4.4 Coherent Threshold Detection

As we have seen, even in the simplest coherent signaling cases, the optimum detector is quite difficult to realize physically, even though we were able to compute a bound on its performance. For this reason most of the work in the past has been to obtain a detector which approaches the true optimum Bayes detector when threshold operations are used, that is, when the desired signal becomes vanishingly small and the number of independent samples becomes large. In this section we will derive two such detectors, one for purely coherent signals and one for the interference discrimination problem. Such detectors are generally termed threshold receivers or locally optimum Bayes' detectors (LOBD).

4.4.1 Binary Purely Coherent Signals

In this section we will derive the threshold receiver for binary purely coherent signals and discuss under what conditions it is asymptotically optimum.

This problem, for arbitrary additive interference, was apparently first attacked by Middleton (1954, 1960), who showed that the optimum threshold receiver must be a nonlinear processor. Later Rudnick (1961) examined the same problem. Using a power series expansion technique similar to Middleton's, he obtained a closed form for the nonlinear optimum receiver and showed, in addition, that it must be adaptive. Later canonical results have been obtained by Middleton (1966), Antonov (1967), Levin and Kushmir (1969), and Levin and Shinakov (1970).

Our problem is to decide optimally between the two hypothesis

$$H_1: \quad X(t) = S_1(t) + Z(t), \ 0 \le t \le T$$

$$H_2: \quad X(t) = S_2(t) + Z(t), \ 0 \le t \le T \quad . \tag{4.52}$$

The likelihood ratio is

$$\Lambda(\underline{X}) = \frac{p(\underline{X}|H_2)}{p(\underline{X}|H_1)} = \frac{p_Z(\underline{X} - \underline{S}_2)}{p_Z(\underline{X} - \underline{S}_1)} \underset{H_2}{\overset{H_1}{\lessgtr}} 1 \quad . \tag{4.53}$$

Using a vector Taylor expansion, we can write

$$p_Z(\underline{X} - \underline{S}_j) = p_Z(\underline{X}) - \sum_{i=1}^{N} \frac{\partial p_Z(\underline{X})}{\partial x_i} s_{ji}$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{\partial^2 p_Z(\underline{X})}{\partial x_i \partial x_k} s_{ji} s_{jk} + \ldots \quad . \tag{4.54}$$

For coherent signals, the normal small signal assumption is

$$p_Z(\underline{X} - \underline{S}_j) \approx p_Z(\underline{X}) - \sum_{i=1}^{N} \frac{\partial p_Z(\underline{X})}{\partial x_i} s_{ji} \quad , \tag{4.55}$$

where we have ignored all signal terms of degree 2 and higher. [This small-signal assumption is really intuitive, and one needs to show that the higher order terms are not required for asymptotic optimality.]

Therefore, we obtain

$$\Lambda(\underline{X}) \approx \frac{p_Z(\underline{X}) - \sum_{i=1}^{N} \frac{\partial p_Z(\underline{X})}{\partial x_i} s_{2i}}{p_Z(\underline{X}) - \sum_{i=1}^{N} \frac{\partial p_Z(\underline{X})}{\partial x_i} s_{1i}} \underset{H_2}{\overset{H_1}{\lessgtr}} 1 \ . \qquad (4.56)$$

Now dividing the numerator and denominator by $P_Z(\underline{X})$, and assuming independent noise samples, we get

$$\Lambda(X) \approx \frac{1 - \sum_{i=1}^{N} \frac{d}{dx_i} \ell n \; p_Z(x_i) \; s_{2i}}{1 - \sum_{i=1}^{N} \frac{d}{dx_i} \ell n \; p_Z(x_i) \; s_{1i}} \underset{H_2}{\overset{H_1}{\lessgtr}} 1 \ . \qquad (4.57)$$

Since the additive constant, 1, will not effect the test, a physical implementation of (4.57) is given by figure 4.13 below:

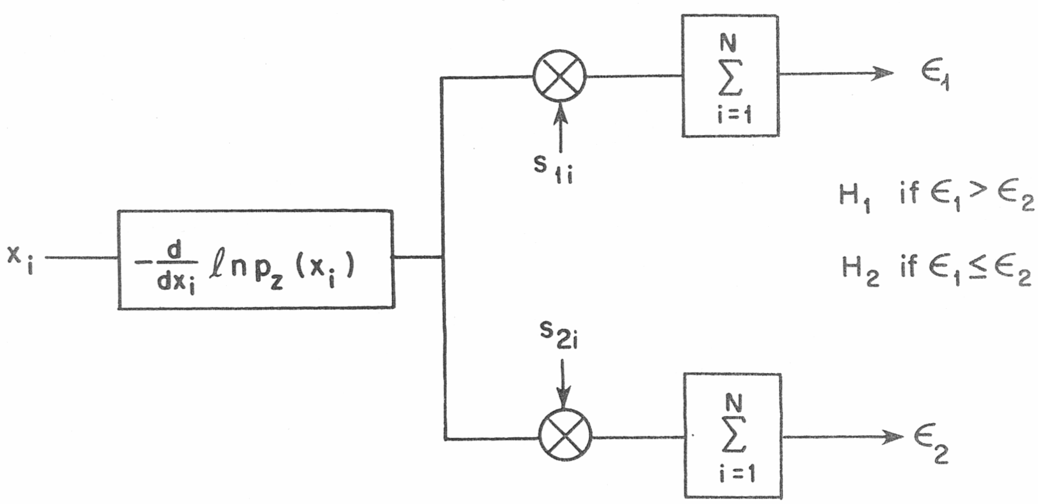

Figure 4.13. Threshold receiver for purely coherent signals.

We see that this is simply the standard coherent receiver for Gaussian noise preceeded by a logarithmic nonlinearity. This same result has been obtained by Algazi and Lerner (1964) and Nirenberg (1975) as well as others. The general result for coherent signals is that in order to choose the most likely signal from among the ensemble of m possible received signals, the optimum threshold receiver is the standard Gaussian receiver preceeded by the above nonlinearity. In chapter 5, we will obtain similar results for some forms of incoherent reception. Note that this receiver must be adaptive; i.e., required to adjust itself for changing interference conditions. It must know, a priori, or estimate $p_Z(x)$. Some approximations to this receiver have been built and tested (Bernstein and McNeill, 1973) for use where the impulsive interference is atmospheric noise (an example of class B interference).

For our class A interference, figure 4.14 shows the required nonlinearity, $-d/dx \ln p_Z(x)$, for our two sample cases, $A = 0.35$, $\Gamma' = 0.5 \times 10^{-3}$, and $A = 0.1$, $\Gamma' = 10^{-4}$. We see that it would be rather difficult to implement physically. [We will denote this nonlinearity by $-\ell(x)$.]

While the nonlinearity does not "Gaussianize" the interference, it does suppress the large "spikes." Since N is large, the signal is small, and the interference has been reduced in its amplitude excursions (fig. 4.14), performance can be estimated via the central limit theorem. Our previous

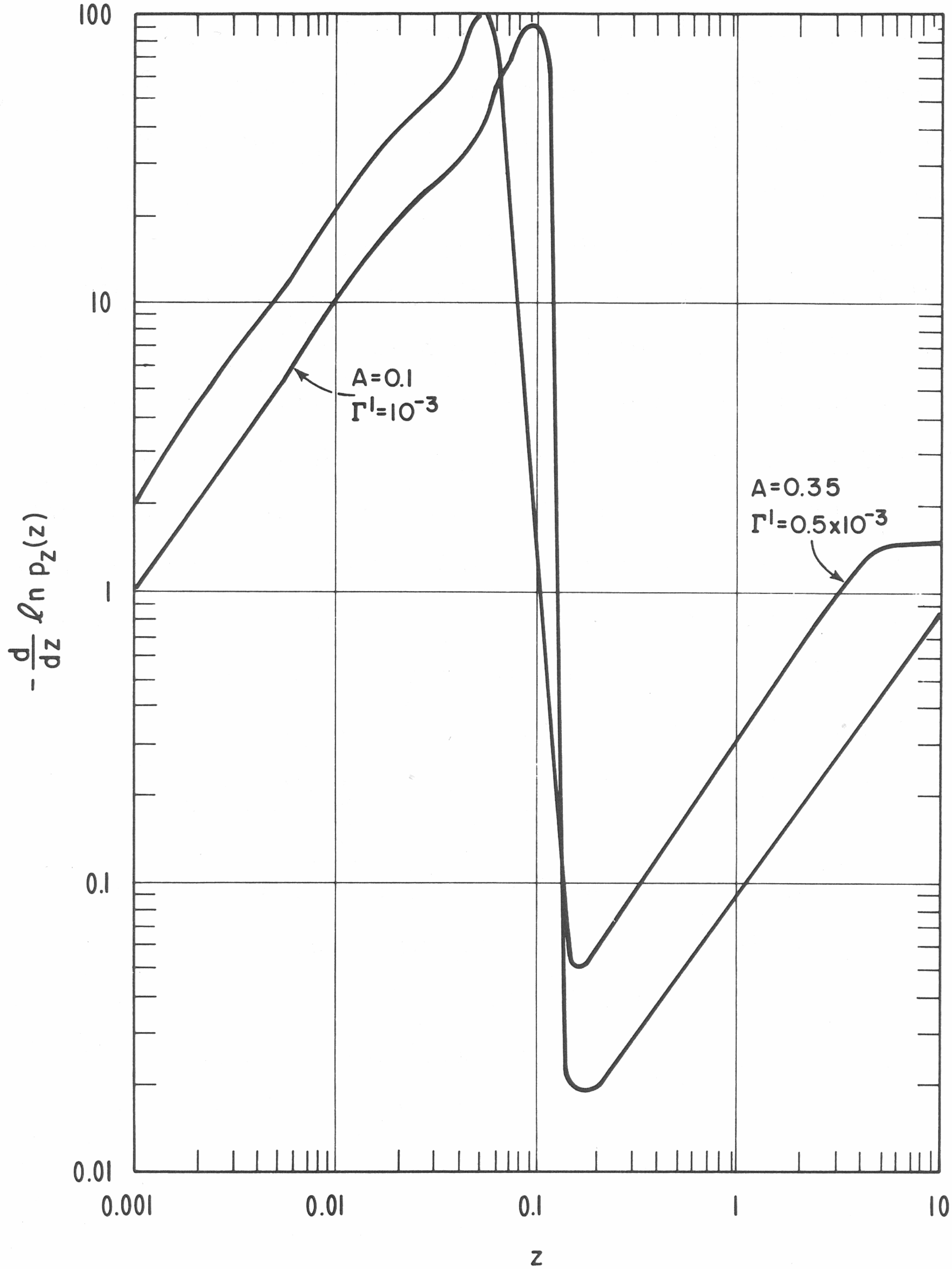

Figure 4.14. The nonlinearity, -d/dz ln $p_Z(z)$, for the two example cases of class A interference.

bound was quite good only for small P_e. Analyzing the performance of the above suboptimum detector will, therefore, give an estimate of performance of the optimum detector for small signal and large N.

We start by rewriting the test (4.57),

$$\sum_{i=1}^{N} \frac{d}{dx_i} \ln p_Z(x_i)(s_{1i} - s_{2i}) \underset{H_2}{\overset{H_1}{\gtrless}} 0 \quad , \tag{4.58}$$

so that our receiver is reduced to the following:

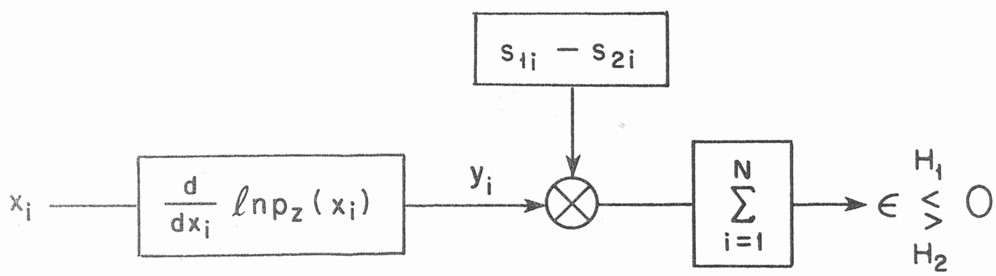

Figure 4.15. Another form of the threshold receiver for purely coherent signals.

We consider, for example, the antipodal signals, so that

$$s_{1i} - s_{2i} = 2\sqrt{2S} \cos \omega_o t_i \quad . \tag{4.59}$$

Suppose H_1 is true; $X(t) = S_1(t) + Z(t)$, then

$$E[y_i|H_1] = \int_{-\infty}^{\infty} y_i \; p(y_i) \; dy_i = \int_{-\infty}^{\infty} dx \; \ln p_Z(x) \; p_Z(x-s_{1i}) \; dx$$

$$= \int_{-\infty}^{\infty} \frac{p_Z'(x)}{p_Z(x)} \{p_Z(x) - s_{1i} p_Z'(x)\} \; dx \quad , \qquad (4.60)$$

where the small signal assumption (4.55) has again been used and y_i is the output of the nonlinearity for the input x_i. Since $p_Z'(x) = -p_Z'(-x)$; i.e., $p_Z(x)$ is symmetric about $x = 0$, (4.60) reduces to

$$E(y_i|H_1) = -s_{1i} \; L \quad , \qquad (4.61)$$

where

$$L = \int_{-\infty}^{\infty} \left[\frac{d}{dx} \ln p_Z(x)\right]^2 p_Z(x) \; dx \quad . \qquad (4.62)$$

Also,

$$E[y_i^2|H_1] = \int_{-\infty}^{\infty} \left[\frac{d}{dx} \ln p_Z(x)\right]^2 \{p_Z(x) - s_i p_Z'(x)\} dx \; . \qquad (4.63)$$

This gives us, since $p_Z(x)$ is even and $p_Z'(x)$ is odd,

$$E[y_i^2|H_1] = L \quad . \qquad (4.64)$$

Therefore,

$$Var[y_i|H_1] = L - s_{1i}^2 L^2 \quad . \qquad (4.65)$$

We obtain, then, after multiplying by $s_{1i}-s_{2i}$ and summing over the N samples, for the decision viable ε (cf. fig. 4.15),

$$E[\varepsilon|H_1] = L \sum_{i=1}^{N} (-s_{1i}^2 + s_{1i}s_{2i}) \quad ,$$

and (4.66)

$$\mathrm{Var}[\varepsilon|H_1] = \sum_{i=1}^{N} (s_{1i} - s_{wi})^2(L - s_{1i}^2 L^2) \quad .$$

For our antipodal signaling case, this gives us

$$E[\varepsilon|H_1] = -4SL \sum_{i=1}^{N} \cos^2 \omega_o t_i$$

$$\approx -2SLN \quad ,$$

and (4.67)

$$\mathrm{Var}[\varepsilon|H_1] = \sum_{i=1}^{N} (8SL \cos^2\omega_o t_i - 16S^2L^2\cos^4\omega_o t_i)$$

$$\approx 4SLN - 6S^2L^2N \quad .$$

In similar manner, we obtain

$$E[\varepsilon|H_2] = -E[\varepsilon|H_1] \quad ,$$

and (4.68)

$$\mathrm{Var}[\varepsilon|H_2] = \mathrm{Var}[\varepsilon|H_1] \quad .$$

Therefore, via the central limit theorem, a performance estimate for small signal and large N is

$$P_e = \mathrm{Prob}[\varepsilon>0] = \frac{1}{2}\,\mathrm{erfc}\left\{\frac{|E[\varepsilon]|}{\sqrt{2\mathrm{Var}[\varepsilon]}}\right\} \quad ,$$

$$\approx \frac{1}{2}\,\mathrm{erfc}\,(\sqrt{SLN/2}),\ (SL << 1) \quad . \qquad (4.69)$$

We see that, first of all, S must be quite small in order for the variance to be positive. This results from the small signal assumption, especially as used in (4.63). Performance is a function of S, N, and the parameter L, where L is given by the integral (4.62). For our two example cases, the integrand in (4.62) is not particularly well behaved. Numerical integration of (4.62) gives for $A = 0.1$ and $\Gamma' = 10^3$,

$$L = 892.7 \quad , \tag{4.70}$$

and for $A = 0.35$ and $\Gamma' = 0.5 \times 10^{-3}$,

$$L = 1340 \quad .$$

The fact that L is large further shows that S must be quite small and N large, as (4.67), for example, shows.

Figure 4.16 shows the estimated performance for the threshold receiver, from (4.67) and (4.69) along with the upper bound on the performance of the optimum detector calculated earlier (from table 4.1) for our case $A = 0.35$. $\Gamma' = 0.5 \times 10^{-3}$.

As Levin et al. (1970) have pointed out, reliable detection when the signal is weak requires us to increase the sample size N. This leads to an increase in the effect of terms of higher order which are discarded in the synthesis of locally optimal detectors. That is, for a given small signal level, the locally optimal detectors become increasingly suboptimum as N increases. When such threshold devices are used in practice, attempts to overcome this problem generally take the form of coding. An information bit is represented by a large code word (usually a pseudo-random word). A decision is made on each element of the code word (usually termed a "chip"), thereby keeping N small and the detector near optimum. The resulting P_e for each chip is, of course, large, and an acceptable low P_e for the information bit is obtained then by the power of the coding.

In order to determine the effectiveness of such schemes, we need some useful criteria to compare the performance of the optimum and the locally optimum detectors, particularly in the limiting situations of large sample size. The concepts of Relative Efficiency (RE) and Asymptotic Relative Efficiency (ARE) as proposed by Pitman (1948) and specified in detail by Capon (1961) and Middleton (1965, 1966) as well as others, provide the needed basis of comparison.

Middleton's generalized definitions are, for our case: The Efficiency of the test H_1 vs. H_2 by a system y with non-zero input signal (S > 0) and finite sample size N is, for a fixed P_e,

$$\epsilon_y(N,S) = \left(\frac{\bar{y}_1}{\sigma_1} - \frac{\bar{y}_2}{\sigma_2}\right)^2 \quad , \tag{4.71}$$

where $\bar{y}_1$ denotes the mean of the test statistic y under hypothesis H_1, σ_1 the corresponding standard deviation, etc.

Relation (4.67) allows us to calculate this quantity for our locally optimum detector.

The <u>Relative Efficiency</u> (RE) of the test H_1 vs. H_2 by system y_1 vs. that by system y_2 is

$$\epsilon_{y_1|y_2}(N_1,N_2,S) = \frac{\epsilon y_1(N_1,S)}{\epsilon y_2(N_2,S)} \quad , \tag{4.72}$$

where N_i is the sample size for system y_i for some required level of performance.

The <u>Asymptotic Relative Efficiency</u> (ARE) is, for given S, given by

$$\lim_{N_1,N_2 \to \infty} \epsilon_{y_1|y_2}(N_1,N_2,S) \quad . \tag{4.73}$$

If $\epsilon_{y_1|y_2}(N_1,N_2,S) \leq 1$, then y_2 is said to be more efficient than y_1, and if the ARE ≤ 1, then y_2 is said to be asymptotically more efficient than y_1.

In our case, a more useful and equivalent relation is simply

$$\epsilon_{y_1|y_2}(N_1,N_2,S) = \frac{N_2}{N_1} \quad . \tag{4.74}$$

That is, the RE simply tells us, for a given small signal size S, and required level of performance P_e, how many more samples (N_1) system y_1 requires than the more efficient system y_2 (Levin and Kushnir, 1970).

We have developed the means to obtain a good estimate of N_1 for any small signal size and required P_e for the locally optimum detector via (4.69). Obtaining N_2 (for the optimum detector), however, is no easy matter, since all we have is an upper bound expression, which is likely to be quite "loose" for small signal levels. However, we can apply a central limit theorem argument to this upper bound to obtain an estimate of performance for the optimum detector, especially for small signals.

Following Gallager (1968, appendix 5A) and Van Trees (1968, sec. 2.7), we have the following: Let

$$\mu(\alpha) = \ln \int_{-\infty}^{\infty} [p(\underline{X}|H_1)]^{\alpha} \, [p(\underline{X}|H_2]^{1-\alpha} \, d\underline{X} \quad , \qquad (4.75)$$

so that our bound, obtained earlier, is

$$P_e \le \frac{1}{2} e^{\mu(\alpha^*)} \quad , \qquad (4.76)$$

where α^* is the value of α for which $\ddot{\mu}(\alpha) = 0$. In our case α^* was equal to $\frac{1}{2}$. Then an estimate on performance, based on the central limit theorem is given by (Van Trees, 1968, sec. 2.7)

$$P_e \approx \frac{1}{4} \exp[\mu(\alpha^*) + \frac{(\alpha^*)^2}{2} \ddot{\mu}(\alpha^*)] \operatorname{erfc}\left[\frac{\alpha^* \sqrt{\ddot{\mu}(\alpha^*)}}{\sqrt{2}}\right]$$

$$+ \frac{1}{4} \exp[\mu(\alpha^*) + \frac{(1-\alpha^*)^2}{2} \ddot{\mu}(\alpha^*)] \operatorname{erfc}\left[\frac{(1-\alpha^*) \sqrt{\ddot{\mu}(\alpha^*)}}{\sqrt{2}}\right] . \qquad (4.77)$$

While (4.77) can be used then to obtain an estimate of N_2, it requires the computation of $\ddot{\mu}(\alpha^*)$. Direct differentiation of $\mu(\alpha)$ results in indeterminate forms for $\ddot{\mu}(\alpha^*)$ for our case. Even so, the numerical evaluation of $\ddot{\mu}(\alpha^*)$, while rather involved, is straightforward.

Since using (4.77) to determine N_2 and (4.69) to determine N_1 for use in (4.74) gives us only estimates (we hope good) of the required sample sizes, with no guarantee that we would thereby obtain a good estimate of $\epsilon_{y_1|y_2}$, the numerical exercise of evaluating $\ddot{\mu}(\alpha^*)$ for our example cases is not carried out here.

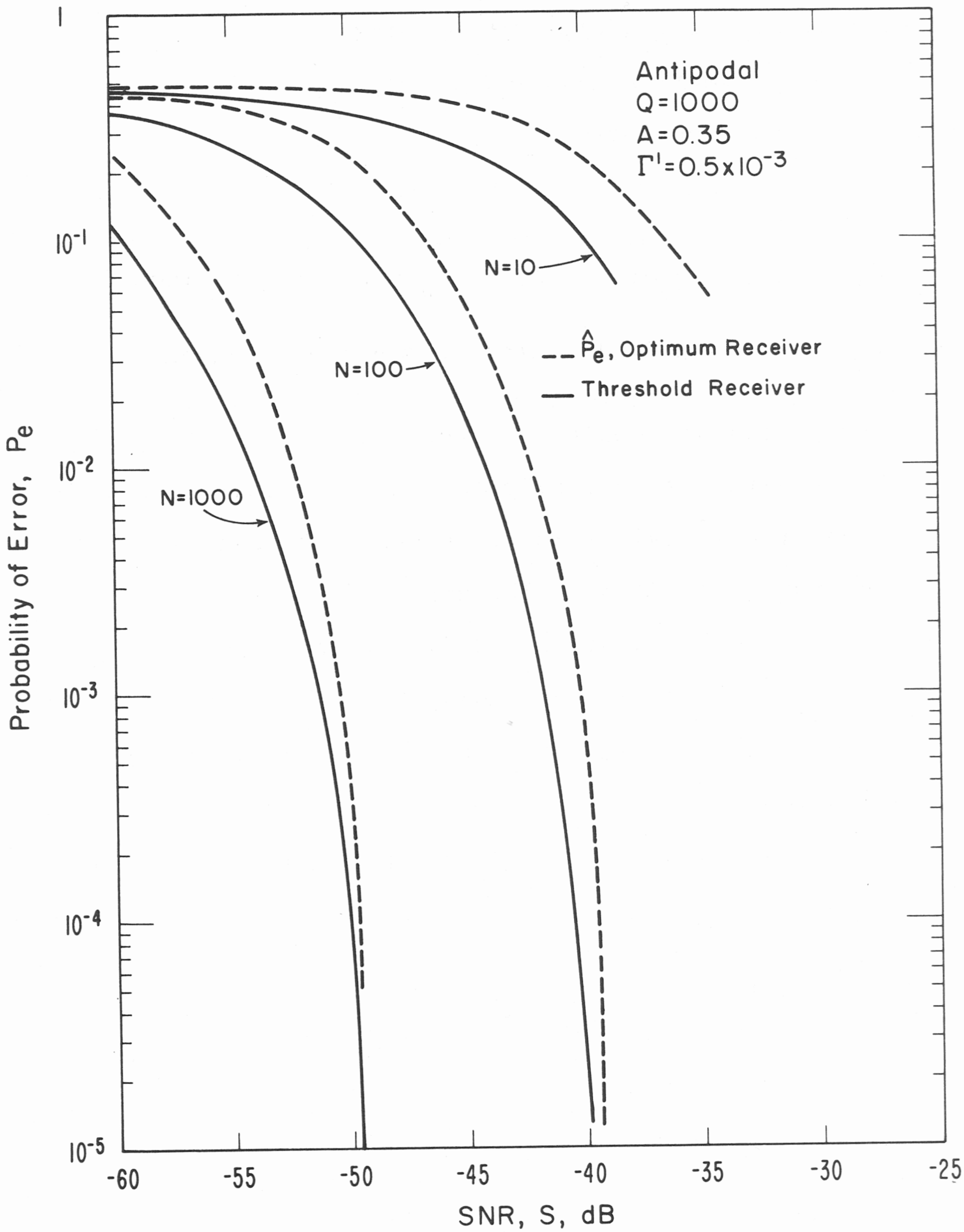

Figure 4.16. Comparison of the estimate on performance for small signal (4.69) with the upper bound on performance (4.44) for Q = 1000, N = 10, 100, and 1000 for antipodal signals.

4.4.2 Interference Discrimination

Another coherent problem of current interest is the interference discrimination problem. Consider the two hypotheses:

$$
\begin{aligned}
H_1: \quad & X(t) = b\,S(t) + Z(t),\ b_o \le b < \infty, \text{ not jammed} \\
H_2: \quad & X(t) = b\,S(t) + Z(t),\ 0 \le b < b_o, \text{ jammed}
\end{aligned}
\tag{4.78}
$$

The signal S(t) is completely known except for its amplitude b. The receiver is optimally to make the binary decision as to whether the signal is large enough to be useful ($b_o \le b < \infty$) or not ($0 \le b \le b_o$). The amplitude b is random, but fixed over a decision period [0,T] with some pdf and b_o is a subjectively determined threshold above which the signal is considered to be useful; that is, large enough so that information can be extracted.

The likelihood ratio for this problem is

$$
\Lambda(\underline{X}) = \frac{\int_0^{b_o} p_Z(\underline{X}-b\underline{S})\,p(b)\,db}{\int_{b_o}^{\infty} p_Z(\underline{X}-b\underline{S})\,p(b)\,db} \ \mathop{\gtrless}_{H_2}^{H_1}\ K \ , \tag{4.79}
$$

where

$$
p_Z(\underline{X} - b\underline{S}) = \prod_{i=1}^{N} p_Z(x_i - bs_i) \quad . \tag{4.80}
$$

Via the above procedures, we find, using only first order terms, that the corresponding threshold (LOBD) receiver is

$$\Lambda(\underline{X}) = \frac{-\text{Prob}[b<b_o] + \sum_{i=1}^{N} \ell(x_i)\, s_i \int_{o}^{b_o} bp(b)\, db}{-\text{Prob}[b>b_o] + \sum_{i=1}^{N} \ell(x_i)\, s_i \int_{b_o}^{\infty} bp(b)\, db} \overset{H_1}{\underset{H_2}{\lessgtr}} K \quad , \tag{4.81}$$

or, in block diagram form:

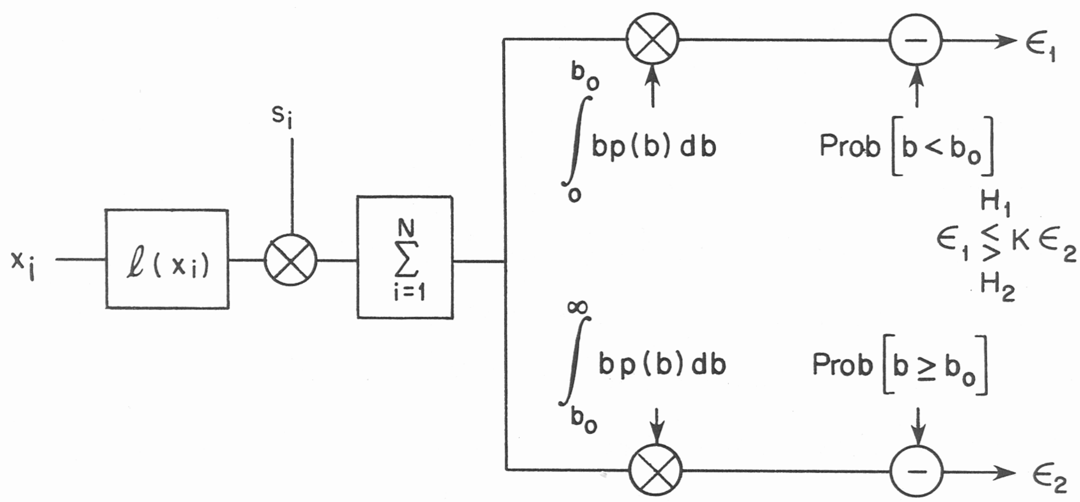

Figure 4.17. LOBD for the coherent interference discrimination problem.

As previously, performance can be analyzed via the central limit theorem. If $y_i = \ell(x_i)\, s_i$, then

$$E[y_i|H_1] = \int_o^{b_o} \int_{-\infty}^{\infty} \ell(x_i)\, s_i\, p_Z(x_i - bs_i)\, p(b)\, dx_i\, db. \tag{4.82}$$

Or, using the small signal assumption (4.55), we get

$$E[y_i|H_1] = -s_i^2\, L \int_o^{b_o} bp(b)\, db \quad . \tag{4.83}$$

Similarly,

$$E[y_i^2|H_1] = s_i^2 \ L \ \mathrm{Prob}[b < b_o] \quad , \tag{4.84}$$

so that

$$\mathrm{Var}[y_i^2|H_1] = s_i^2 \ L \ \mathrm{Prob}[b < b_o] - L^2 s_i^4 \left[\int_o^{b_o} bp(b)\ db\right]^2 \quad . \tag{4.85}$$

Let

$$\mathrm{Prob}[b < b_o] = P_o^- \quad ,$$

$$\mathrm{Prob}[b > b_o] = P_o^+ \quad ,$$

$$\int_o^{b_o} bp(b)\ db = B_o^- \quad , \text{ and}$$

$$\int_{b_o}^{\infty} bp(b)\ db = B_o^+ \quad . \tag{4.86}$$

Then, we have ε_1 normally distributed with

$$E[\varepsilon_1|H_1] = -L(B_o^-)^2 \sum_{i=1}^{N} s_i^2 - P_o^- \quad ,$$

and (4.87)

$$\mathrm{Var}[\varepsilon_1|H_2] = P_o^-(B_o^-)^2 L \sum_{i=1}^{N} s_i^2 - (B_o^-)^4 \ L^2 \sum_{i=1}^{N} s_i^4 \quad .$$

Similarly, we have ε_2 normally distributed with

$$E[\varepsilon_2 | H_1] = -L\, B_o^- B_o^+ \sum_{i=1}^{N} s_i^2 - P_o^+ \quad ,$$

and (4.88)

$$\mathrm{Var}[\varepsilon_2 | H_1] = P_o^- (B_o^+)^2 L \sum_{i=1}^{N} s_i^2 - (B_o^-)\,(B_o^+)^2 L^2 \sum_{i=1}^{N} s_i^4 \quad .$$

Now, we obtain

$$P_e | H_1 = \mathrm{Prob}[\varepsilon_1 | H_1 > K\varepsilon_2 | H_1] \quad , \qquad (4.89)$$

with corresponding results for $P_e | H_2$. Our performance can now be estimated for the receiver in figure 4.17, once p(b) and b_o are given so that the constants P_o^-, P_o^+, B_o^-, and B_o^+ can be evaluated.

The above coherent interference discrimination problem is one of composite hypothesis testing, requiring an averaging operation in the likelihood ratio. In section 5, where incoherent detection is treated, we will indicate a technique that, under some circumstances at least, can be used to obtain more general results than those above for the coherent (or incoherent) interference discrimination problem (sec. 5.2).