit size in the x,y-plane that would be called the "polarization of the antenna." For the two vectors **E**, **P** we can find the respective normalized Stokes parameters and plot them on the Poincaré sphere. It then turns out that

$$|V| = G\ |E|\ \cos(\Delta/2)\quad , \tag{57}$$

where $\Delta$ is the angle between the two plotted points. Thus maximum efficiency of reception occurs when **E** and **P** differ by only a multiplicative scalar, and V becomes 0 (the two vectors are "orthogonal") when they appear at opposite points on the Poincaré sphere.

There is an alternative way to measure polarization that has been especially championed by Beckmann (1968). Given the components $E_x$, $E_y$ as before, one defines the complex number

$$p = E_y/E_x\ . \tag{58}$$

From (54) we have immediately

$$p = (g_2 + ig_3)/(g_o + g_1) \tag{59}$$

and, when the Stokes parameters are normalized,

$$g_1 = (1 - |p|^2)/(1 + |p|^2) , \qquad g_2 + ig_3 = 2p/(1 + |p|^2) . \tag{60}$$

Thus, the one complex number p provides the same information as the three real normalized Stokes parameters. In particular, the real p-axis corresponds to linear polarization, the upper half-plane to right-hand polarization, and the lower half-plane to left-hand polarization. The point $p = i$ corresponds to right circular polarization, $p = -i$ to left circular polarization, $p = 0$ to linear polarization along the x-axis, and $p = \infty$ to linear polarization along the y-axis.

The advantage of Beckmann's notation is its simplicity and the fact that this seemingly complicated subject has been reduced to single number. The disadvantage is a certain loss of symmetry between small values of p (near where $g_1 = 1$) and large values (near $g_1 = -1$). Indeed, the fact that we need to introduce the point at infinity shows that we are really using the Riemann sphere of complex numbers; in fact the transformation (59) is simply a stereographic projection of the Poincaré sphere onto the complex plane.

### 3.5 Symmetries and Other Properties of the Characteristic Waves

There are several properties of the characteristic waves that should be mentioned either because they help us understand results or because they are useful in simplifying engineering calculations. We shall simply list them here, leaving their derivation as exercises.

As in Section 3.3, we begin with the tensor operator $\mathbf{N}^e$. We suppose it has eigenvalues, $\rho_1$, $\rho_2$, and corresponding eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$.

*Property 1.* The eigenvalues $\rho_1$, $\rho_2$ have positive imaginary parts, thus ensuring that the characteristic waves are attenuated with distance. This follows from the fact that the $N_\alpha$ have positive imaginary parts and that the $\mathbf{Q}_\alpha^m$ of (35) are positive semidefinite Hermitian symmetric matrices.

*Property 2.* The two eigenvectors satisfy

$$v_{1x}v_{2x} - v_{1y}v_{2y} = 0 \tag{61}$$

This is an analog to the orthogonal relation satisfied by eigenvectors of a symmetric matrix. It comes about because the two off-diagonal elements of $\mathbf{N}^e$ are negatives of each other.

*Property 3.* A corollary to Property 2 is that if $\mathbf{v} = (v_x, v_y)$ is an eigenvector, then the second eigenvector can be represented by $(v_y, v_x)$.

*Property 4.* A corollary to Property 3 is that if an eigenvector has normalized Stokes parameters $(g_1, g_2, g_3)$ then the second eigenvector has the normalized Stokes parameters $(-g_1, g_2, -g_3)$. Thus $g_2$ remains fixed while the other two parameters change sign. The two eigenvectors are orthogonal if and only if $g_2 = 0$.

*Property 5.* Recall that $\phi$ is the angle between the geomagnetic field and the direction of propagation. Let us replace it by $\phi' = \pi - \phi$ and then consider the resultant eigenvalues $\rho'$ and eigenvectors $\mathbf{v}'$. Since the sine of this angle remains the same while the cosine changes sign, the effect on $\mathbf{N}^e$ will be only to change the signs of the two off-diagonal elements. Thus the trace and determinant are unchanged so $\rho_j' = \rho_j$ and

$$\mathbf{v}_j' = (-v_{jx}, v_{jy}) \ . \tag{62}$$

If $(g_1, g_2, g_3)$ are the normalized Stokes parameters for one of the original eigenvectors, then $(g_1, -g_2, -g_3)$ are normalized Stokes parameters for the corresponding new eigenvector.

*Property 6.* A corollary to Property 5 is that propagation in this medium does not satisfy the laws of reciprocity. If we reverse the direction of propagation, we change the angle $\phi$ to $\pi - \phi$, and although the eigenvalues remain unchanged, the eigenvectors do not. An initial field vector is resolved into different components, thus leading to different results.

Actually, we must be cautious in making these statements because another effect here is to change the coordinate system. In reversing the direction of propagation, we have replaced $\mathbf{e}_z$ by $\mathbf{e}_z' = -\mathbf{e}_z$. It will then follow that $\mathbf{e}_x' = -\mathbf{e}_x$ and $\mathbf{e}_y' = \mathbf{e}_y$, and that therefore (62) may be written $\mathbf{v}_j' = \mathbf{v}_j$. As vectors showing magnitude and direction in three-dimensional space, the eigenvectors are also unchanged.

Nevertheless, because the direction of propagation is an important part of the definition of polarization, our original statements are still valid. Consider, for example, two right-handed helical antennas pointed at each other along a line parallel to the geomagnetic field. In the direction of the field, radio waves are attenuated at a rate proportional to $Im[N_+]$. In the opposite direction, the rate of attenuation is proportional to $Im[N_-]$.

*Property 7.* Let $\Delta f$ be the deviation of the frequency from the unsplit line center frequency; in the notation of Section 2,

$$\Delta f = f - \nu_o \ . \tag{63}$$

The refractivities $N_\alpha$ and all consequent objects may then be treated as functions of $\Delta f$. From (12) and Table 3, it may be shown that

$$N_o(-\Delta f) = -N_o(\Delta f)^*, \quad N_+(-\Delta f) = -N_-(\Delta f)^*, \quad N_-(-\Delta f) = -N_+(\Delta f)^*. \tag{64}$$

Let $\rho_j$ be the eigenvalues and $\mathbf{v}_j$ the corresponding eigenvectors when the frequency deviation has the value $\Delta f$, and consider the case when the frequency deviation equals $-\Delta f$. It will turn out that now $-\rho_j^*$ are the eigenvalues and that $v_j^*$ are the corresponding eigenvectors.

Note that we cannot state an equality between particular eigenvalues but only between sets of the two eigenvalues. For example, consider the case when $\Delta f = 0$. Then we cannot say $\rho_1 = -\rho_1^*$. We can only say that either this is true or that $\rho_1 = -\rho_2^*$.

*Property 8.* To find simpler formulas for the eigenvalues and eigenvectors, we write

$$\mathbf{N}^e = (N_+ + N_-)I + (N_+ - N_-)\mathbf{S}, \tag{65}$$

where

$$\mathbf{S} = \begin{bmatrix} 2s \sin^2\phi & -i\cos\phi \\ i\cos\phi & 0 \end{bmatrix} \tag{66}$$

and

$$s = \frac{N_o - N_+ - N_-}{2(N_+ - N_-)} \ .$$

Let $\sigma_1$, $\sigma_2$ be the two eigenvalues of $\mathbf{S}$ and let $\mathbf{v}_1$, $\mathbf{v}_2$ be the corresponding eigenvectors. Then the $\mathbf{v}_j$ are also eigenvectors of $\mathbf{N}^e$ and they correspond to the eigenvalues

$$\rho_j = (N_+ + N_-) + (N_+ - N_-)\sigma_j. \tag{67}$$

We have

$$\sigma_1\sigma_2 = \det(\mathbf{S}) = -\cos^2\phi \qquad (68)$$
$$\sigma_1+\sigma_2 = \mathrm{trace}(\mathbf{S}) = 2s\ \sin^2\phi$$

and, for example,

$$\mathbf{v}_j = \begin{bmatrix} \sigma_j \\ i\cos\phi \end{bmatrix} \qquad (69)$$

Of course, it is still true that $\mathbf{v}_2$ can be obtained by interchanging the components of $\mathbf{v}_1$.

*Property 9.* Until now we have implied that there are always two different, linearly independent eigenvectors so that, for example, (47) always has a solution. As it turns out, this is not true. To see how this might happen, we first look for conditions when the eigenvalues are equal.

The discriminant involved in solving (68) for the eigenvalues of **S** has the form

$$(\sigma_1-\sigma_2)^2/4 = s^2\sin^4\phi + \cos^2\phi \qquad (70)$$

and this vanishes if

$$\frac{\cos\phi}{\sin^2\phi} = \pm is. \qquad (71)$$

The left side here is real and so a solution can exist only if s is pure imaginary. Let us assume this condition is satisfied and then let $\phi_o$ be the unique solution to (71) for which $0 < \phi_o < \pi/2$. The solution with the opposite sign will have the angle $\pi - \phi_o$. For the first solution, the resultant single eigenvalue is

$$\sigma = s\ \sin^2\phi_o = \pm i\cos\ \phi_o \qquad (72)$$

and there is only one eigenvector,

$$\mathbf{v} = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix} \qquad (73)$$

where in both equations, the ambiguous sign equals the sign of the imaginary part of s. Note that the eigenvector is linearly polarized and is tilted 45° to both the x- and y-axes.

We are still left with the question of whether s can be pure imaginary. The quantity s as defined in (66) is a function of pressure, temperature, and particularly of the frequency deviation $\Delta f$. It would then not seem surprising to find that

$$Re[s(\Delta f)] = 0 \tag{74}$$

has one or more solutions. Indeed, from (64) there follows

$$s(-\Delta f) = -s(\Delta f)^* \tag{75}$$

so that $\Delta f = 0$ is always one such solution. As it turns out, however, there are (depending on pressure, temperature, and line number) quite likely to be additional solutions.

To summarize, we first solve (74) for frequency deviations $\Delta f$ and then (71) for particular angles $\phi_o$. For such special pairs of frequency and propagation direction, the problem of characteristic waves becomes degenerate. We can still evaluate the exponential in (40) but the process must be somewhat different.

*Property 10.* Let us consider the eigenvalues and eigenvectors as functions of the angle $\phi$, all other parameters being held constant. When $\phi$ = 0, the two eigenvalues of S are $\sigma = \pm 1$. For small $\phi$ we can expand these functions in powers of $\sin\phi$ and we find

$$\begin{aligned} \sigma_1 &= 1 + (s-1/2)\sin^2\phi + \ldots \\ \sigma_2 &= -1 + (s+1/2)\sin^2\phi + \ldots \end{aligned} \tag{76}$$

Corresponding eigenvectors are given by (69) and the normalized Stokes parameters of, say, $v_1$ are

$$\begin{aligned} g_1 + ig_2 &= s\ \sin^2\phi + \ldots \\ g_3 &= 1 - (1/2)|s|^2\sin^4\phi + \ldots . \end{aligned} \tag{77}$$