2. MODELS OF IMPULSIVE INTERFERENCE

In order to be able to determine the optimum receiving system for a given class of signals and analyze its performance, a mathematical model for the random interference process is required. That is, for optimal system studies and also for determining the performance of some of the existing suboptimum systems, more information about the interference process is required than can generally be obtained by measurement alone. The interference process as seen by the receiver is, for almost all cases of interest, a narrowband process in that it can be characterized by an envelope and a phase. Narrowband noise processes arise whenever the receiver bandwidth is substantially less than the receiver center frequency. The problem is to develop a model for the interference that fits all the available measurements; is physically meaningful when the nature of the noise sources, their distributions in time and space, propagation, etc., are considered; is directly relatable to the physical mechanisms giving rise to the interference; and is still simple enough so that the required statistics can be obtained for solving signal detection problems. While various models have been proposed in the past (to be summarized later) that meet these requirements in particular instances, the only general (canonical) model proposed to date that meets all the above requirements is that proposed by Middleton (1974).

We want to distinguish between two classes of interference:

Class A--Interference arising from sources whose emission spectra are narrower than the bandpass of our receiver (to be termed narrowband interference), and

Class B--Interference arising from sources whose emission spectra are much broader than the bandpass of our receiver (to be termed broadband interference).

Both classes produce "narrowband" (i.e., envelope and phase) interference in the receiver. We can also, of course, consider a class C interference as one which is composed of a combination of class A and class B. Examples of class A interference include collections of unwanted signals (unwanted by our receiver, but wanted by someone else) and the emissions of various man-made devices (e.g., radio frequency dielectric heaters, soldering machines, plastic welders, etc.), while examples of class B include atmospheric noise, automotive ignition noise, arc welders, etc. All models proposed in the past have considered only class B interference. The only exception to date is Middleton's model, which treats both class A and class B.

Models that have been developed to date can be categorized into two basic types. The first type (and earliest models) are empirical models which do not represent the interference process itself but which propose various mathematical expressions designed only to fit the measured statistics of the interference. The second type of model is that which is designed to represent the entire random interference process itself. The majority of these models represents the received interference waveform as a summation of filtered impulses.

We want to summarize briefly these two types of models and then treat the Middleton model in more detail. It is the Middleton model (class A) we will use in the remainder of this report where we investigate optimum detection structures and the performance of these structures.

2.1 Summary of Empirical Models

Most of the empirical models have concentrated on the amplitude probability distribution (APD) of the noise envelope, $P(\epsilon > \epsilon_o)$. This distribution has been measured extensively for both atmospheric and man-made noise (see the bibliographies by Thompson, 1971, and Spaulding et al., 1975).

The first "model" for the noise envelope was the Rayleigh distribution

$$P(\epsilon > \epsilon_o) = e^{-a\epsilon_o^2} \quad . \qquad (2.1)$$

This simply assumes that the interference is Gaussian, and was quickly recognized to be quite inappropriate, since the envelope distribution of atmospheric and man-made noise exhibits large impulsive tails.

In 1954, Hesperper, Kessler, Sullivan, and Wells independently proposed the log normal distribution for atmospheric noise (see Furutsu and Ishida, 1961),

$$p(\epsilon) = \frac{1}{\sigma\sqrt{2\mu}} e^{-\frac{1}{2}\left(\frac{\log \epsilon - \log \mu}{\sigma}\right)^2} . \qquad (2.2)$$

This approach gave reasonable approximations to the impulsive tail of the distribution, but did not match the Rayleigh (Gaussian) character of the interference at the lower amplitude levels.

Likhter (1956) used a combination of two Rayleigh distributions for atmospheric noise:

$$P(\epsilon > \epsilon_o) = (1 - c)\, e^{-a\epsilon_o^2} + c\, e^{-b\epsilon_o^2} . \qquad (2.3)$$

This distribution gave poor agreement with actual data.

Also in 1956, Watt proposed a variation on the Rayleigh distribution (see Furutsu and Ishida, 1961),

$$P(\epsilon > \epsilon_o) = e^{-x^2} , \qquad (2.4)$$

where

$$x = a_1\epsilon_0 + a_2\epsilon_0^{(b+1)/2} + a_3\epsilon_0^{b} ,$$

$$b = 0.6[20 \log(\epsilon_{rms}/\epsilon_{ave})] .$$

This distribution was designed for atmospheric noise and was claimed to give better results at high and low probabilities than the previously proposed distributions.

Ishida (1956) proposed a combination for atmospheric noise,

$$P(\epsilon > \epsilon_o) = (1 - c)\, e^{-a\epsilon_o^2} + c(\text{log normal distribution}). \tag{2.5}$$

Nakai (1960) recommended this same combination. Ibukun (1966) found good agreement with some measured data for this log normal, Rayleigh combination.

In 1956 Horner and Harwood (see Ishida, 1969) used

$$P(\epsilon > \epsilon_o) = \frac{\gamma^2}{(\epsilon_o^{q} + \gamma^2)} \tag{2.6}$$

to represent the APD of atmospheric noise, and also in 1956 the Department of Scientific and Industrial Research of Great Britain proposed the following distribution (see Ibukun, 1966):

$$P(\epsilon > \epsilon_o) = \frac{1}{\left[1 + \left(\frac{\alpha\epsilon_o}{\bar{\epsilon}_o}\right)^{r}\right]^{-1}} \quad , \tag{2.7}$$

obtaining experimental values for α and r of 2.7 and 1.4, respectively, using atmospheric noise data from Nakai (1960).

Crichlow et al. (1960) represented the APD of atmospheric noise by a Rayleigh distribution at the lower amplitude levels and a "power" Rayleigh distribution at the higher levels

$$P(\epsilon > \epsilon_o) = e^{-(a\,\epsilon_o^2)^{1/s}} \quad , \quad y > B \quad , \tag{2.8}$$

with these two distributions being joined by a third expression for the middle range of amplitudes. These APD's were found to fit data very well over a wide range of bandwidths and are still the "standard" representation for atmospheric radio noise (CCIR, 1963). Means of obtaining the distribution for bandwidths other than the measurement bandwidth was also obtained (Spaulding, et al., 1962). It has been this empirical representation that has generally been used in determining the performance of digital systems in atmospheric noise (see Akima, 1972, and the bibliographies by Thompson, 1971, and Spaulding, et al., 1975).

Mertz (1961) used the distribution

$$P(\epsilon > \epsilon_o) = \frac{h^n}{(\epsilon_o + h)^n} \quad , \tag{2.9}$$

with h = 3, 4, 5 to represent impulsive noise on telephone circuits. He presented no comparison with data, however.

Kneuer (1964) used

$$p(\epsilon) = \frac{C}{\epsilon^{(2/q)+1}} \quad , \tag{2.10}$$

with q varying between 1/2 and 1. He offered no comparison with data either.

Engel (1965) used a variation of Mertz's formula,

$$P(\epsilon > \epsilon_o) = \frac{(k_o)^{2\alpha}}{\epsilon_o^{\,2\alpha}} \tag{2.11}$$

He found good agreement in comparing this APD with the data of Fennick (1964) for impulsive noise in telephone circuits.

Galejs (1966) used a variation of the Rayleigh distribution

$$P(\epsilon > \epsilon_o) = (1 - \delta)\, e^{-\alpha_1 \epsilon_0} + \delta e^{-\alpha_2 \epsilon_0} \quad . \qquad (2.12)$$

He reported satisfactory agreement with atmospheric noise data using appropriate values of the parameters δ, α_1, and α_2. Galejs (1967) also used a more complicated version for atmospherics

$$P(\epsilon > \epsilon_o) = \left[1 - \left(\frac{b}{a}\right)^2 - \left(\frac{d}{a}\right)^2\right]^{-\left(\frac{\epsilon_o}{\sigma}\right)^2} + \frac{b^2}{(\epsilon_o{}^2 + a^2) + \epsilon_o(\epsilon_o{}^2 + a^2)^{\frac{1}{2}}}$$

$$+ \left(\frac{d}{a}\right)^2 e^{-s\epsilon_o} \quad . \qquad (2.13)$$

Nakai and Nagatani (1970) recommended a divided log normal distribution,

$$p(\epsilon) = \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\log\,\epsilon - \log\,\mu_1}{\sigma_1}\right)^2} \, , \quad B < \epsilon < \infty$$

$$(2.14)$$

$$p(\epsilon) = \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\log\,\epsilon - \log\,\mu_2}{\sigma_2}\right)^2} \, , \quad 0 < \epsilon < B \quad .$$

This distribution showed good agreement with their atmospheric noise data.

Finally, Ponhratov and Antonov (1967) used a variation of the normal distribution with mean μ to represent the instantaneous amplitude,

$$p(z) = \frac{\nu}{2\sqrt{2}\ \Gamma(\frac{1}{2})\mu} \exp\left(- \frac{|z|^{\nu}}{2^{\nu/2}\mu^{\nu}}\right), \quad -\infty < z < \infty, \quad (2.15)$$

with $1/2 < \nu < 1$. They found this to be a good approximation for the probability density of atmospheric noise.

2.2 Summary of Models of the Interference Process

In the last section we gave a short, but fairly complete, summary of empirical models. Almost all these models were for atmospheric noise and concentrated on the envelope of the received noise process. Such models, while useful in determining the performance of "idealized" digital systems using matched filter or correlation receivers (i.e., those optimum for white Gaussian noise), give no insight into the physical processes that cause the interference. Neither can they be used to determine performance of "real" systems which employ various kinds of nonlinear processing, nor can they be used in optimum signal detection problems. Various investigators have developed models for the entire interference process, and we will summarize the most significant of these models in this section.

Furutsu and Ishida (1960) represented atmospheric noise as a summation of filtered impulses and considered two cases: (I) Poisson noise, consisting of the superposition of independent, randomly occurring impulses and (II) Poisson-Poisson noise, consisting of the superposition of independent, randomly occurring Poisson noise, each Poisson noise forming a wave packet of some duration. They represent the response of the receiver for a elementary pulse to be

$$r = r(t,a)\cos(\omega t + \psi) \quad , \qquad (2.16)$$

express this response as a vector, and take the summation of n (n random) such vectors. They obtain, for the envelope amplitude for Poisson noise,

$$p(\epsilon) = \int_0^\infty \lambda\epsilon J_0(\lambda\epsilon)\; f(\lambda,T)\; d\lambda \quad , \qquad (2.17)$$

where the characteristic function $f(\lambda,T)$ is given by

$$f(\lambda,T) = \exp\left[\nu \int_0^T dt \int da\; p(a)\{J_0(\lambda r) - 1\}\right] \quad , \qquad (2.17a)$$

and ν is the mean rate of occurrence of pulses in the Poisson distribution, T is the total time period of interest, and p(a) is the pdf of a. They also obtain

$$P(\epsilon < \epsilon_0) = \epsilon_0 \int_0^\infty J_1(\lambda\epsilon_0)\; f(\lambda,T)\; d\lambda \quad . \qquad (2.18)$$

Corresponding results are obtained for Poisson-Poisson noise (ν becomes Poisson distributed) and for second order distributions; i.e., $p(\epsilon_1, \epsilon_2)$ and $f(\lambda_1, \lambda_2, T)$. Furutsu and

Ishida (1960) proceeded to evaluate (2.18) for two "typical" cases of discrete and continuous spatial distributions of sources [using $f(\lambda,\infty)$]. Their results showed good agreement with measurements.

Beckmann (1962, 1964) developed a theoretical model for the received envelope of atmospheric noise and related his results to the number of sources (atmospheric discharges) and the properties of the propagation paths from these sources to the receiver. He assumed that the shape of the envelope of an individual atmospheric, attaining its peak value E_p at time t_o, was of the form

$$u_o(t) = \begin{cases} E_p \exp\left(-\frac{t-t_o}{a}\right) & \text{for } t > t_o \\ E_p \exp\left(\frac{t-t_o}{b}\right) & \text{for } t < t_o \end{cases} \quad . \tag{2.19}$$

The total signal at time t_o is given as

$$\hat{U} = \hat{u}_o + \sum_{k=1}^{\infty} \hat{u}_k + \sum_{k=1}^{\infty} \hat{s}_k \quad , \tag{2.20}$$

where the circumflex accents denote uniformly distributed phase vectors, the u_k are atmospherics of the form (2.16) that have reached their peak values at times previous to t_o, and the s_k are atmospherics that have not yet reached their peak values. For any arbitrary time, t (between two successive peaks), the amplitude is

$$\epsilon = U\ e^{-t/a} \quad . \tag{2.21}$$

A Poisson distribution is assumed for the occurrence times of the atmospherics and a log-normal distribution is postulated for the peak amplitude, E_p; i.e.,

$$E_p = e^{\Delta} \quad , \tag{2.22}$$

where Δ is normally distributed with mean μ and variance σ^2. Bechmann's results from the above are:

$$\epsilon_{rms} \approx \sqrt{Nc\ \ln(1/Nc)}\ e^{\sigma^2+\mu} \quad , \tag{2.23}$$

where N is the number of discharges per unit time and c = (a+b)/2, and

$$P\left(\frac{\epsilon}{\epsilon_{rms}} > \epsilon_o\right) = \frac{2}{Nc\sigma\sqrt{2\pi}} \int_{\epsilon_o}^{\infty} dx \int_{0}^{\infty} dy\ \frac{x}{y} \times \exp\left[-\frac{x^2+y^2}{Nc} - \frac{(\ln\ y+\sigma^2)}{2\sigma^2}\right] I_o\left(\frac{2xy}{Nc}\right) \quad , \tag{2.24}$$

which reduces for large and small values of ϵ_o to

$$P\left(\frac{\epsilon}{\epsilon_{rms}} > \epsilon_o\right) \approx \frac{1}{2}\left[1 - \text{erf}\left(\frac{\ln\ \epsilon_o+\sigma^2}{\sigma\sqrt{2}}\right)\right] \quad , \tag{2.25}$$

for large ϵ_o, and

$$P\left(\frac{\epsilon}{\epsilon_{rms}} > \epsilon_o\right) \approx e^{-\epsilon_o{}^2/Nc} \quad , \tag{2.26}$$

for small ϵ_o, respectively.

These results showed good agreement with measurements and were the first results which related measurements to the physical properties giving rise to the noise. The parameter Nc depends on the properties of atmospheric discharges, and μ and σ^2 are the mean and variance of the total attenuation, which is determined by the properties of the propagation path. Bechmann's analysis, however, gave no consideration to the characteristics of the receiver.

Ottesen (1968) applied the same techniques (summation of uniformly distributed phase vectors) to develop a model for the interference process, considering man-made noise sources which are scattered in space and overlapping in time.

Hall (1966) applied work on the applicability of a class of "self similar" random processes as a model for certain intermittent phenomena to signal detection problems considering LF atmospheric noise. The concept introduced is that of a random process that is controlled by one "régime" for the duration of observation, while this régime is itself a random process. The model that Hall proposed for received impulsive noise is one that takes the received noise to be a narrowband Gaussian process multiplied by a weighting factor that varies with time. Thus, the received atmospheric noise y(t) is assumed to have the form

$$y(t) = a(t)\, n(t) \quad , \tag{2.27}$$

where n(t) is a zero-mean narrowband Gaussian process with covariance function $R_n(\tau)$, and a(t), the régime process, is a stationary random process, independent of n(t), whose statistics are to be choosen so that y(t) is an accurate description of the received atmospheric noise. For a(t), Hall chose the "two sided" chi distribution, $\chi_2(m,\sigma)$, for the reciprocal of a(t), resulting in

$$p_a(a) = \frac{(m/2)^{m/2}}{\sigma^m \Gamma(m/2)} \frac{1}{|a|^{m+1}} \exp\left[-\frac{m}{2a^2\sigma^2}\right] , \qquad (2.28)$$

and

$$p_n(n) = \frac{1}{\sqrt{2\pi\sigma_1{}^2}} \exp\left[-\frac{n^2}{2\sigma_1{}^2}\right] . \qquad (2.29)$$

Using (2.28) and (2.29), Hall found the pdf of the noise to be given by

$$p_y(y) = \frac{\Gamma(\frac{\theta}{2})}{\Gamma(\frac{\theta-1}{2})} \frac{\gamma^{\theta-1}}{\sqrt{\pi}} \frac{1}{[y^2+\gamma^2]^{\theta/2}} , \qquad (2.30)$$

where $\gamma \equiv m^{\frac{1}{2}}\sigma_1/\sigma$ and $\theta \equiv m+1 > 1$. For the special case $\sigma_1 = \sigma$, $p_y(y)$ is Student's "t"-distribution. Hall terms (2.30) the generalized "t"-distribution with parameters θ and γ. Hall shows that θ in the range $2 < \theta \leq 4$ is appropriate to fit measured data of atmospheric noise and that $\theta \approx 3$ is appropriate to fit a large body of data at VLF and LF. [Unfortunately, for θ in the range $2 < \theta \leq 3$, y(t) has infinite variance and therefore cannot be a model for physical noise, although it fits the data very closely.]

Hall then considers the envelope and phase of the received noise; i.e.,

$$y(t) = V(t) \cos[\omega_o t + \phi(t)] \quad . \tag{2.31}$$

Using

$$p_{V,\phi}(V,\phi) = V \, p_{y,\tilde{y}}(V \cos\phi, V \sin\phi) \quad , \tag{2.32}$$

where $V = (y^2 + \tilde{y}^2)^{\frac{1}{2}}$, $\phi = \tan^{-1}(\tilde{y}/y)$, and $\tilde{y}(t)$ is the Hilbert transform of $y(t)$, Hall showed that the phase in uniformly distributed and that the envelope distribution is given by

$$p_V(V) = (\theta - 1) \, \gamma^{\theta - 1} \frac{V}{[V^2 + \gamma^2]^{(\theta+1)/2}} \quad . \tag{2.33}$$

For his model Hall also obtains expressions for the average rate of envelope level crossings and the distribution of pulse widths and pulse spacings. The envelope distributions and level crossing rates show good agreement with measurements but poor agreement with measurements of pulse width and pulse spacing distributions (Hall, 1966; Spaulding et al., 1969).

Hall uses his model to determine the optimum receiver for coherent ON-OFF signaling and analyzes its performance. [We will derive Hall's receiver in chapter 3].

While the Hall model results in expressions that are mathematically simple enough for solving detection problems, the parameters of the model, θ and γ, have no relation to the physical processes causing the interference.

Omura (1969) presented a noise process similar to that of Hall (1966). He defined

$$n(t) = A\, X(t) \sin(\omega_o t + \phi(t)) \quad , \qquad (2.34)$$

where

$$X(t) = \text{a log normal process} = e^{b(t)} \quad ,$$

where b(t) is a stationary Gaussian process with zero mean and autocorrelation $R_b(\tau)$ and A is a constant to be determined from noise power estimates. This results in the phase being uniformly distributed and

$$p_x(AX) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{\log(\frac{X}{A})}{\sigma}\right]^2\right\} \quad , \qquad (2.35)$$

where $\sigma \equiv \sigma_{\log X}$. Omura also obtained expressions for the average rate of envelope level crossings and pulse width and pulse spacing distributions. The model showed agreement with measurements only at the higher envelope levels. Omura used his model to calculate the performance of various LF and VLF digital modems.

Giordano (1970, 1972) used a filtered impulse model to obtain results similar to Furutsu and Ishida (1960) for the envelope distribution of atmospheric noise. He obtained

$$p(\epsilon) = \epsilon \int_0^\infty d\lambda\; H(\lambda)\; J_1(\lambda\epsilon) \quad , \qquad (2.36)$$

where

$$H(\lambda) = \exp\left(-\mu \int_0^\infty da_i \; p(a_i) \int_{t-T}^{t} dt_i \; \{1 - J_o[\lambda a_i b(t-t_i)]\}\right) , \tag{2.37}$$

and

$b(t)$ = envelope of the receiver impulse response,

a_i = strength of an input pulse (random variable),

$p(a_i)$ = pdf of a_i,

t_i = occurrence time of any input pulse (random variable),

T = observation interval,

t = observation time, and

μ = rate at which pulses arrive at the receiver.

He then evaluates (2.36) for various spatial distributions of sources and propagation situations. Each such assumption results in a different "model." One case of interest that Giordano treats is:

(1) Uniform spatial distribution of sources.

(2) Field strength that varies inversely with distance, $a_i = c/r_i$, r_i distance to ith source, and r_m = mean value of r_i.

(3) Arbitrary receiver envelope response.

The result is

$$p(\epsilon) = \frac{K}{(K^2 + \epsilon^2)^{\frac{1}{2}}} , \tag{2.38}$$

where

$$K = \frac{\mu C}{r_m} \int_0^T b(t)\ dt \quad .$$

The result (2.38) is a distribution of the Hall (1966) form, and so Giordano gave a physical rationale to the Hall model.

Giordano considered numerous other cases of propagation and source distributions and also developed expressions for the average rate of envelope crossings and pulse spacing distributions.

In addition to the above developments, there have been many studies that develop impulsive noise models to analyze system performance. Bello and Esposito (1969, 1971) use the receiver impulse characteristic (RIC) defined by

$$R_K(\Gamma,\delta) = \frac{1}{(2\pi)^K} \int_0^{2\pi} \cdots \int_0^{2\pi} p_K(E|\Gamma,\delta,\psi)\ d\psi \quad , \qquad (2.39)$$

where $R_K(\Gamma,\delta)$ is the kth order RIC, Γ is a K-dimensional noise pulse amplitude vector, δ is a K-dimentional pulse occurrence time vector, ψ is a K-dimensional noise phase vector, and $p_K(E|\Gamma,\delta,\psi)$ is the conditional probability that K noise impulses occur per bit. They use the customary model in which the noise takes the form of a summation of filtered impulses, the arrival times of the impulses being Poisson distributed. Bello and Esposito evaluate error rates for PSK and DPSK with and without hard-limiting. In their analysis the impulses are assumed to be nonoverlapping.

Ovchinnikov (1973) and Richter and Smits (1974) present analyses which include the intermediate case where impulses overlap, but not so frequently as to approach Gaussian noise. Richter and Smits (1974) also evaluate the case of "smear-desmear" filtering.

Shaver et al. (1972) used a "Markov Regrime Model" to represent man-made noise. They represent the interference as

$$\underline{z}(t) = \underline{n_o}(t) + \underline{\gamma_o}(t) \quad , \qquad (2.40)$$

where $\underline{n_o}(t)$ represents Gaussian background noise and $\underline{\gamma_o}(t)$ represents man-made noise, which may or may not have a Gaussian component but which when added to $\underline{n_o}(t)$ does represent the complex noise envelope at time t. They allow $\underline{\gamma_o}$ to be a process that can be described as a two-state Markov chain,

$$\begin{aligned} \underline{\gamma_o} &= 0 \text{ with probability } p(a), \text{ (state a)} \\ \underline{\gamma_o} &= \underline{n_1}(t) \text{ with probability } p(b), \text{ (state b)} \quad , \end{aligned} \qquad (2.41)$$

where $\underline{n_1}$ is a complex Gaussian random process representing man-made noise, and σ_{n_1} can be large and p(b) very small. They give no comparisons with data and proceed to use this "little Gauss-big Gauss" model to analyze the performance of various digital modems.

The above is a short summary of the most significant models proposed to date. There have been numerous others similar to one or more of the above. With the exception of the Hall model, none of the above models has ever been used to attempt to determine optimum detection algorithms.

In the next section we will summarize Middleton's recently proposed "physical-statistical" model for impulsive interference.

2.3 Middleton's Physical-Statistical Model for Impulsive Interference

Recent work by Middleton has led to the development of a physical-statistical model for radio noise. In this section we wish briefly to summarize this model, presenting the main results which we use in subsequent chapters for the solution of various signal detection and system performance problems. The Middleton model is the only general one proposed to date in which the parameters of the model are determined explicitly by the underlying physical mechanisms (e.g., source density, beam-patterns, propagation conditions, emission waveforms, etc.). It is also the first model which treats narrowband interference processes (class A) as well as the traditional broadband processes (class B). As we shall see, the model is also canonical in nature in that the mathematical forms do not change with changing physical conditions. We will also show some comparisons of the model with measurements for both class A and class B interference.

As in past models, Middleton's model postulates the familiar Poisson mechanism for the initiation of the interfering signals that comprise the received waveform X(t). The received interfering process is

$$X(t) = \sum_j U_j(t,\underline{\theta}) \quad , \tag{2.42}$$

where U_j denotes the jth received waveform from an interfering source and $\underline{\theta}$ represents the random parameters that describe the waveform scale and structure. It is next assumed that only one type of waveform, U, is generated, with variations in the individual waveforms taken care of by appropriate statistical treatment of the parameters $\underline{\theta}$.

With the assumption that the sources are Poisson distributed in space and emit their waveforms independently according to the Poisson distribution in time, the first-order characteristic function of X(t) is well known to be

$$F_1(i\xi,t)_P = \exp\left[\int_\Lambda \rho(\underline{\lambda}) \langle e^{i\xi U(t;\underline{\lambda},\underline{\theta})} - 1\rangle_{\underline{\theta}} \, d\underline{\lambda}\right] \quad , \tag{2.43}$$

where $\underline{\lambda}$ are coordinates of the source-receiver geometry and Λ is the physical domain in which the sources are located. The $\rho(\underline{\lambda})$ is the process density (which has been defined by Middleton, 1967). The quantity $\int_\Lambda \rho(\underline{\lambda})\, d\underline{\lambda} \equiv A$ is one of the basic parameters of the model and $\int_\Lambda \rho(\underline{\lambda})\,(\cdot)\, d\underline{\lambda}$ is a "counting" functional which adds up the contributions of the individual sources. The quantity A (a basic parameter) is called the Impulsive Index. Specifically A can be shown to be equal to $\nu_T \overline{T}_s$, where ν_T is the average rate of "signal" generation and $\overline{T}_s$ is the mean duration of a typical interfering signal. The Impulsive Index measures the amount of temporal overlap among the waveforms of the interfering signals (outside the receiver). Large A means large overlap with a corresponding approach to Gauss while small A means highly "impulsive" interference.

The next step is to obtain the generic waveform U(t) explicitly from the underlying physical mechanisms (see Middleton, 1972, for this development). The waveform U(t) is written in envelope and phase form, with $B_o(t,\underline{\lambda},\underline{\theta})$ denoting the envelope. This gives us (see Middleton, 1974, sec. 3)

$$F_1(i\xi)_P = \exp[\langle A\ J_o(B_o\xi) - A\rangle] \quad , \qquad (2.44)$$

where $\langle\cdot\rangle$ denotes required statistical averages over the random epoch representing the time at which the typical jth source emits, doppler velocities, if any, and the random signal parameters $\underline{\theta}$. The characteristic function (2.44) is of the same form as that obtained previously by various investigators (e.g., see Fruratsu and Ishida, 1960, Giordano, 1970, etc.). These past investigators have made various assumptions for the distributions required to perform the averages indicated in (2.44), performed these averages, and then transformed the resulting characteristic function to obtain their model. Each different assumption, of course, leads to a different model (mathematical form). In our present case, we have B_o explicitly related to the physics causing the interference. A unique approach of Middleton's model is to develop expressions for the transform of $F_1(i\xi)_p$ above without performing the indicated averages explicitly, thereby obtaining a canonical model.

We now must make the distinction between our class A and class B interference. For narrowband interference, the signal duration, T_s, inside the receiver, is finite, allowing us to write in (2.44) $\langle A\, J_o(B_o\xi) - A\rangle$ as $A\langle J_o(B_o\xi)\rangle - A$. On the other hand, for broadband sources, the impulse response of the receiver results in the signal duration, T_s, being infinite. Now $\langle A \rangle$ is infinite, so that $\langle A\, J_o(B_o\xi)-A\rangle$ must be considered as a whole. Another consequence of the above is that for the pure Poisson process, class A results in "gaps in time;" i.e., periods of time during which there is no interference in the receiver, whereas for class B, there are no gaps in time because the receiver responses always overlap.

To obtain a canonical reduction of the characteristic function for class A interference, Middleton (1974, sec. 3) shows that using the steepest-descent approximation for $\langle J_o(B_o\xi)\rangle$ gives correct behavior of the pdf for both large and small values of the amplitude; i.e.,

$$\langle J_o(B_o\xi)\rangle \approx \exp[-\langle B_o^2\rangle \xi^2/4] \quad . \qquad (2.45)$$

Using (2.45), Middleton obtains the exact expansion

$$F_1(i\xi,t)_P = \exp\{A\, H_1(i\xi,t)\} \quad , \tag{2.46}$$

where

$$H_1(i\xi,t) = e^{-\xi^2\langle B_o{}^2\rangle/4}\left[1 + \sum_{\ell=2}^{\infty} \frac{C_{2\ell}(-1)^{\ell}\xi^{2\ell}\langle B_o{}^2\rangle^{\ell}}{2^{2\ell}(\ell!)^2}\right] -1 \ , \tag{2.47}$$

with the coefficients $C_{2\ell}$ given by

$$C_{2\ell} = \ell!(-1)^{\ell}\ \langle {}_1F_1(-\ell;1;B_o{}^2/\langle B_o{}^2\rangle)\rangle \quad , \tag{2.48}$$

where ${}_1F_1$ is a confluent hypergeometric function, terminating after ℓ+1 terms. The above gives us finally

$$F_1(i\xi)_P = \exp[-A + A\langle J_o(B_o\xi)\rangle]$$

$$= e^{-A}\sum_{m=0}^{\infty}\frac{A^m}{m!}\, e^{-m\xi^2\langle B_o{}^2\rangle/4}\left[1 + \frac{A\langle B_o{}^2\rangle^2\, C_4\, e^{-\xi^2\langle B_o{}^2\rangle/4}}{4^3} + \ldots\right] . \tag{2.49}$$

The resulting pdf may now be obtained term by term by transforming the above characteristic function. Since it has been shown that the coefficients C_4, C_6, etc. have little effect (for our purposes) on the resulting pdf (Middleton, 1974), we will use only the first term of (2.49).

A more general model of the man-made noise environment includes an additive independent Gaussian background process. The additive background is due either to receiver noise, the limit of a high density Poisson process representing the contributions of the nonresolvable background sources, or both. For this we have

$$F_1(i\xi,t)_{P+G} = F_1(i\xi,t)_P \, F_1(i\xi,t)_G \quad , \tag{2.50}$$

where

$$F_1(i\xi,t)_G = e^{-\xi^2 \sigma_G{}^2/2} \quad . \tag{2.51}$$

The result corresponding to (2.49), with only the first term, is

$$F_1(i\xi)_{P+G} \approx e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} e^{-c_m{}^2 \xi^2/2} \quad , \tag{2.52}$$

where

$$c_m{}^2 = m\langle B_o{}^2\rangle/2 + \sigma_G{}^2 \quad . \tag{2.53}$$

We now define the second basic parameter of the model, Γ', as the ratio of the power in the Gaussian portion of the interference to the power in the Poisson portion,

$$\Gamma' = \overline{X_G{}^2}/\overline{X_P{}^2} = \frac{\sigma_G{}^2}{A\langle B_o{}^2\rangle/2} \quad . \tag{2.54}$$

We now want, for computational and discussion purposes generally, to consider the standardized variable

$$z = \frac{X}{\sqrt{\overline{X_G{}^2} + \overline{X_P{}^2}}} \quad . \tag{2.55}$$

We obtain, upon transforming (2.52) for the standardized variable z,

$$p_Z(z)_{P+G} \approx e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-z^2/2\sigma_m^2} , \qquad (2.56)$$

where

$$\sigma_m^2 = \frac{c_m^2}{\frac{A\langle B_o^2\rangle}{2}(1+\Gamma')} = \frac{m/A + \Gamma'}{1 + \Gamma'} . \qquad (2.57)$$

It is the result (2.56) for the pdf of the instantaneous amplitude of the received interference process which we will use in subsequent chapters for the solution of signal detection problems.

A special case of (2.56) occurs when there is no Gaussian background ($\Gamma' = 0$). This gives us

$$p_Z(z)_P = e^{-A}\,\delta(z-0) + e^{-A} \sum_{m=1}^{\infty} \frac{A^m}{m!\sqrt{2\pi\sigma_m^2}} e^{-z^2/2\sigma_m^2} . \qquad (2.58)$$

The $e^{-A}\delta(z-0)$ gives the probability of no interference ("gaps in time").

Figure 2.1 shows (2.58) for various values of the parameter A ($\Gamma'=0$). Note that for small A we have large impulsive "tails" and as A becomes large ($\sim$10) we approach the limiting case of Gaussian interference (still narrow-band interference, however). Figure 2.2 shows (2.56) for $\Gamma' = 0.001$ and various A while figure 2.3 shows (2.56) for $\Gamma' = 0.1$ and various A.

In the above, note that $p_Z(z)$ is given by a weighted sum of Gaussian distributions with increasing variance. The above summarizes the results that have been published to date (Middleton, 1974). We want now to summarize the corresponding results for the envelope distribution and the results for class B interference (Middleton, 1975).

The class A envelope distribution is obtained by a similar expansion of the characteristic function after averaging out the uniformly distributed phase. The result is, not surprisingly, for the standardized envelope

$$P(\epsilon > \epsilon_o)_A = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} e^{-\epsilon_o^2/\sigma_m^2} \quad ; \qquad (2.59)$$

i.e., a weighted sum of Rayleigh distributions with increasing variance. Figure 2.4 shows the envelope APD (2.59) for $A = 0.1$ for various Γ', while figure 2.5 shows (2.59) for $\Gamma' = 10^{-4}$ for various A. The coordinates used in figures 2.4 and 2.5 are such that a Rayleigh distribution (envelope of Gauss) plots at a straight line of slope - 1/2. Note the impulsive "tails" departing from the low level Gaussian background at the lower probabilities.

For the distribution of the envelope of class A interference plus a signal $\sqrt{2S}\cos\omega_o t$, the following result is obtained:

$$p(\epsilon) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \left[\frac{2\epsilon}{\sigma_m^2} e^{-\frac{\epsilon^2+2S}{\sigma_m^2}} I_o\left(\frac{2\epsilon\sqrt{2S}}{\sigma_m^2}\right) \right] , \qquad (2.60)$$

i.e., a weighted sum of Nakagami-Rice distributions.

For the class B case, where in expanding the characteristic functions, the $\langle A\, J_o(B_o) - A \rangle$ cannot be simplified, but must be used as is; the results are (Middleton, 1975):

$$p_Z(z)_B = \frac{e^{-A}}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} A_\alpha^m \, \Gamma\left(\frac{m\alpha+1}{2}\right) \, {}_1F_1\left(-\frac{m\alpha}{2};\frac{1}{2};z^2\right) \quad ,(2.61)$$

where ${}_1F_1$ is a confluent hypergeometric function. The model (2.61) has the two parameters α and A_α. Both these parameters are intimately involved in the physical processes causing the interference. That is, the class B model is sensitive to sourse distributions and the propagation law, whereas the class A model is insensitive to these parameters. Specifically

$$\alpha = \frac{2-\mu}{\gamma}, \; 0 \le \alpha < 2 \quad , \qquad (2.62)$$

where

$$\text{source density} \sim 1/\lambda^\mu \quad ,$$

and

$$\text{propagation law} \sim 1/\lambda^\gamma \quad .$$

The parameter A_α includes the Impulsive Index A, the parameter α, and other terms depending on the physical mechanisms. For class B interference there are no "gaps in time;" i.e., a background is always present, arising from the overlap of interference in the receiver. The normalization in (2.61) is to the power in the Gaussian portion of the distribution, since as with the Hall (1966) model or the Furutsu and Ishida (1960) model, we obtain infinite variance for some values of the parameters α and A_α. For the case $\alpha = 1$, (2.61) reduces to a distribution of the Hall form (2.30).

The corresponding results for the envelope APD for class B are

$$P(\epsilon > \epsilon_o)_B = e^{-\epsilon_o^2}\left[1 - \epsilon_o^2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} A_\alpha^m \times \Gamma\left(1 + \frac{m\alpha}{2}\right) {}_1F_1\left(1 - \frac{m\alpha}{2};2;\epsilon_o^2\right)\right] . \quad (2.63)$$

In (2.61) and (2.62), the confluent hypergeometric functions are not well behaved for large values of the amplitude, resulting in numerical complexities in evaluating $p_Z(z)_B$ or $P(\epsilon > \epsilon_o)_B$. Figure 2.6 shows $p_Z(z)_B$ (2.61) for $\alpha = 1.0$ for various A_α and figure 2.7 shows $P_Z(z)_B$ for $A_\alpha = 1.0$ for various α. Figure 2.8 shows $P(\epsilon > \epsilon_o)_B$ for $\alpha = 1.0$ for various A_α while figure 2.9 shows $P(\epsilon > \epsilon_o)_B$ for $A_\alpha = 1.0$ for various α. A comparison of figures 2.1-2.5 with 2.6-2.9 shows the distinct differences between class A and class B impulsive interference. The most striking difference is that for class A interference the impulsive "tail" departs abruptly and rapidly from the Gaussian background (e.g., fig. 2.4) while for class B interference this departure is much more gentle (e.g., fig. 219), due to the overlap of the receiver responses.

Now that we have summarized Middleton's model, we want to present a few comparisons of the model with typical measured data for both class A and class B interference.

For class A we present two comparisons (almost the only detailed class A measurements available to date). Figure 2.10 shows the measured envelope distribution, $P(\epsilon > \epsilon_o)$, of a narrowband impulsive interference (from Bolton, 1972, fig. 17) along with the envelope distribution (2.59) for $A = 0.35$ and $\Gamma' = 0.5 \times 10^{-3}$. The comparison is seen to be quite good and we will consistently use this example, when, in subsequent chapters, we compute optimum system performance. Figure 2.11 shows the measured envelope distribution (from Adams et al., 1974, fig. 4-17) of narrowband interference from ore crushing machinery along with the envelope distribution (2.59) for $A = 10^{-4}$ and $\Gamma' = 50$. Again, agreement between experiment and theory is quite excellent. Figure 2.12 shows a measured envelope distribution of broadband impulsive interference (from Adams et al., 1974, fig. 4-42) along with the envelope distribution (2.63) for $A_\alpha = 10^{-2}$ and $\alpha = 1.85$. The interference shown on figure 2.12 was probably primarily due to fluorescent lights. Figure 2.13 shows an example of the measured envelope distribution of broadband man-made noise (primarily automotive ignition noise, from Spaulding and Espeland, 1971, fig. 41) along with the envelope distribution (2.63) for $A_\alpha = 1.0$ and $\alpha = 1.5$. Finally, figure 2.14 shows an example of the measured envelope distribution of atmospheric noise (from Espeland and Spaulding, 1970, p. 89) along with the envelope distribution (2.63) for $A_\alpha = 1.0$ and $\alpha = 1.2$.

As the above comparisons show, the Middleton model shows extremely good comparisons with typical measured results for both class A and class B interference.