4.2 Performance Calculations for Optimum Coherent Receivers

In this section we return to (4.6) to compute a bound on the performance of the purely coherent optimum detector. This performance then applies to whatever test statistic is used; i.e., (4.6) or (4.7) or the series algorithm.

Hellman and Raviv (1970) made the elegant observation that the average probability of error is given by

$$P_e = \displaystyle{\int\!\!\!\!\!-}_{\underline{X}} \min\,\{q_1 p(\underline{X}|H_1),\ q_2 p(\underline{X}|H_2)\}\ d\underline{X} \quad , \qquad (4.15)$$

where the integral sign with a bar denotes a multidimensional integral (N dimensions). The above cannot be attacked directly for large N because of the high dimensionality and because the regions of integration are quite awkward. However, we can use the inequality

$$\min\{a,b\} \le a^{\alpha} b^{1-\alpha} \quad , \qquad (4.16)$$

where $0 \le \alpha \le 1$, $a \ge 0$, $b \ge 0$, to obtain

$$P_e \le q_1^{\alpha}\, q_2^{\,1-\alpha} \displaystyle{\int\!\!\!\!\!-}_{\underline{X}} [p(\underline{X}|H_1]^{\alpha}\ [p(\underline{X}|H_2)]^{1-\alpha}\ d\underline{X} \quad . \qquad (4.17)$$

The inequality (4.16) is valid for any α, $0 \le \alpha \le 1$, but we require a tight upper bound. That is, we want to use the value of α, α^*, that minimizes the right hand side of (4.17). In the one dimensional case, the use of $\alpha = 1/2$ gives the Bhattacharyya bound (Van Trees, 1968) and the use of α^* gives the Chernoff bound (Van Trees, 1968). The usual procedure is to obtain the Chernoff bound via the moment generating function for the one dimensional case; next, to assume all signal samples are the same, so that, if β is the bound for N = 1, β^N is the bound for N samples (see, for example, Hall, 1966, and Spaulding et al., 1967). This procedure does not necessarily correspond to the actual physical signals we have choosen, however.

In our case, since $q_1 = q_2 = 1/2$, we have

$$P_e \leq \frac{1}{2} \int_{-\infty}^{\infty} \left[\prod_{n=1}^{N} \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(x_n - s_{1n})^2/2\sigma_m^2} \right]^{\alpha} \cdot$$

$$\left[\prod_{n=1}^{N} \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(x_n - s_{2n})^2/2\sigma_m^2} \right]^{1-\alpha} , \tag{4.18}$$

and we are interested in large N. This reduces to

$$P_e \leq \frac{1}{2} \prod_{n=1}^{N} \int_{-\infty}^{\infty} \left[\sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(x_n - s_{1n})^2/2\sigma_m^2} \right]^{\alpha} \cdot$$

$$\left[\sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(x_n - s_{2n})^2/2\sigma_m^2} \right]^{1-\alpha} dx_n \ . \tag{4.19}$$

Let

$$\rho_n = \frac{s_{1n} - s_{2n}}{2} ,$$

and

$$y = x_n - \frac{s_{1n} + s_{2n}}{2} . \tag{4.20}$$

Now we have

$$P_e \le \frac{1}{2} \prod_{n=1}^{N} I_\alpha(\rho_n) \quad , \tag{4.21}$$

where

$$I_\alpha(\rho_n) = \int_{-\infty}^{\infty} \left[\sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(y-\rho_n)^2/2\sigma_m^2} \right]^\alpha .$$

$$\left[\sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(y+\rho_n)^2/2\sigma_m^2} \right]^{1-\alpha} dy \quad . \tag{4.22}$$

We require

$$I_{\alpha^*}(\rho_n) = \inf_{0\le\alpha\le 1} I_\alpha(\rho_n) \quad . \tag{4.23}$$

Now, we know $I_\alpha(\rho)$ is convex U on [0,1] (Hellman and Raviv, 1970), and we see that the integral in (4.22) is symmetrical about $\alpha = 1/2$. Therefore, $\alpha^* = 1/2$. Our problem has now been reduced to the computation of the single integral in (4.22) with $\alpha = 1/2$. [We also note that for any α, $I_\alpha(0) = 1$.]

Before continuing with the evaluation of our bound on P_e, we find it constructive to evaluate this bound for the known Gaussian noise case for our three sets of signals. This will give us a feeling for the exponential tightness of the bound. Also, as the impulsive index A becomes large ($\sim$10), our interference distribution approaches the Gaussian. That is, as $A \gg 1$,

$$p_Z(z) \to \frac{1}{\sqrt{\pi}} e^{-z^2} \quad . \tag{4.24}$$

The coherent receiver which is optimum for Gauss is the well-known correlation or matched filter receiver. It's performance is given in terms of the signal energy, E, and the correlation, ϕ, between $S_1(t)$ and $S_2(t)$,

$$P_e = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E(1-\phi)}{2}} \quad , \tag{4.25}$$

where erfc denotes the complimentary error function. That is,

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx \quad . \tag{4.26}$$

In (4.25) $\phi = -1$ for antipodal signals and $\phi = 0$ for orthogonal signals and ON-OFF keying. Generally, for ON-OFF keying we use E = average signal energy, which is given by one half the energy of either signal in the orthogonal case.

The result (4.25) assumes that we can sample at any rate we please and still maintain independence between samples. While this is the case for white Guassian noise, we are dealing with narrow band interference. That is, our Gaussian noise (4.24) is colored. To use the result (4.25) we must, accordingly, modify the receiver by preceeding the matched filter receiver by a pre-whitening filter. A physically realizable pre-whitening filter is given by the triangular matrix W, where

$$W^T W = \Phi_{zz}^{-1} \quad , \tag{4.27}$$

and Φ_{zz} is the NxN covariance matrix of our N noise samples. The noise samples after the filter (4.27) are now independent. The appropriate signal energy to use in (4.25) is the energy after the pre-whitener,

$$E = \underline{S}_1^T \Phi_{zz}^{-1} \underline{S}_1 = \underline{S}_2^T \Phi_{zz}^{-1} \underline{S}_2 \quad . \tag{4.28}$$

The receiver must now be matched to the "whitened" signals $W\underline{S}_1$ and $W\underline{S}_2$.

Our performance bound (4.21) is, for (4.24),

$$P_e \leq \frac{1}{2} \prod_{n=1}^{N} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\pi}} e^{-(x-s_{1n})^2}\right]^{\alpha} \left[\frac{1}{\sqrt{\pi}} e^{-(x-s_{2n})^2}\right]^{1-\alpha} dx \quad , \tag{4.29}$$

or

$$P_e \leq \frac{1}{2} \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left[x^2+2(-\alpha s_{1n}-\beta s_{2n})x + (\alpha s_{1n}^2+\beta s_{2n}^2)\right]\right\} dx \quad , \tag{4.30}$$

where $\beta = 1-\alpha$, and s_{1n} and s_{2n} are now samples of the <u>whitened</u> signals. Evaluating (4.30) gives

$$P_e \leq \frac{1}{2} \exp\left[\sum_{n=1}^{N} \alpha^2 s_{1n}^2 + 2\alpha\beta s_{1n} s_{2n} + \beta^2 s_{2n}^2 - \alpha s_{1n}^2 - \beta s_{2n}^2\right] . \tag{4.31}$$

Now, from (4.27) and (4.28)

$$\sum_{n=1}^{N} s_{1n}^2 = \sum_{n=1}^{N} s_{2n}^2 = E \quad , \tag{4.32}$$

and

$$\sum_{n=1}^{N} s_{1n} s_{2n} = \phi E \quad . \tag{4.33}$$

This gives us, then

$$P_e \le \frac{1}{2} \exp[E(2\alpha^2 - 2\alpha - 2\alpha^2\phi - 2\alpha\phi] \quad . \tag{4.34}$$

Also, since our result is normalized to the noise energy, E is the signal-to-noise ratio.

The right hand side of (4.34) is, not surprisingly, minimized by using $\alpha = 1/2$, for any value of ϕ. We obtain, therefore, the simple bounds on performance

$$P_e \le \frac{1}{2} \exp[-E] \ , \quad \text{antipodal} \quad ,$$

$$P_e \le \frac{1}{2} \exp[-E/2] \ , \quad \text{orthogonal} \quad ,$$

and

$$P_e \le \frac{1}{2} \exp[-E/4] \ , \quad \text{ON-OFF} \quad . \tag{4.35}$$

Figure 4.1 shows the actual theoretical performance given by (4.25) along with the bounds obtained in (4.35). [Of course, the exponential function is well known to be an exponentially tight bound for the erfc.]

In order to evaluate the error bound for our optimum detector, we need to evaluate the basic integral (4.22) with $\alpha = 1/2$. We have, from (4.22),

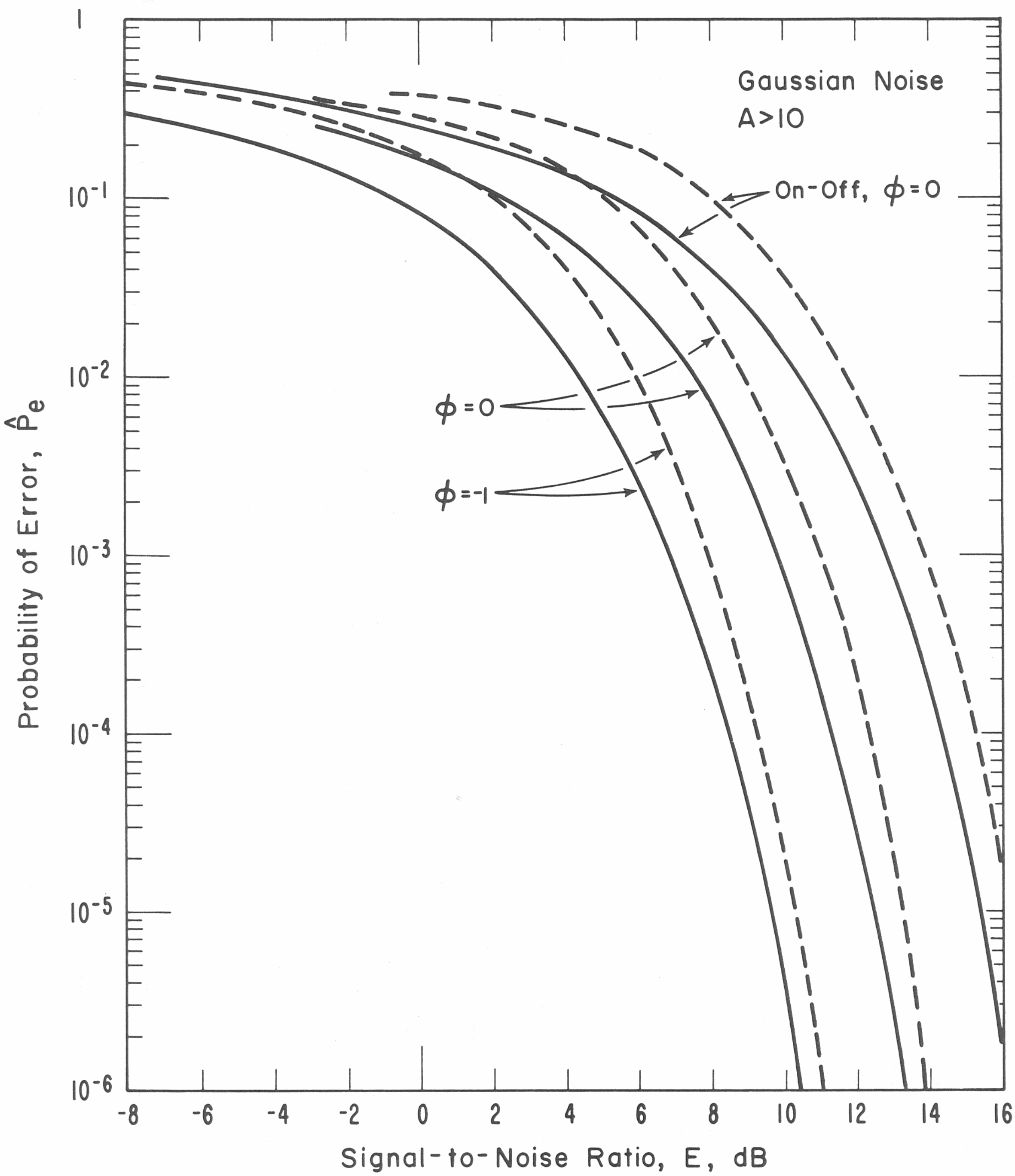

Figure 4.1. Comparison of theoretical performance in Gaussian noise (from 4.25) and the upper bound (4.35).

$$I(\rho) = 2 \int_0^\infty \left[\sum_{m=0}^\infty e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(y+\rho)^2/2\sigma_m^2} \right]^{\frac{1}{2}}$$

$$\cdot \left[\sum_{m=0}^\infty e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-(y+\rho)^2/2\sigma_m^2} \right]^{\frac{1}{2}} dy \ . \qquad (4.36)$$

This integral can be evaluated, for arbitrary ρ, only by numerical techniques. One approach would be to use the approximation given in the Appendix, but we can evaluate (4.36) directly very rapidly. By looking at figure 2.2, we see that the integral is not particularly well behaved in the neighborhood of $y = \rho$, and that the normal Gaussian quadrature approach to infinite integrals will not apply. We therefore truncate the integral; i.e., integrate from 0 to U. For the integral from 0 to U, we will use an adaptive Romberg quadrature developed to handle integrands of our type (Miller, 1970).

We need a bound on the integral from U to ∞, so that we can select where to truncate and still achieve a given degree of accuracy.

Because of the nature of σ_m, it is difficult to obtain a simple usable bound on the truncation error. Let ε denote the truncation error; thus, using the inequality between the geometric and arithmetic means, i.e., $(a+b)/2 \geq \sqrt{ab}$, we find that

$$\varepsilon \leq \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} \int_U^{\infty} e^{-(y-\rho)^2/2\sigma_m^2} dy$$

$$+ \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} \int_U^{\infty} e^{-(y+\rho)^2/2\sigma_m^2} dy \quad . \qquad (4.37)$$

The integrals in (4.37) are erfc's, and we use the inequality (Abramowitz and Stegun, 1964, 7.1.13, p. 298)

$$e^{x^2} \int_x^{\infty} e^{-t^2} dt \leq \frac{1}{x+\sqrt{x^2+4/\pi}} \leq \frac{1}{\sqrt{4/\pi}} , \qquad (4.38)$$

so that

$$\int_U^{\infty} e^{-(y-\rho)^2/2\sigma_m^2} dy = \sqrt{2}\sigma_m \int_{\frac{U-\rho}{\sqrt{2}\sigma_m}}^{\infty} e^{-t^2} dt$$

$$\leq \frac{\sqrt{2}\sigma_m}{\sqrt{4/\pi}} e^{-(U-\rho)^2/2\sigma_m^2} , \text{ etc.} \qquad (4.39)$$

We obtain, therefore,

$$\varepsilon \leq \frac{1}{2} \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!} e^{-(\frac{U-\rho}{\sqrt{2}})/\sigma_m^2} + \frac{1}{2} \sum_{m=0}^{\infty} e^{-A} \frac{A^m}{m!} e^{-(\frac{U+\rho}{\sqrt{2}})/\sigma_m^2} . \qquad (4.40)$$

While this bound on ε is not particularly simple, we note that the summations are of the form already computed for the envelope distribution of class A interference, and so (4.40) is readily usable.

Figure 4.2 shows $I(\rho)$ for the two cases; $A = 0.35$, $\Gamma' = 0.5 \times 10^{-3}$, and $A = 0.1$, $\Gamma' = 10^{-3}$, and for $\rho \geq 0$.

Our bound on performance can now be obtained from (4.21). In order to use (4.21) for large N, we require the signal samples, ρ_n, and a means of obtaining the values of $I(\rho_n)$ for each ρ_n. For our three signal sets given by (4.2), (4.3), and (4.4), let

$$t_n = \frac{T}{N-1}(n-1); \; 1 \leq n \leq N, \text{ and } Q = \omega_o T \quad . \qquad (4.41)$$

We obtain, then, from (4.20),

$$\rho_n = \sqrt{2S} \cos\left[\frac{Q}{N-1}(n-1)\right], \text{ antipodal}$$

$$\rho_n = \sqrt{S} \cos\left[\frac{Q}{N-1}(n-1) - \frac{3\pi}{4}\right], \text{ orthogonal, and}$$

$$\rho_n = \sqrt{S/4} \cos\left[\frac{Q}{N-1}(n-1)\right], \text{ ON-OFF keyed ,} \qquad (4.42)$$

where for the ON-OFF keyed system, we have also followed the convention of using the average signal power in the two signals; i.e., one-half the power of either signal in the other cases.

We now develop a simple upper bound to $I(\rho)$, $\hat{I}(\rho)$, which can be quickly evaluated by the computer. For the case $A = 0.35$, $\Gamma' = 0.5 \times 10^{-3}$, we use

$$\hat{I}(\rho_n) = 0.42 + 0.58\, e^{-1200\,\rho_n^{\,2}} \quad , \; 0 \le |\rho_n| \le 0.1$$

$$\hat{I}(\rho_n) = 0.42\, e^{-0121|\rho_n|^{1.69}} \quad , |\rho_n| > 0.1 \quad . \qquad (4.43)$$

The expression (4.43) was obtained by adjusting (via a computer) functions of the form $K_1 \exp K_2\, \rho_n^{\,K_3}$ until a tight upper bound was achieved. This bound for $I(\rho)$ is also shown on figure 4.2, but cannot be distinguished from $I(\rho)$. Now, using (4.42) and (4.43), our performance bound, $\hat{P}_e$, is given by

$$P_e \le \hat{P}_e = \frac{1}{2} \prod_{n=1}^{N} \hat{I}(\rho_n) \quad , \qquad (4.44)$$

and we use S for the SNR.

Table 4.1 gives $\hat{P}_e$ for our three signal sets for Q = 10, 1000, 100,000 and for N = 10, 100, 1000, for the case A = 0.35, $\Gamma' = 0.5 \times 10^{-3}$. By varying the parameter Q we vary the number of oscillations of our desired signal in the detection period T. As table 1 shows, performance is weakly dependent on Q.

Figure 4.3 shows three results for Q = 10 and N = 10; figure 4.4 for N = 10 and Q = 100,000. Figure 4.5 gives these results for N = 100 and Q = 100,000, while figure 4.6 shows the results for the ON-OFF system for Q = 1000 and N = 10, 100, and 1000. We note that, as it should, the $P_e \to 0$ as N becomes large. That is, we can make P_e as small as we like by making N large enough, which corresponds to making T large, or equivalently, by making the time-bandwidth product large.

For our other sample case, $A = 0.1$, $\Gamma' = 10^{-3}$, we use

$$\hat{I}(\rho_n) = 0.165 + 0.835\, e^{-450|\rho_n|^{1.95}}, \quad 0 \le |\rho_n| \le 0.77 ,$$

$$\hat{I}(\rho_n) = 0.175\, e^{-0.091|\rho_n|^{1.71}}, \quad |\rho_n| \ge 0.77 . \qquad (4.45)$$

Figure 4.7 shows the results for Q = 1000, N = 10,100 for our two sample cases for the antipodal signals while table 4.2 gives the results for Q = 10, 1000, 100,000 and N = 10, 100, 1000 for our three signaling sets and the case $A = 0.1$, $\Gamma' = 10^{-3}$.