## 3.2 The Assumption of Independent Sampling

As we have seen, the detection problem must be formulated using discrete random variables which are time samples from the random process (waveform) representing the interference. For non-Gaussian situations it is also almost always necessary to assume that these random variables (samples) are effectively independent so that only first order pdf's are required to determine the nth order pdf's for the vectors of these samples.

We first note that by assuming independent samples we obtain an upper bound on performance for the truely optimum detector, since if the samples were dependent, the optimum detector would make use of the information contained in this dependence to "reduce" the interference. That is, the performance of the optimum detector for dependent samples can be no worse than the performance of the optimum detector for independent samples (for the same continuous detection time from which the discrete samples are taken).

We now want to determine some criteria for "effectively" independent samples for our two classes of interference, but concentrating on class A here.

For class B interference, the receiver responces always overlap so that, in principle, we can never have true independence. Numerous measurements of the normalized autocovariance function, $\rho(t)$, have been made which indicate the following:

For atmospheric noise the results are varied. Sometimes we have practical independence [$\rho(t) < 0.1$, say] for $t = t_2 - t_1$ ($t_i$ the sampling instants) on the order of 1/B, where B is the impulse bandwidth* of the receiver. [See, for example, figure 61 of Disney and Spaulding, 1970.] Other times, effective independence requires a much larger sampling interval. For example, figure 62 of Disney and Spaulding (1970) shows an example in which $\rho(t) < 0.1$ only for $t \gtrsim 200/B$. In short, the situation with atmospheric noise is quite varied.

For man-made noise the situation is also varied. Interference from a single source, e.g., one automobile, shows obvious dependence due to the existing structure of the source waveform. In a general urban environment, however, where the interference is caused by many sources, primarily automobiles and power lines, we have effective independence for t on the order of 1/B [see, for example, figures 46 and 48 of Spaulding and Espeland, 1971].

For class A interference there have been no measurements of $\rho(t)$ and we must resort to the model itself to attempt to gain insight into how large the sampling interval, t, must be to insure effective independence.

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* The impulse bandwidth, B, is given by $1/B = 1/v_{max} \int_0^\infty v(t)dt$, where v(t) is the impulse responce at the receiver.

The following analysis is based on the results of Middleton (1960, sec. 11.2-2 and 11.2-3) and Middleton (1975). We may write the $n^{th}$ order characteristic function for our classes of interference (A or B) as

$$F_n(i\xi_1,t_1;\cdots;\xi_n,t_n) = \exp\langle \int_\Lambda \rho_N(\underline{\lambda},\hat{\varepsilon})(e^{i\underline{\varepsilon}\cdot\underline{U}} -1)\, d\underset{\sim}{\lambda} d\hat{\varepsilon}\rangle_{\underline{\theta},\hat{\varepsilon}} \; , \tag{3.33}$$

with $\underline{U} = [U(t_\ell-\lambda;\, \underline{\lambda},\hat{\varepsilon},\underline{\theta}] = [U_\ell]$ (= column vector of received waveform); $\underline{\xi} = \{\xi_1, \ldots \xi_n\}$ (row vector); $1 \le \ell \le n$ and $\underline{\theta}$ are a set of random parameters of the basic received waveform, U. Equation (3.33) is the generalization of (2.1) of Middleton (1975) [see also, sec. 2.3]. We rewrite (3.33) more explicitly using the envelope-phase form for the received waveform, U, as

$$F_n = \exp\{\hat{I}_n(i\underline{\xi},\underline{t})\} \quad , \tag{3.34a}$$

where

$$\hat{I}_n = A\,\langle \int_0^{z=T_s/\overline{T}_s} \left[e^{i\underline{\xi}[\hat{z}_o(z+z_\ell;\underline{\lambda};\underline{\theta}')\cos\mu\hat{\psi}(z+z_\ell;\underline{\lambda};\underline{\theta}')]} -1\right] dz\rangle_{\underline{\lambda},z_o,\underline{\theta}'} \; , \tag{3.34b}$$

in which

$$\hat{B}_o = \hat{B}_{o\ell} = |\mathcal{a}_T\mathcal{a}_R|\, g(\underline{\lambda})\; A_o e_{or} U_o(z+z_\ell) \quad , \tag{3.34c}$$

with $U_o = 0$ when $z + z_\ell > T_s/\overline{T}_s$, and when $z + z_\ell < 0$. For the details of the representations (3.34b) see Middleton

(1975), (2.33), and the associated discussion.

For class A interference, the time duration of the interference waveform, U, is $T_s$ with $T_s < \infty$. Therefore, for some $z + z_\ell$, the interference waveform U is zero, so $B_{o\ell} = 0$ when $z + z_\ell > T_s$ and we have no overlap between waveforms. In other words, for class A interference it is possible to sample independently strictly, if the sample intervals are made large enough. Practically, however, such intervals are so large as to be impractical. Therefore, it is necessary to see if one can sample at closer intervals and still achieve nearly independent samples.

Accordingly, let us look at 2nd order independence, n = 2, for class A noise [see p. 496 and (11.81) of Middleton (1960)]. From (3.34b) we can write, for n = 2,

$$\hat{I}_2 = A[\rho(t)\ \hat{J}_{12} + \hat{I}_1(i\xi_1) + \hat{I}_1(i\xi_2)] \quad , \qquad (3.35)$$

where

$$\hat{J}_{12} = \hat{J}_{12}(i\xi_1, i\xi_2, |t|)$$

$$\equiv \left\langle \int_0^{z=\beta(\tau-|t|)} \left(e^{i\xi_1 B_{o1} + i\xi_2 B_{02}} - 1\right) dz \right\rangle_{\tau = T_s, \underline{\lambda}, +\underline{\theta}'} \cdot \rho^{-1}(t)$$

(3.36a)

and

$$\hat{I}_1(i\xi_i) = \left\langle \int_0^{z_o=\beta\tau} \left(e^{i\xi_i B_{oi}} - 1\right) dz \right\rangle_{z_o = T_s/\overline{T}_s, \underline{\lambda}, \underline{\theta}'} \quad ;(3.36b)$$

$$B_{oi} = \hat{B}_o(z + z_i;\ \underline{\lambda};\underline{\theta}')\ \cos\mu\psi, \quad i = 1,2 \quad .$$

Here $\beta = \overline{T}_s^{-1}$ $(= \overline{\tau}^{-1})$. The normalized correlation function for overlap is [see (11.81c) of Middleton, 1960],

$$\rho(t) = \beta \int_{|t|}^{\infty} (\tau-|t|)\; p(\tau)\; d\tau > 0,\; \tau \le \tau_{max}\;, \qquad (3.37)$$
$$= 0,\; \tau > \tau_{max}$$

where $T_s \equiv \tau$, $t = t_2-t_1$, $\tau_{max}$ is the maximum possible signal duration ($<\infty$, since class A) and $p(\tau)$ is the pdf of the signal duration $\tau$. It is clear that

$$|\hat{J}_{12}| \le 1,\; |\hat{I}_1|,\; |\hat{I}_2| \le 1, \text{ and } |\rho(t)| \le 1! \quad . \qquad (3.38)$$

Another way of writing (3.37) is

$$\rho(t) = \langle \overline{T}_s^{-1} \int_0^{\overline{T}_s^{-1}(\tau-|t|)} dz \rangle_\tau = \beta \int_{|t|}^{\infty} (\tau-|t|)\; p(\tau)\; d\tau \quad . \qquad (3.39)$$

Now let us put (3.35) into (3.34a), $n = 2$, to get

$$F_2(i\xi_1,t_1;\; i\xi_2,t_2) = \exp\{A[\rho\; \hat{J}_{12}+\hat{I}_1(i\xi_1) + \hat{I}_2 i\xi_2]\} \qquad (3.40a)$$

$$= F_1(i\xi_1)\; F_1(i\xi_2)\; e^{A\rho(t)\; \hat{J}_{12}(i\xi_1,i\xi_2,|t|)} \qquad (3.40b)$$

$$= F_1(i\xi_1)\; F_1(i\xi_2) \sum_{m=0}^{\infty} \frac{A^m}{m!}\; \rho^m(t)\; \hat{J}_{12}^m \quad , \qquad (3.41c)$$

where

$$F_1(i\xi_1) = e^{A\; \hat{I}_1(i\xi_1)} \quad , \qquad (3.41d)$$

i.e., the 1st order characteristic function of class A noise. When $\rho(t) = 0$, i.e., for $|t| \geq \tau_{max}(<\infty)$, $F_2 = F_1(i\xi_1)F_1(i\xi_2)$, and we have statistical independence at time $t_2 \geq t_1 + \tau_{max}$, <u>in the second order</u>. [There may still be statistical dependencies of $0(n \geq 3)$, however.]

Let us look further at n=2 above, before generalizing to $n > 2$. The controlling quantity for 2nd order independence (n=2), or conversely, for 2nd order overlap (e.g., statistical dependence), is

$$A\rho(t) \equiv A\rho_{12}(t) = A\beta \int_{|t|}^{\infty} (\tau - |t|)\, p(\tau)\, d\tau \quad . \qquad (3.42)$$

For, from (3.41c), it is clear that if $A\rho \to 0$, or is very small, we have practical 2nd order independence; e.g.,

$$F_2(i\xi_1,t_1;\, i\xi_2,t_2) = F_1(i\xi_1)\, F_1(i\xi_2)\, \{1 + \sum_{m=1}^{\infty} \frac{(A\rho_{12})^m}{m!} \hat{J}_{12}{}^m\}$$

$$\approx F_1(i\xi_1)\, F_1(i\xi_2) \qquad (3.43)$$

provided

$$|A\, \rho(t)| << 1 \quad , \qquad (3.44)$$

since $|\hat{J}_{12}^m| \leq 1$. So to insure 2nd order practical statistical independence, we must make the sampling interval, $t_2-t_1$, such that the condition (3.44) is meet.

Let us take a specific example. Let $A = 10^{-1}$ and let $\rho(\tau)$ be uniform, $0 \leq \tau \leq \tau_{max}$, so that $\beta = \overline{\tau}_s^{-1} = 2/\tau_{max}$. From (3.37),

$$A\ \rho(t) = \frac{2A}{\tau_{max}} \int_{|t|}^{\tau_{max}} \left(\frac{\tau-|t|}{\tau_{max}}\right) d\tau \qquad (3.45a)$$

$$= A\left(1 - \frac{|t|}{\tau_{max}}\right)^2, \quad |t| \le \tau_{max} \qquad (3.45b)$$

$$= 0 \quad , \quad |t| > \tau_{max} \quad .$$

We see that there is no overlap (2nd order) when $|t| > \tau_{max} = 2\ \overline{T}_s$. However, we can sample more closely as long as $A\rho$ is small, e.g. (3.44). Let us try $\rho = 0.1$, e.g. $\rho(t_{0.1}) = 0.1$, where $t_{0.1} = t$ for which $\rho = 0.1$, etc. We have the condition $A\rho = 10^{-1} \cdot 10^{-1} = 10^{-2} << 1$, or overlap at the one percent level. From (3.45b),

$$10^{-1} = \left(1 - \frac{|t_{0.1}|}{max}\right)^2 \quad , \text{ or}$$

$$t_{0.1} = \tau_{max}\left(1 - \frac{1}{\sqrt{10}}\right) = 1.38\ \overline{T}_s = 0.69\ \tau_{max} \quad . \qquad (3.46)$$

Thus, for nearly independent sampling (2nd order), at 1 percent effective overlap level $(A_\rho = 10^{-2})$, we should take samples at 1.38 × average interfering signal duration for this postulated uniform distribution of interfering signal durations.

We can generalize (i.e., $n \ge 2$) the above (3.40b) directly and write

$$F_n(i\underline{\xi},\underline{t}) = F_1(i\xi_1)F_1(i\xi_2)\ldots F_1(i\xi_n) \exp A\{\rho_{123\ldots n}\hat{J}_{12\ldots n}$$

$$+ \rho_{12\ldots n-1,n}\hat{J}_{12\ldots n-1,n} + \cdots$$

$$+ \rho_{12,\ldots n}\hat{J}_{12,\ldots n} + \rho_{13,2\ldots n}\hat{J}_{13,\ldots}$$

$$+ \ldots + \text{etc.}\} \quad . \tag{3.47}$$

The exponential contains the effects of all possible combinations of overlap: pairs, triples, etc..

Let us consider

$$\rho_{123} = \rho(t_1,t_2,t_3) = \langle \int_0^{T_s} u_o(\tau)\, u_o(\tau+t_2)\, u_o(\tau+t_3)\, d\tau \rangle_{T_s} \quad , \tag{3.48}$$

where $t_1 = 0$ (stationarity). The waveform $u_o$ has a constant amplitude 1 and time duration $T_s$ (rectangular waveform). [See Middleton, 1960, section 11.2-3.] Now

$$u_o(\tau) = 1 \quad ,$$

$$u_o(\tau+t_2) = 1 \text{ iff } 0 < \tau + t_2 < T_s \quad ,$$

$$u_o(\tau+t_3) = 1 \text{ iff } 0 < \tau + t_3 < T_s \quad . \tag{3.49}$$

We use the convention $t_n > t_{n-1}$, etc., i.e. $t_3 > t_2$, so the integrand in (3.48) is nonzero only for $0 < \tau < T_s - t_2$, and therefore

$$\rho_{123} = \langle \int_0^{T_s - t_3} d\tau \rangle_{T_s} = \langle T_s - |t_3| \rangle_{T_s} \quad . \tag{3.50}$$

Similarly, we have

$$\rho_{123\ldots n} = \langle T_s - |t_n| \rangle_{T_s} \quad , \qquad (3.51)$$

and $\rho_{123\ldots n-1,n} = \rho_{123\ldots n-1}$ for stationary cases. Next looking at (3.47) we see that, again, for all $\hat{J}$'s, $|\hat{J}| \leq 1$. We now need to determine how many J's there are in the exponent in (3.47); i.e., the total number of possible overlaps of all types. For n overlaps, n-1 overlaps, etc., we have

$$123\ldots n: \quad \binom{n}{n} = 1 \text{ (n overlaps)}$$

$$123\ldots n-1,n: \quad \binom{n}{n-1} = n \text{ (n-1 overlaps)}$$

$$123\ldots n-2,n-1,n: \quad \binom{n}{n-2} = n(n-1)/2 \text{ (n-2 overlaps)}$$

$$\vdots$$

$$12,3\ldots n: \quad \binom{n}{1} = n \text{ (single overlaps)} \; . \qquad (3.52)$$

Therefore the total number of n,n-1,...,2 (pair) overlaps is given by

$$\sum_{k=0}^{n-2} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{1} - \binom{n}{0} \qquad (3.53)$$

$$= 2^n - n - 1 \quad .$$

Therefore, from (3.47)

$$\exp A\{\ldots\} \leq \exp\{A(2^n-n-1)\ \rho_{123\ldots n}\} \quad , \qquad (3.54)$$

or

$$\exp A\{\ldots\} \le 1 + \sum_{m=1}^{\infty} \frac{[A(2^n-n-1)]^m}{m!} \rho_{123\ldots n}^m |\hat{J}|^m, \quad n \ge 2 \quad . \tag{3.55}$$

Consequently, our nth order condition for practical independence is

$$|A(2^n-n-1)\ \rho_{123\ldots n}| \ll 1, \quad n \ge 2 \quad , \tag{3.56}$$

with

$$\rho_{123\ldots n} = \langle T_s - |t_n| \rangle_{T_s} \quad , \tag{3.57}$$

or, as before (3.37),

$$\rho_{123\ldots n} = \beta \int_{|t_n|}^{\infty} (\tau - |t_n|)\ p(\tau)\ d\tau = \rho(t) \quad . \tag{3.58}$$

We note that the condition becomes stricter as n becomes larger due to the factor $2^n$-n-1.

Notice that (3.56) is a very strict upper bound, since it assumes that, in any data sample, all possible combinations of overlapping waveforms will occur, when in fact, the higher order overlaps become increasingly rare effects. Thus, a suitable less strict (and more realistic) bound would include a multiplicative factor (much less than 1 as n increases) that takes into account the probability of occurance of these rare events. On the basis that the most probable form of overlapping is pair overlap, we will assume that 2nd order practical independence is sufficient.