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Median-voter Equilibria in the Neoclassical Growth Model under Aggregation

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Abstract

We study a dynamic version of Meltzer and Richard's median-voter model where agents differ in initial wealth. Taxes are proportional to total income, and they are redistributed as equal lump-sum transfers. Voting takes place every period and each consumer votes for the current tax rate that maximizes his or her welfare. We characterize time-consistent (differentiable) Markov-perfect equilibria in three ways. First, by restricting the class of utility functions, we show that independently of the number of wealth types, the economy's aggregate state can be summarized by two statistics: mean and median wealth. Second, we derive the median voter's first-order condition and interpret it in terms of a tradeoff between distortions and net wealth transfers. Third, we illustrate the key endogenous-taxation mechanisms using 1- and 2-period versions of the model. Quantitatively, we find that the model in its baseline form cannot explain the large wealth inequality that we observe in most economies.

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1 Introduction

Income taxes are arguably important determinants of aggregate economic performance, and they are fundamentally endogenous. What determines taxes? A widely held belief is that the desire to redistribute is a key explanatory factor, and furthermore that the amount of redistribution is one of the central elements over which elections are decided. One way to evaluate this theory is to construct a reasonably calibrated macroeconomic model and to compare its politico-economic equilibrium to data. Dynamic models of political economy are complex objects of analysis, however, and the development of the theory and the associated numerical analysis and empirical methods are still in their infancy. The goal of this paper is to contribute to this methodological development. We consider a theory based on endogenous redistribution between consumers of different wealth types: agents vote on general income tax rates, with associated equal-per-capita lump-sum transfers. The setting is a standard decentralized version of the one-sector neoclassical growth model, and the political aggregation mechanism is majority voting. Taxes are proportional to total income, and they are redistributed as equal lump-sum transfers. Voting takes place every period and each consumer votes for the current tax rate that maximizes his or her welfare, and we characterize time-consistent (differentiable) Markov-perfect equilibria.

The two most closely related papers in the literature are Meltzer and Richard (1981) and Krusell and Ríos-Rull (1999). The former paper describes a static setup where distortionary labor taxes are used to fund the transfers. Though conceptually constituting the core of modern median-voter models, Meltzer and Richard's setup is arguably not well-suited for quantitative analysis in that it does not deal with the taxation of capital income and the associated distortions. Krusell and Ríos-Rull (1999), on the other hand, is a fully dynamic model that considers the taxation of capital income in a Meltzer-Richard kind of setting. Krusell and Ríos-Rull define equilibrium in the dynamic model, provide a method for numerically finding equilibrium, and report quantitative findings for a calibration of the model. Compared to that paper, in terms of theory, the present work makes three contributions: (i) it develops an aggregation result that is useful for simplifying the analysis, (ii) it derives a first-order condition for the median voter that allows interpretation of the results, and (iii) by studying two "baby versions" of the fully dynamic setup considered last—both a 1-period and a 2-period model—it offers a better understanding of some of the intuitive mechanisms behind tax determination. In terms of implementation, we also use an entirely different method for computing equilibria, one that is based directly on the median voter's first-order condition.

In addition, finally, this paper also studies a slightly different setting than the one used in Krusell and Ríos-Rull (1999), who assume that there is an implementation lag for taxes, i.e., in their setting, the tax voted on at t is implemented at $t + k$, with $k > 0$. This assumption means that as far as its impact on capital accumulation, there is no immediate impact at all of raising the tax rate at t—capital income is completely inelastic. The absence of an implementation lag in the present work leads to significantly different quantitative results. In particular, the model predicts that only a very narrow range of wealth distributions can be observed as a long-run outcome, and the wealth inequality observed in most developed countries is outside this range. In short, given the large wealth inequality observed, our model predicts that the median voter would tax away most of these differences and that the economy would subsequently converge to a new steady state with much lower inequality, i.e., we find that the model is unable to account for the observed combination of taxes and inequality—the marginal benefit to the median voter of further taxation by far exceeds the marginal cost. Therefore, we learn that a model that would have greater quantitative success would need a larger cost of taxation or a smaller benefit. The setting in Krusell and Ríos-Rull (1999) can better explain the data because their implementation lag is one way of making taxation more costly, however, the nature of implementation lags in actual tax constitutions is not apparent. One implication of the findings here is indeed that we need quantitative measurement of any lags between political decision making and implementation: these lags really matter quantitatively for what our theories predict.

Our aggregation characterization roughly is as follows: given consumer preferences of the appropriate form, Markov-perfect equilibrium outcomes which arise as limits of finite-horizon equilibria depend on the mean and the median asset holdings, and on nothing else. This finding is useful because it simplifies both the theoretical and numerical analysis. It is not surprising: with preferences that allow aggregation (for simplicity, we look at constant-elasticity-of-intertemporal-substitution preferences here), so long as taxes do not depend on anything but mean and median wealth, neither can prices nor aggregate quantities: marginal propensities to save and work are equal across all consumers, so aggregates can be arrived at by summing up across individuals; moreover, the propensities cannot depend on higher moments through taxes, since taxes cannot depend on any other other moments by backwards induction.

The explicit first-order conditions for the median voter that we use in our analysis are significantly more complex, both conceptually and mathematically, than standard first-order conditions from optimal control theory. They express how the median voter trades off the marginal benefits against the marginal costs of changes in income taxes. The tradeoff is expressed in terms of distortions to labor-leisure and consumption-savings choices—"gaps"—on the one hand and net transfer effects on the other. By measuring the size of each of the gaps in the calibrated politico-economic equilibrium, one could provide an assessment of which tradeoffs matter the most. Prior analysis focusing on first-order conditions in similar contexts include the work on individual saving under time-inconsistent preferences, where reminiscent first-order conditions have been derived (so-called "generalized Euler equations"; see Laibson (1997)), and some recent work on dynamic public finance (e.g., see Klein, Krusell, and Ríos-Rull (2003)). Azzimonti, de Francisco, Krusell, and Ríos-Rull (2005) surveys these methods and their use in different applications, one of which is the present setup. That paper derives the first-order condition for the present model from first principles; here, we show how to derive the condition only in the 1- and 2-period models we consider; for the infinite-horizon version, we simply state the result and focus on its contents.

Numerical solution of models of the kind studied in this paper is not straightforward either. Steady states are impossible to find the way they are found in growth models with exogenous policy, because the level of capital and the tax rate on income depend, via the first-order condition of the median voter, on the derivatives of the equilibrium decision rules. This means that one cannot specify a finite set of equations in levels only: levels depend on decision rule derivatives, which in turn depend on higher-order derivatives of these same decision rules. One approach for computation is to specify decision rules of some flexible parameterized functional form over some domain and evaluate all the equilibrium conditions, including first-order condition of the median voter, on this domain. Thus, one can iterate on the parameters of the functions in order to meet some criterion. This approach seems feasible but cumbersome to implement, since we are dealing with decision rules with more than one argument and since, in our recursive equilibrium definition there are several such functions whose shape are unknown. In this paper, we instead continue the development of an approach suggested in Krusell and Smith (2003) and later applied elsewhere which is fast and which allows a systematic search over all possible steady states.[1] This method, which builds on approximating the decision rules with polynomials evaluated at the steady-state point only, generates more equilibrium conditions—which are expressed as functional equations (the unknown decision rules being the unknowns and their arguments being mean and median wealth)—by successive differentiation and evaluation of these conditions at steady state.

Our focus on (differentiable) Markov-perfect equilibria, i.e., those where the state of the economy consists of payoff-relevant information only (see, e.g., Maskin and Tirole (2001)), by design rules out the study of the possible role of “reputation” in influencing political outcomes, whereby voters would collectively make their current voting behavior depend on historical voting/policy outcomes (see, e.g., Bernheim and Nataraj (2002)). Thus, we study the “fundamental” endogenous-tax equilibrium. Even if basic, such a framework, however, almost by definition is very complex: it involves both economic and political dynamics. Most existing papers on the topic are either based on computational work without much theoretical characterization (see, e.g., Krusell and Ríos-Rull (1999)) or on models which are not quantitatively satisfactory. In the latter category, physical capital accumulation is typically ignored or studied in frameworks that do not allow decreasing returns to capital (see, e.g., Persson and Tabellini (1994), Alesina and Rodrik (1994), and Hassler, Rodriguez-Mora, Storesletten, and Zilibotti (2003)).

The paper is organized as follows. A 1-period model is analyzed in Section 2, a 2-period model in Section 3, and an infinite-period model is finally discussed in Section 4. In each of these sections, both equilibria with exogenous and endogenous taxes are discussed in turn, and aggregation is explored. There is also emphasis, for each version of the model, on the first-order condition from the median voter's tax problem, since it changes nature depending on the time

[1] See, e.g., Krusell (2002) and Klein, Krusell, and Ríos-Rull (2003).

horizon. Moreover, each of the sections concludes with a functional-form example that admits "almost" closed-form solutions. Section 5 concludes. Finally, an Appendix contains some formal definitions and proofs.

2 The 1-period model

The 1-period model is a simple extension of Meltzer and Richard (1981) to include capital. The 1-period model with capital makes taxation more beneficial, on net, for the median voter: capital is inelastically supplied as the tax is decided upon. One can argue that by only focusing on labor income, Meltzer and Richard ignored a key determinant behind large governments: the existence of a large stock of unequally distributed wealth that can be taxed away and redistributed at low cost. The taxation of capital in an intertemporal setting is more complex—taxation at t distorts saving in earlier time periods, and because it redistributes it influences future taxation as well—so it is important to build slowly toward the fully dynamic analysis. This goal is accomplished here by the use of first a 1-period, then a 2-period, and finally and infinite-horizon model.

In the present section, we will (i) set up the basic environment, which will then be considered in a dynamic context as well and (ii) to use a simple example to discuss aggregation, existence, and how taxes are determined as a function of the primitives in this environment.

2.1 The environment: exogenous taxes

We describe the basic environment first, and then the decentralized equilibrium.

2.1.1 Environment

There is a set of agents who only differ in initial asset holdings. We will assume for simplicity that the number of "types" is finite with measure μ_i for type $i \in \{1, 2, \ldots, I\}$. Population size is normalized to one: $\sum_{i=1}^{I} \mu_i = 1$.

Utility of each agent is $u(c, l)$, where c is consumption and l is leisure. Consumption and leisure both have to be nonnegative.

Production takes place according to a production function which depends on capital and labor and has constant returns to scale: $Y = F(K, N)$ (we use capital letters to denote aggregates). In the 1-period model, all of output is (privately) consumed: $Y = C$.

There is a constraint on the amount of time for each agent: each consumer has one unit, so that $l_i + n_i = 1$ for all i, where n_i denotes the amount of hours worked. We will make assumptions on primitives so that agents' decision problems are strictly concave; hence, all agents of the same type will make the same decisions and we can also write $L_i + N_i = 1$, where L_i and N_i reflect common decisions regarding leisure and labor of all agents of type i. Therefore the aggregate labor input is $N = \sum_{i=1}^{I} \mu_i N_i$.

2.1.2 Equilibrium

In a decentralized economy, consumers buy consumption goods and rent capital and sell their labor services to firms under perfect competition. The rental rate for capital is denoted r and the wage rate w, both in terms of consumption goods in the same period. In addition, we now consider a government which taxes income at a proportional rate τ—capital and labor income are taxed at the same rate—and makes equal lump-sum transfers T back to all consumers under a balanced budget. Thus, a typical consumer i's budget set $B(A_i)$ reads

$$B(A_i) \equiv \left\{(c,l) \in R_+^2 : \, c = (1-\tau)[A_i r + w(1-l)] + T\right\}.$$

In equilibrium, consumers' holdings of assets have to add up to the total capital stock: $\sum_{i=1}^{I} \mu_i A_i = K$. Consumer heterogeneity thus originates in $a_i \neq a_j$ for all $i \neq j$. We define a competitive equilibrium for a given government policy as follows:

Definition 1 *Given a policy τ, a* ***competitive equilibrium*** *is a set of prices (w, r) together with an allocation $(K, N, T, (C_i, A_i, L_i)_i)$ satisfying the following conditions.*

1. *For all i, (C_i, L_i) solves $\max_{(c,l)\in B(A_i)} u(c,l)$.*
2. *$w = F_N(K,N)$ and $r = F_K(K,N)$, where $N = \sum_{i=1}^{I} \mu_i(1-L_i)$ and $K = \sum_{i=1}^{I} \mu_i A_i$.*
3. *$T = \tau\,(Kr + Nw)$.*

2.1.3 Aggregation

The equilibrium outcome for prices and output in this economy generally depends on the entire distribution of assets and not just on the total capital stock. However, if the utility function has a simple form—$u(c,l)$ is homothetic—and consumers choices are interior, then only the total capital stock matters. In other words, no matter how any given amount of capital is distributed in the population, equilibrium prices and aggregate quantities are the same: there is "aggregation".

The essential insight in the proof is simple, and the algebra that represents the following discussion will be left to the reader. Under homotheticity, independently of a consumer's wealth level, the marginal rate of substitution between the two goods—consumption and leisure—is a function only of the ratio of the two goods (homogeneity) or of the ratio of two given affine functions of each of the goods (homotheticity).[2] When the consumer equalizes the marginal rate of substitution (assuming interiority) to the net-of-tax relative price, consumption can be expressed as an affine function of leisure, and the associated coefficients depend on prices but not on wealth. Therefore, when budget balance is imposed, it follows that (i) leisure as well as consumption must respond linearly to wealth and thus that (ii) the associated marginal propensities (coefficients in the linear expression) are the same for all agents, though of course they depend on prices. Given that the resulting demand functions have the same marginal propensities to consume for all agents, any distribution of a given amount of capital leads to the same total demand. This demand function can thus equivalently be derived from the assumption that there is only one type of agent in the economy: an agent with utility function u and total wealth K. In conclusion, given any equilibrium $\{(w, r), (K, N, T, (C_i, A_i, L_i)_i)\}$ we can find another equilibrium $\{(w, r), (K, N, T, (\hat{C}_i, \hat{A}_i, \hat{L}_i)_i)\}$, with $\sum_i \mu_i \hat{A}_i = \sum_i \mu_i A_i = K$ (and, of course, $\sum_i \mu_i \hat{C}_i = \sum_i \mu_i C_i$ as well as $\sum_i \mu_i \hat{L}_i = \sum_i \mu_i L_i$). In all the examples we consider, we use logarithmic utility—utility is a weighted average of $\log c$ and $\log l$—which satisfies homotheticity.

Given the aggregation theorem, we can define *competitive equilibrium outcome functions* for exogenous tax rates, namely the functions $\tilde{C}_i(K, \tau)$, $\tilde{L}_i(K, \tau)$, $\tilde{N}_i(K, \tau)$, $\tilde{N}(K, \tau)$, and $\tilde{T}(K, \tau)$. These functions will be used below.

[2] More generally, aggregation applies if the marginal rate of substitution is a function of the ratio of two affine functions of the consumption vectors.

2.2 Endogenous taxes and aggregation: politico-economic equilibrium

In order to focus as much as possible on the economics implied by the endogeneity of taxes, we will simply assume that the "median consumer" is the politically pivotal one. We will, moreover, take the median consumer to be the consumer with median wealth holdings. Later in this section, we will briefly discuss whether these assumptions are warranted, i.e., whether there is an explicit voting game with an equilibrium where these assumed properties are endogenous outcomes. We can now proceed to the definition of a median-voter equilibrium. We use subscripts m to denote the median agent.

Definition 2 *Given $(A_i)_i$, a **median-voter equilibrium** is a $\tau^* \leq 1$ and an associated competitive equilibrium $\{(w^*, r^*), (K, N^*, T^*, (C_i^*, A_i, L_i^*)_i)\}$ such that there is no other $\tau \leq 1$ and associated competitive equilibrium $\{(w, r), (K, N, T, (C_i, A_i, L_i)_i)\}$ for which $u(C_m, L_m) > u(C_m^*, L_m^*)$.*

Suppose that u is homothetic. Then given a vector $(A_i)_i$, can a median-voter equilibrium, or more precisely its prices and aggregate quantities, depend on anything but A_m and K? If all equilibria are characterized by interior solutions for all agents, the answer is no. Formally, one needs to show that for any median-voter equilibrium associated to a vector $(A_i)_i$, one could construct another median-voter equilibrium with identical prices and aggregate quantities associated with any other vector $(\hat{A}_i)_i$ as long as $\hat{A}_m = A_m$ and $\sum_i \mu_i \hat{A}_i = \sum_i \mu_i A_i = K$. But it is now straightforward to check that this must be the case, because (i) for any given K, the set of prices and aggregate quantities in any equilibria have to coincide, and thus (ii) median utility has to be identical across these equilibria, since prices are the same and the median agent's wealth is the same in the two cases. We summarize this finding as follows.

Claim 3 *Suppose that u is homothetic and that in any competitive equilibrium all agents' solutions are interior. Then given values for A_m and K, prices and aggregate quantities in a median-voter equilibrium are independent of $(A_i)_i$.*

We can therefore define *politico-economic equilibrium outcome functions* as functions of (K, A_m): $\tau = \Psi(K, A_m)$, $N = N(K, A_m)$, and $T = T(K, A_m)$.[3] They satisfy $N(K, A_m) = \tilde{N}(K, \Psi(K, A_m))$ and $T(K, A_m) = \Psi(K, A_m)[r(\Psi(K, A_m))K + w(\Psi(K, A_m))\tilde{N}(K, \Psi(K, A_m))]$.

We now briefly discuss whether a median-voter equilibrium can be supported using majority voting. First, note that in any equilibrium consumers differ only in one dimension: asset wealth. Second, we restrict attention to the case that delivers aggregation, i.e., homothetic utility. Now consider the indirect preferences over taxes of any agent:

$$u\left(\tilde{C}_i(K,\tau), \tilde{L}_i(K,\tau)\right) = u\left(b_c(K,\tau) + d_c(K,\tau)A_i, b_l(K,\tau) + d_l(K,\tau)A_i\right),$$

given homotheticity; it is crucial here to note that homotheticity implies that the coefficients (b_c, b_l, d_c, d_l) do not depend on i. Hence, the derivative with respect to τ reads

$$u_{1i}\left(\frac{\partial b_c}{\partial \tau} + A_i \frac{\partial d_c}{\partial \tau}\right) + u_{2i}\left(\frac{\partial b_l}{\partial \tau} + A_i \frac{\partial d_l}{\partial \tau}\right).$$

This expression is positive if and only if

$$\frac{u_{1i}}{u_{2i}}\left(\frac{\partial b_c}{\partial \tau} + A_i \frac{\partial d_c}{\partial \tau}\right) + \frac{\partial b_l}{\partial \tau} + A_i \frac{\partial d_l}{\partial \tau} > 0.$$

[3] Similarly, we can define $C_i(K, A_m)$ and $L_i(K, A_m)$.

Noting that u_{1i}/u_{2i} is equal to $w(1-\tau)$ for all i, because it has to be in any competitive equilibrium, this expression is affine in A_i, with coefficients that are the same for all agents (and that depend on prices and taxes). In particular, this means that if the agent with *median* asset holdings has the first-order condition for taxes met with equality, so that he is indifferent in terms of changing the tax rate at the margin; anyone poorer will prefer a higher tax and anyone richer would prefer a lower tax. More generally, if an agent prefers a marginal tax increase over an unchanged tax (in which case the inequality above is met), then so does anyone with a lower (or higher, depending on the sign of $w(1-\tau)\frac{\partial d_c}{\partial \tau} + \frac{\partial d_l}{\partial \tau}$) value for A_i. Thus, half (or more) of the population would support the median's preferred choice over any alternative tax choice.

2.3 Analysis: the median voter's first-order condition

In this model, there are several kinds of costs and benefits of taxing for the median voter. On the cost side, income taxes generate well-known distortions to the decisions of the agents, and these are taken into account by the median voter. The distortion to the median agent's labor-leisure decisions obviously matters directly to the median voter, but the distortions to the behavior of other agents matter as well, though indirectly. In particular, the latter influence the provision of inputs, which influence prices, and it is for this reason that the median agent cares about them. This situation is parallel to that considered in Meltzer and Richard's (1981) economy, though we are casting the discussion more directly in terms of distortions here.

On the benefit side, the median agent seeks to use the gap in wealth between himself and the mean agent to obtain transfers. In addition, the effect of taxation is to alter prices of capital and labor, which also influences the median agent: he may be more or less dependent on one of the sources of income than the mean agent, and thus would stand to benefit from some amount of price distortion. We will discuss in turn each of the different effects of taxation that are relevant to the median voter.

Given the median-voter theory, we can now derive the median voter's first-order condition, which is a necessary condition for a median-voter equilibrium (assuming that $\tau = 1$ will not be chosen; this will be guaranteed in the examples we look at). The median voter's indirect utility is given by

$$u\left(\tilde{C}_m(K,\tau), 1 - \tilde{N}_m(K,\tau)\right),$$

where

$$\tilde{C}_m(K,\tau) = \left(r(K,\tau)A_m + w(K,\tau)N_m(K,\tau) + \tau\left\{r(K,\tau)[K - A_m] + w(K,\tau)[\tilde{N}(K,\tau) - \tilde{N}_m(K,\tau)]\right\}\right)$$

using $r(K,\tau) \equiv F_k(K, \tilde{N}(K,\tau))$ and $w(K,\tau) \equiv F_n(K, \tilde{N}(K,\tau))$. Taking derivatives with respect to τ, we can write the resulting expression symbolically as

$$GAP_n \frac{d\tilde{N}_m}{d\tau} + GAP_{red} = 0.$$

The expression says that the optimal tax rate is chosen so that the weighted sum of distortions ("gaps" for the labor-leisure choice and for optimal redistribution) created by such a policy is equal to zero. The weights simply involve the induced changes in behavior, i.e., changes in labor supply.

More precisely, GAP_n is associated with the pure distortion on labor supply to the median voter:

$$GAP_n \equiv wu_{1m} - u_{2m}.$$

From the first-order condition of the consumer, this gap is zero if and only if taxes are zero; moreover, in a Pareto optimum this gap must be zero, since w is the marginal product of labor.

The other gap in the median voter's first-order condition is GAP_{red}, which measures how an increase in the marginal tax raises "redistribution", and thus utility. It reads

$$GAP_{red} = u_{cm} \underbrace{\left\{ r\left(K - A_m\right) + w\left(\tilde{N} - \tilde{N}_m\right) + \right.}_{\text{direct}}$$

$$\underbrace{\left. \tau\left(w \frac{d\left[\tilde{N} - \tilde{N}_m\right]}{d\tau} + \frac{dr}{d\tau}\left[K - A_m\right] + \frac{dw}{d\tau}\left[\tilde{N} - \tilde{N}_m\right]\right) + \frac{dr}{d\tau}A_m + \frac{dw}{d\tau}\tilde{N}_m \right\}}_{\text{indirect}}.$$

There are two sorts of redistribution that take place. One is through the change in the direct net redistribution that occurs. The direct net redistribution is

$$\tau\left\{ r(\tau)\left[K - A_m\right] + w(\tau)\left[\tilde{N}(\tau) - \tilde{N}_m(\tau)\right]\right\}.$$

It is influenced by taxes both through direct changes in demands from raising τ—the effects referred to as "direct" in the expression—and via changes in prices—the remaining "indirect" effects except for the last two terms.

Therefore, the median agent sees a net direct gain from taxation if he has lower asset holdings than the mean agent has. We will, in line with all available data, indeed assume that median asset holdings are lower than average asset holdings. Moreover, the mean agent is richer also in an overall wealth sense, since he only differs from the median agent in his asset holdings (recall that labor productivity, and thus the value of the sequence of time endowments, is equal among agents in the benchmark model). Therefore, if leisure is a normal good, he would buy more leisure and therefore work less than the median agent. This result means that the second direct effect of taxation is detrimental to the median agent: he loses, on net, by redistribution of labor income.

The indirect effects include a standard, Meltzer-Richard channel: increased redistribution lowers the gap between the median and mean labor supply, because it moves the net-present-value wealth of the two agents closer to each other. In this case, this is an effect in the median's favor, because the labor redistribution channel works against the median.

The indirect effects also include price effects. Here, a tax increase in the current period will not affect the total capital stock, but it will reduce work effort, leading to a lower rental rate and a higher wage. The median views both of these negatively: the lower rental rate is negative because his asset holdings are lower than mean asset holdings, and the higher wage rate is also negative because median labor supply exceeds mean labor supply.

Finally, the last two terms come from a second form of redistribution that occurs: through changes in the composition of income due to price changes. Even in the absence of transfers, a tax increase would lower N and thus increase w (and, given that we use a Cobb-Douglas production function in our application, decrease r). Thus, consumers whose income has a larger wage share in relative terms see an increased relative income share. This is the case for a median consumer with less than average assets: such a consumer has below-average capital income and above-average labor income, since poorer consumers buy less leisure and therefore work more. An agent with mean wealth obtains no gain at all from the change in income composition.[4]

[4]The proof of this statement is as follows. The derivative of $wN + rK = F_n(K,N)N + F_k(K,N)K$ with

It is apparent from the expression that if $A_m = K$, i.e., if the median consumer has wealth that exactly matches mean wealth, the first-order condition is met for a zero tax: the labor-leisure distortion is minimized at this point, there is no change in the net transfer from changing the tax, since the net transfer is always zero in this case, and finally, as argued in the previous paragraph, there is no gain (or loss) to agents with mean asset holdings from changing the composition of income through price changes.

2.4 Numerical results for the example economy

We now consider an example: $u(c,l) = \alpha \log c + (1-\alpha) \log l$ and Cobb-Douglas production with capital share θ.

2.4.1 Exogenous taxes

Consider competitive equilibria for different exogenous tax rates. In such case, we can find total labor supply in closed form:

$$\tilde{N}(\tau) = \frac{\alpha(1-\tau)(1-\theta)}{1-\alpha+\alpha(1-\tau)(1-\theta)}.$$

Moreover, one can show that

$$\tilde{C}_m(\tau) = \tilde{N}(\tau)^{1-\theta} K^{\theta} \left((1-\tau)(1-\alpha\theta(1-x)) + \tau \right),$$

respect to N equals both $F_{nn}N + F_n + F_{kn}K$ and, due to Euler's theorem since total factor income equals total production for a production function that is homogeneous of degree 1, F_n. This, in turn, means that $F_{nn}N + F_{kn}K$, i.e., the change in total income for the agent with mean asset and labor income only taking price effects into account, must equal zero. Thus, since a tax change operates through the change in N, we have the desired result.

where $x \equiv A_m/K$, whereas

$$\tilde{N}_m(\tau) = \tilde{N}(\tau)\left((1-\alpha)\frac{1-\theta x}{1-\theta} + \alpha\right).$$

As for consumption and labor supply of agents with assets A_i, the same formulas apply, with $x_i \equiv A_i/K$ in the place of x.

Notice that these expressions reflect interiority of all choices; corner solutions will never be optimal here unless a consumer cannot obtain positive consumption even while working all the time (setting leisure to zero). For an equilibrium to exist, therefore, the poorest consumer has to have enough net-of tax wealth. As is clear from the expression above, for consumption (and leisure) to be positive, we need $(1-\tau)(1-\alpha\theta(1-\underline{x}))+\tau > 0$, where $\underline{x} \equiv \frac{\min_i A_i}{K}$. Thus, no matter what $\underline{x}$ is, there is always a low enough tax rate for which this expression is violated. Given any asset distribution, this puts a bound on the set of feasible tax rates:

$$\tau > \underline{\tau} \equiv 1 - \frac{1}{\alpha\theta(1-\underline{x})}.$$

Notice, thus, that although aggregation holds for all feasible values of the tax rate in this economy, there is another kind of distribution dependence: whether an equilibrium exists or not depends on the distribution of wealth, and not just on average wealth. More generally, if preferences were such that corner solutions could apply, aggregation would break down at the point where one or more types of (poor) agents chooses a corner for leisure: at that point, the marginal propensity to work from decreasing taxes is 0, whereas the agent with mean wealth will have a strictly positive propensity to work.

2.4.2 The median voter's choice

First, note that since (i) consumption (of the median agent) is proportional to K^{θ}, (ii) labor supply is a function of the ratio x of median to mean income, and (iii) utility is logarithmic, the objective function for the median voter in terms of its choice variable, τ, does not depend on K: the level of wealth factors out.[5] This means that the tax choice will depend on x only. We will use $\psi(x)$ to denote this function $\psi(A_m/K) = \Psi(K, A_m)$ whenever these assumptions are employed.

The first-order condition for the median can be expressed as a second-order polynomial equation in the unknown: τ, and since one of the resulting solutions violates non-negativity of consumption or leisure, it can be ignored throughout the analysis. Looking first at the case $x < 1$, i.e., the empirically relevant case where median wealth is less than mean wealth, so that the median voter is poorer than average, we note that a positive tax rate is called for: it is strictly better with a small positive tax than with a zero tax. A slight deviation from a zero tax rate to a positive rate gives no first-order distortion loss, whereas it gives a first-order utility gain from the transfer it induces and from the change in income composition it leads to. A hundred per cent tax rate, on the other hand, is too high, because it leads to zero production and thus zero consumption for the median consumer. Thus, an interior, positive tax rate will be chosen, and it will be the one that solves the first-order condition stated.

For the case where the median consumer is richer than the mean, $x > 1$, we have the slightly odd outcome that the lower bound on taxes may bind. To see this, notice that when $x > 1$, median consumption as a fraction of mean output goes to infinity as the tax rate goes to minus infinity

[5]Had we included in total resources a stock of undepreciated capital that could be "eaten directly", without involving any labor input, this result would not have been possible to establish. In fact, in the multi-period versions of the present model, the assumption that depreciation is 100% precisely fills the role of making the level effect on taxes disappear.

(ignoring the lower bound on the tax rate). With large subsidies to working, the working hours of the agent with mean asset holdings approach 1 in this case, with leisure going to zero. Because $x > 1$, however, the formula for median working hours reveals that median leisure stays strictly bounded below and away from 0, so median utility would increase without bound as taxes go to minus infinity. Thus, were the median to be able to choose an infinite subsidy rate, he would. This is the case where a corner solution may apply: taxes are not allowed to be below $\underline{\tau}$. Thus, whether this lower bound is indeed the solution or an interior solution gives the highest utility depends on parameter values; we found straightforward examples of both.

Figure 1 depicts the equilibrium tax as a function of mean to median asset holdings for two alternative values of the capital share, θ; we set α at 0.3. The graph reveals that the tax rate is indeed decreasing in x: the lower is median wealth relative to mean wealth, the higher is the equilibrium tax rate. It also suggests that the equilibrium level of income taxes, for any wealth distribution, increases with the capital share. This can be understood by analyzing the median's first-order condition. As θ increases, capital income is relatively more important in the agents' budget constraints. Keeping in mind that the stock of capital is given and completely inelastic in this static economy, taxes are perceived as being less distortionary when θ is high. The labor gap is thus smaller for any given tax rate and net redistribution larger, inducing the median so set taxes at a higher level. In other words, when the capital share is high, the median can extract resources from richer agents at a low cost in terms of foregone labor income.

2.4.3 Concluding comments on existence

The nonexistence of a competitive equilibrium under exogenous taxes is not surprising: though the exact conditions for existence are not entirely trivial to find, in essence nonexistence arises

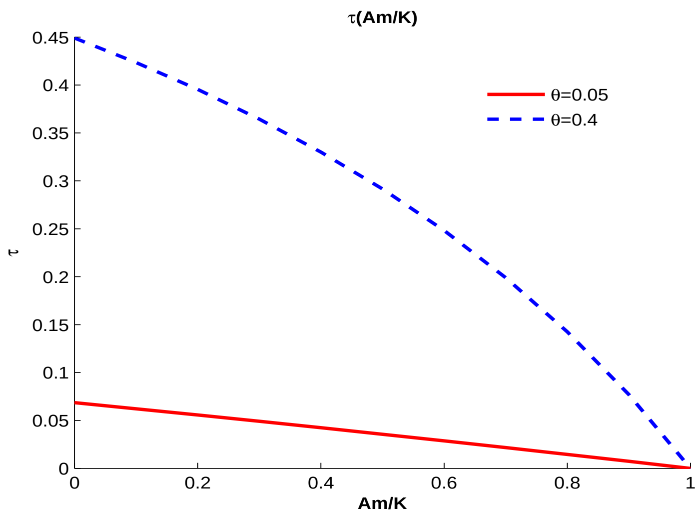

Figure 1: Taxes as a function of the median to mean wealth

because the initial wealth distribution is too unfavorable to the poorest agents to allow positive consumption. However, what about a median-voter (political) equilibrium: are there similar existence problems? We will not discuss this issue here in detail, but in essence the answer is no. The argument is that one can construct a set of tax rates that is nonempty and compact, and as long as the median voter's utility is continuous in the tax rate, a solution must exist to his choice problem. Continuity is evident in the example economy due to the functional forms assumed. Non-emptiness means that there are always tax rates that are feasible to choose for the median agent—that are associated with competitive equilibria. In particular, we always assume that a tax rate of 0 is feasible—a laissez-faire allocation is always feasible. Furthermore, the choice set is, or can be made, closed and bounded.[6]

[6]Note that $\underline{\tau}$ would lead to a maximum utility of minus infinity for the poorest agent. Therefore taxes that are associated with the existence of competitive equilibrium need to be strictly greater than $\underline{\tau}$. Similarly, the choice $\tau = 1$ is not, strictly speaking, consistent with competitive equilibrium. Thus, the set of taxes that are associated with competitive equilibria is an open set. The upper limit point will never be chosen by a median voter, and one can therefore without loss of generality impose a slightly lower bound than 1. The lower limit point, in contrast, would often be chosen, but in this case it does not seem restrictive to arbitrarily raise the lower bound ever so little in order to render the set closed.

We will again note the possibility of existence problems in the 2-period model when taxes are given exogenously, now for different reasons.

3 The 2-period model

The 2-period environment is a straightforward extension of what was described above. There is production in both periods, with the same technology, and capital can be accumulated in the usual neoclassical fashion. First-period output can be either consumed directly or installed as second-period capital. Feasibility in this economy is summarized as follows.

$$I_1 + \sum_i \mu_i C_{1i} = F(K_1, N_1),$$

$$K_2 = I_1,$$

$$\sum_i \mu_i C_{2i} = F(K_2, N_2),$$

$$N_t = 1 - \sum_i \mu_i L_{ti}, t = 1, 2$$

$$C_{ti}, L_{ti} \geq 0, t = 1, 2; \text{ all } i.$$

Notice that we are assuming full depreciation, i.e., that capital takes one period to install and then cannot be used again.[7] Utility is assumed to be time-additive and stationary for simplicity: $u(c_1, l_1) + \beta u(c_2, l_2)$.

The competitive environment will be discussed next.

[7]It is conceptually straightforward to consider the more general case of less than full depreciation; the reason we do not look at it here is that all the examples (numerical and analytical) rely on this assumption, since it admits closed forms.

3.1 Exogenous taxes in the first period

In the model with fully endogenous taxes presented below, we will assume that taxes in period t are voted on in period t, for $t = 1, 2$. That is, in period 1, the median voter *cannot commit to a period-2 tax rate.* This assumption means that in period 1, the median voter has to consider two effects of his choice of a τ_1: (i) the effect on current utility and (ii) the effect on capital accumulation and, thus, on period-2 utility. The second of these effects, moreover, involves how τ_2 will change in response to a change in the period-2 state variable induced by the choice of τ_1 in period 1.

We will immediately restrict attention to a homothetic utility function $u(c, l)$, as above, since this allows us to write the second-period outcome for tax rates as a function of K_2 and A_{2m} only: $\tau_2 = \Psi_2(K_2, A_{2m})$. The function Ψ_2 thus summarizes the endogenous tax-determination mechanism described in Section 2.2. Similarly, we can summarize utility outcomes in the second period with value functions where the aggregate state is (K_2, A_{2m}), and we can summarize aggregate equilibrium labor decisions with $N_2(K_2, A_{2m})$, and aggregate equilibrium transfers with $T_2(K_2, A_{2m})$, as defined above.

We begin by discussing the 2-period economy with an exogenously given first-period tax rate τ_1, along with the given initial asset distribution, $(A_{1i})_i$. Thus, we can define a competitive equilibrium in the two-period model recursively: we define equilibrium for any given τ_1 and an endogenously determined τ_2.

Definition 4 *Given a policy τ_1, a **competitive equilibrium** for period 1 of a 2-period economy is a set of prices (w_1, r_1) together with an allocation $(K_1, N_1, T_1, (C_{1i}, A_{1i}, L_i, A_{2i})_i)$ satisfying the following conditions.*

1. *For all* i, (C_{1i}, L_{1i}, A_{2i}) *solves*

$$\max_{(c,l,a')\in B_1(A_{1i})} u(c,l) + \beta V(a', K_2, A_{2m})$$

where $K_2 = \sum_{i=1}^{I} \mu_i A_{2i}$,

$$B_1(A_{1i}) \equiv \left\{(c,l,a') \in R_+^2 \times R : \quad c + a' = [A_{1i}r_1 + w_1(1-l)](1-\tau_1) + T_1\right\},$$

and $V(a', K_2, A_{2m})$ *is the appropriate indirect utility function for period 2, i.e.,*

$$V(a, K, A_m) \equiv$$

$$\max_{(c,l)\in R_+^2} u([aF_k(K, N_2(K, A_m)) + (1-l)F_n(K, N_2(K, A_m))](1-\Psi_2(K, A_m)) + T_2(K, A_m), l),$$

where N_2 *and* T_2 *are equilibrium functions, as solved for in a 1-period economy.*

2. $w_1 = F_n(K_1, N_1)$ *and* $r_1 = F_k(K_1, N_1)$, *where* $N_1 = \sum_{i=1}^{I} \mu_i(1 - L_{1i})$ *and* $K_1 = \sum_{i=1}^{I} \mu_i A_{1i}$.

3. $T_1 = \tau\,(K_1 r_1 + N_1 w_1)$.

3.1.1 Aggregation

Aggregation in this economy holds if present-value utility is homothetic jointly in all the goods that the consumer derives utility from: (c_1, l_1, c_2, l_2). For this to be true, it is not sufficient that u be homothetic in (c, l). However, most of the utility functions employed by applied macroeconomists satisfy the required criterion. In particular, if period utility is a power (including logarithmic, or exponential, or quadratic) function of a function of (c, l) that is homogeneous of degree 1, then the criterion is satisfied. In our example economy below, we use $u(c, l) = \alpha \log c + (1 - \alpha) \log l$, which is in this class. This formulation leads to a recursive formulation where consumers can be regarded as deriving utility in period 1 from three goods: period-1 consumption and leisure and assets left for period 2, (c_1, l_1, a_2), where the assets indirectly give utility through their provision of second-period consumption and leisure. Under the stated assumptions, V will be such that present-value utility is homothetic in (c_1, l_1, a_2).[8] Therefore, with the same kind of argument as was used for the 1-period model, we can deduce that equilibria will not depend on the distribution of capital but merely on its mean K_1 and on its median A_{1m}. The reason why A_{1m} matters is that it influences A_{2m}, and hence period-2 taxes.

As a result of these facts, we obtain period-1 equilibrium outcome functions that parallel those we defined for the 1-period model: $\tilde{C}_{1i}(K_1, A_{1m}, \tau_1)$, $\tilde{L}_{1i}(K_1, A_{1m}, \tau_1)$, $\tilde{N}_{1i}(K_1, A_{1m}, \tau_1)$, $\tilde{N}_1(K_1, A_{1m}, \tau_1)$, and $\tilde{T}_1(K_1, A_{1m}, \tau_1)$. We also obtain, in this case, the equilibrium outcome functions for savings: $A_{2i} = \tilde{H}_i(K_1, A_{1m}, \tau_1)$.

3.2 Politico-economic equilibrium

We can now state the following definition.

Definition 5 *Given $(A_{1i})_i$, a* ***median-voter equilibrium*** *in the 2-period economy is*

- *a $\tau_1^* \leq 1$ and an associated competitive equilibrium $\{(w_1^*, r_1^*), (K_1, N_1^*, T_1^*, (C_{1i}^*, A_{1i}, L_{1i}^*, A_{2i}^*)_i)\}$ such that there is no other $\tau \leq 1$ and associated competitive equilibrium $\{(w_1, r_1), (K_1, N_1, T_1, (C_{1i}, A_{1i}, L_{1i}, A_{2i})_i)\}$ for which $u(C_{1m}, L_{1m}) + \beta V(A_{1m}, K_2, A_{2m}) > u(C_{1m}^*, L_{1m}^*)+$*

[8] The algebraic proof of this claim is straightforward.

$\beta V(A^*_{1m}, K^*_2, A^*_{2m})$; and

- *a τ^*_2 which is a median-voter equilibrium, as defined above, for $(A^*_{2i})_i$.*

Again, this definition is a general one that applies for any period utility function u, and thus in general politico-economic outcomes could depend nontrivially on the whole initial distribution of asset holdings. Fortunately, if u has certain properties, we can extend our aggregation result from the 1-period model—that outcomes depend only on mean and median wealth—to the 2-period model by use of recursive methods. In particular, given that the result applies in the last period, and given intertemporal preferences in a certain class, the proof techniques for the last period can be used for the period immediately prior to it, and so on. So define

Assumption 6 *Suppose that $u(c, l) = f_1(f_2(c, l))$, where f_1 is a power function, logarithmic, exponential, or quadratic and f_2 is homogeneous of degree 1.*

We thus have the following.

Claim 7 *For the 2-period model, suppose that $u(c, l)$ satisfies Assumption 6 and that in any median-voter equilibrium all agents' solutions are interior. Then with A_{1m} and K_1 fixed, prices and aggregate quantities in a median-voter equilibrium are independent of $(A_{1i})_i$.*

The arguments underlying the proof can be used for models with any time horizon, including an infinite one. For infinite-horizon models, one can also imagine that other kinds of equilibria exist, namely those where other moments of the asset distribution matter due to "self-fulfilling expectations." We are not aware of models where such equilibria in fact do exist, however.

Aggregate and median outcomes in the 2-period median-voter equilibrium are summarized by the functions $\Psi_t(K, A_m)$, $N_t(K, A_m)$, $N_{tm}(K, A_m)$, $H(K, A_m)$, $H_m(K, A_m)$, and $T_t(K, A_m)$, for $t = 1, 2$. These are defined similarly to those in the 1-period model, e.g., $H_m(K, A_m) = \tilde{H}(K, A_m, \Psi_1(K, A_m))$.

The definition of a median-voter equilibrium in the 2-period model is stated in terms of sequences. Alternatively, an equilibrium can be defined directly as a set of functions. The infinite-horizon equilibrium definition below will be stated in terms of functions only, and it can be thought of as the limit of outcome functions dated t obtained from a sequence of finite-horizon models as those described here, using the same recursive technique, i.e., solving backwards from the last period.

Finally, though not in focus here, one needs to verify that the median-voter theorem also applies in the 2-period model. We will not develop this argument in detail, but it follows the idea behind the argument in the 1-period model: in consumption unit terms, if someone prefers to increase the tax rate slightly, so does anyone with a lower asset level, due to the linearity in A_i that follows from the functional forms we employ.

3.3 The median voter's first-order conditions

The problem faced by the second period's median voter is analogous to the one presented for the one-period economy. Since the economy only lasts two periods, and given the levels of K_2 and A_{m2}, the trade-offs faced are exactly the same than those faced in the static case. We can solve for the implied tax function in period two by using the GEE presented in section 2.3, which as discussed above will take the form $\Psi_2(K_2, A_{m2})$.

The median voter in the first period chooses taxes taking into account how the winner of the next election—his future self in this case—will choose taxes tomorrow (the Ψ_2 function).

Therefore, when finding the optimal level for τ_1, he must consider how this will affect current period savings (summarized by the functions $\tilde{H}_m$ and $\tilde{H}$), which by modifying the level of assets that the next incumbent inherits will influence next period's economic as well as political (tax) outcomes.

Like in the one-period economy, the median voter will trade off distortions away from the first-best—gaps—that are introduced by redistributive policies. There are a larger *number* of distortions in a 2-period economy than in a 1-period economy. In particular, a wedge is introduced in the first period's savings decision. The final result is a first-order condition for the median voter in the first period—a "generalized Euler equation", or GEE—that can be written as a weighted sum of gaps that involve wedges in both periods:

$$\underbrace{GAP_{1n}\frac{d\tilde{N}_{1m}}{d\tau_1} + GAP_{red} + GAP_{a'_{1m}}\frac{d\tilde{H}_m}{d\tau_1}}_{\text{t=1}} + \beta\underbrace{\left[GAP_{2n}\frac{d\tilde{N}_{2m}}{d\tau_1} + GAP_{2red}\right]}_{\text{t=2}} = 0$$

The derivation is straightforward. An increase in the tax rate (τ_1) will, first, decrease labor supply in the first period which has a per-unit cost of GAP_{1n}: the static labor-leisure distortion (recall that the gaps are distortions from the perspective of the median agent). Second, it has a static redistributive gain like that discussed in the context of the 1-period model: GAP_{1red}. Third, by changing savings—since less time will be spent working and savings under our assumptions will be increasing in income/wealth: $\frac{d\tilde{H}_m}{d\tau_1} < 0$—there is also an intertemporal distortion, since the presence of 2nd-period taxes on total income will distort savings in the direction of being too low: $GAP_{a'_{1m}}$, defined as $u_{t1} - \beta u_{t+1,1}r_{t+1}$, all evaluated for the median agent and for $t = 1$.

Fourth, because the increase in τ_1 will induce changes in assets and thus in second-period tax rates, we also have static costs and benefits for the median voter in period 2. The GAP_{2red} is

$$GAP_{2red} = u_{cm2} \underbrace{\left\{ \frac{d\Psi_2}{d\tau_1} \left[r_2 \left(\tilde{H} - \tilde{H}_m \right) + w_2 \left(N_2 - N_{m2} \right) \right] + \right.}_{\text{direct}}$$

$$\underbrace{\left. \Psi_2 \left[w_2 \frac{d\left[N_2 - N_{m2}\right]}{d\tau_1} + \frac{dr_2}{d\tau_1} \left[\tilde{H} - \tilde{H}_m \right] + \frac{dw_2}{d\tau_1} \left[N_2 - N_{m2} \right] + r_2 \left(\frac{d\tilde{H}}{d\tau_1} - \frac{d\tilde{H}_m}{d\tau_1} \right) \right] + \frac{dr_2}{d\tau_1} \tilde{H}_m + \frac{dw_2}{d\tau_1} N_{m2} \right\}}_{\text{indirect}}.$$

Changes in τ_1, by affecting savings in the first period, trigger changes in second period taxes (since the optimal choices of the government at that point depend on the states inherited). The median voter in period one realizes that this will cause a direct effect on redistribution next period. The direct effect is then the change in net redistribution (keeping asset holdings constant) due to induced changes in future taxes.

The first three terms in the indirect effect are analogous to those in the static one period economy: there is a decrease in the labor gap between the mean and the median, plus an effect through changes in prices. The fourth term appears because next period's asset holding is elastic, which in this case is a negative effect of raising current taxes (savings of the median and the mean move closer to each other, thus lowering the net transfer to the median). The last two terms, as in the one period case, reflect changes in the median's current income due to changes in prices.

3.3.1 Tax manipulation

The lack of commitment and the fact that only current taxes can be chosen implies that the government today perceives tomorrow's capital to be more elastic than what future governments will. Hence, it has an incentive to strategically influence next period's taxation decision. This effect appears in the GEE but is only visible indirectly: by changing the current tax rate, the current median influences savings, and thus future taxes, in order to change current expectations of consumers. In this class of models, time inconsistency of fiscal policy is limited to the effect on private agents' expectations: if these expectations (which determine current behavior) could be controlled separately, there would be no disagreement between the current and the future governments/median voters, i.e., conditional on arriving at a state (A'_m, K') tomorrow, there is no remaining disagreement. As an example, taxes on capital income next period lowers savings this period. If the government could affect expectations today so as to make consumers believe that taxes next period will be very low, that would be desirable: it would increase savings and reduce the distortion. However, the assumption that the consumers' expectations are rational is a constraint on the government, and in this model it can thus only partially influence these expectations by altering the current tax, and hence future variables, in the direction of making the distortions less costly.

3.4 The 2-period example economy

We consider the same parametric example as in the 1-period model, with the only new parameter being β, the discount factor.

3.4.1 Exogenous taxes in the first period: preliminaries

For the second-period outcomes, we inherit all the results from the earlier analysis. In the first period, we obtain savings that satisfy

$$K_2 = h_2 K_1^{\theta} N_1^{1-\theta}$$

$$A_{2m} = h_{2m} K_1^{\theta} N_1^{1-\theta},$$

where thus h_2 and h_{2m} are the *fractions* of total first-period output saved by the average agent and the median agent, respectively, and where

$$N_1 = \frac{\alpha(1-\theta)(1-\tau_1)}{(1-\alpha)(1-h_2)+\alpha(1-\theta)(1-\tau_1)}.$$

It is straightforward to solve for h_2 given τ_2:

$$h_2 = \frac{\beta\theta(1-\tau_2)}{1+\beta\theta(1-\tau_2)}.$$

However, τ_2 here depends on h_{2m}/h_2: it is endogenous. In particular, we know that $\tau_2 = \psi(h_{2m}/h_2)$. Furthermore,

$$\frac{h_{2m}}{h_2} = 1 - \frac{1+\beta\theta(1-\tau_2)}{1+\beta}\frac{(1-\tau_1)(1-x_1)}{(1-\tau_2)},$$

where x_1 stands for A_{1m}/K_1. That is, we have

$$\frac{A_{2m}}{K_2} = 1 - \frac{1+\beta\theta(1-\tau_2)}{1+\beta}\frac{(1-\tau_1)(1-\frac{A_{1m}}{K_1})}{(1-\tau_2)} = 1 - S + S\frac{A_{1m}}{K_1},$$

where $S \equiv \frac{1+\beta\theta(1-\tau_2)}{1+\beta}\frac{1-\tau_1}{1-\tau_2}$. Note now that if tax rates are (positive and) decreasing over time, i.e., $\tau_2 \leq \tau_1$, we find $0 < S < 1$, so that inequality—as measured by the ratio of median to mean asset holdings—must *decrease* over time.

Because ψ does not have a convenient closed form, there is no closed-form solution for h_2 or

h_{2m}/h_2 here. One finds, in addition, that

$$N_{1m} = \frac{(1-\alpha)[h_{2m} - h_2] + (1-\tau_1)\left[1 - \theta\left(\alpha + (1-\alpha)x_1\right)\right]}{(1-\theta)(1-\tau_1)} N_1.$$

What is actually possible to derive is the difference in savings:

$$h_{2m} - h_2 = -(1-\tau_1)\frac{\beta\theta(1-x_1)}{1+\beta}.$$

Here, we see that the difference between median and mean savings does not depend on the period-2 tax rate, so this is indeed a closed form. This expression can then be used in turn to solve, in closed form, for

$$\frac{N_{1m}}{N_1} = \frac{(1+\beta)(1-\theta(\alpha+(1-\alpha)x_1)) - \beta\theta(1-x_1)(1-\alpha)}{(1+\beta)(1-\theta)}.$$

3.4.2 Exogenous taxes in the first period: existence and uniqueness

Reexpressing the solution for h_2 above as a fixed-point problem, we see that we need to find a solution to

$$LHS(h_2) \equiv h_2 = \frac{1}{\frac{A}{1-\psi \;\; 1+\frac{B}{h_2}} + 1} \equiv RHS(h_2)$$

in the interval $[0,1]$, where we recall that $A = 1/(\beta\theta)$ and $B = (1-\tau_1)\frac{\beta\theta(x_1-1)}{1+\beta}$. Three cases appear. We presume throughout what we found for the 1-period model, i.e., that ψ_2 is decreasing.

<u>The case with $x_1 = 1$</u>. Here, we have $B = 0$ and since $\psi(1) = 0$ (no taxation when the median equals the mean), we obtain $RHS(h_2) = 1/(1+A) = \beta\theta/(1+\beta\theta)$ so that is what savings must be.

<u>The case with $\underline{x}_1 > 1$.</u> In this case, first we need to deal with the issue of non-existence in period 2. So suppose that we "resolve" that problem by, whenever the median voter would want to choose a corner solution $\tau_2 = \underline{\tau}$, we define this as the outcome (even though strictly speaking the poorest agent does not have a well-defined utility maximization problem in this case). Notice here that $\underline{\tau}$ actually depends on the period-2 wealth ratio between the poorest agent and the mean agent, $\underline{x}_1$, which in turn is influenced by mean savings: the absolute (level) gap is constant, so the higher the savings of the mean, the lower is the poor-to-mean asset ratio in period 2. Specifically, we have that $h_2 - \underline{h}_2 = \underline{B} \equiv (1-\tau_1)\frac{\beta\theta(1-\underline{x}_1)}{1+\beta}$, where $\underline{h}_2$ is the savings of the poorest and $\underline{x}_1$ is the period-1 ratio of the asset holdings of the poorest to those of the mean. Since $1 - \underline{x}_1 = (h_2 - \underline{h}_2)/h_2 = (1-\tau_1)\frac{\beta\theta(1-\underline{x}_1)}{h_2(1+\beta)}$, this means that $\underline{\tau}$ is decreasing in h_2, because it satisfies

$$\underline{\tau} = 1 - \frac{1}{\alpha\theta(1-\underline{x}_1)} = 1 - \frac{1}{\alpha\theta(1-\tau_1)\frac{\beta\theta(1-\underline{x}_1)}{h_2(1+\beta)}} = 1 - h_2 \cdot \frac{1+\beta}{\alpha\beta\theta^2(1-\tau_1)(1-\underline{x}_1)}.$$

In fact, for $h_2 = 0$, we must have $\underline{x}_1 = -\infty$, which implies $\underline{\tau} = 1$: the only feasible tax rate in order for the poorest to have non-negative consumption is a tax rate of 1. The implied expression for $\underline{RHS}(h_2)$, which is the value for $RHS(h_2)$ in the case the corner constraint for taxes binds in the second period, is

$$\underline{RHS}(h_2) = \frac{1}{\frac{A}{h_2 \cdot \frac{1+\beta}{\alpha\beta\theta^2(1-\tau_1)(1-\underline{x}_1)}} + 1} = \frac{1}{\frac{A\alpha\beta\theta^2(1-\tau_1)(1-\underline{x}_1)}{h_2(1+\beta)} + 1}.$$

This is an increasing concave function in h_2 whose value is strictly less than 1. Moreover, it is easy to see that its derivative at $h_2 = 0$ equals $\frac{1+\beta}{A\alpha\beta\theta^2(1-\tau_1)(1-\underline{x}_1)} = \frac{1+\beta}{\alpha\theta(1-\tau_1)(1-\underline{x}_1)} > 1$, unless τ_1 is a large enough subsidy. This is an important feature: $\underline{RHS}(h_2)$ starts out at zero with a derivative above the 45-degree line, i.e., with $\underline{RHS}(h_2) > LHS(h_2)$ for small h_2. Because $\underline{RHS}(h_2)$ is concave and less than 1 it then intersects $LHS(h_2)$ at a unique value of h_2 less than 1 (this value can be solved for analytically).

Next, consider the issue of when the corner constraint will actually bind in period 2. For small enough h_2s, it has to bind, because then inequality in the second period is extreme: we saw above that the only feasible period-2 tax rate at $h_2 = 0$ is 1. So then either it ceases to bind for some $h_2 < 1$, or it binds for all $h_2 \leq 1$. In the former case, we can see that, since a higher value for h_2 decreases $h_{2m}/h_2 = 1 + \frac{B}{h_2}$ (since $B > 0$), $\psi(1 + \frac{B}{h_2})$ must be increasing in h_2 if the tax in the second period is interior, so $RHS(h_2)$ is a decreasing function, and again there must be a unique solution for h_2. Thus, graphically, either of the following cases must apply.

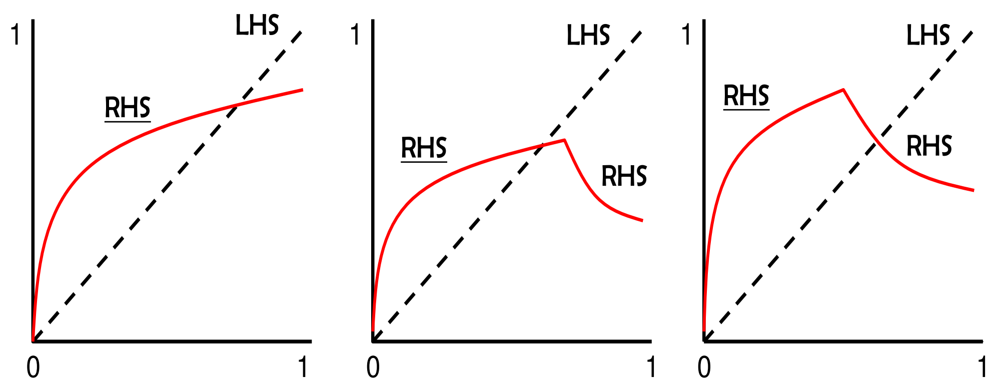

Figure 2: Three cases

Finally, here, one of course needs to check not only that period-2 consumption is positive for the poorest consumer but that period-1 consumption is positive as well. However, because of the form of preferences considered here, period-1 consumption is positive if and only if period-2 consumption is positive, and if and only if leisure (in either period) is positive.

The case with $x_1 < 1$. Now, since $B < 0$, an increase in h_2 must raise h_{2m}/h_2, and $\psi(1+\frac{B}{h_2})$ is consequently decreasing, leading $RHS(h_2)$ to be an increasing function globally. Here as well, $RHS(0) = 0$, but for a different reason than in the $x_1 > 1$ case: here, the median consumer is infinitely poor relative to the mean, and chooses a tax equal to 100%; in the $x_1 > 1$ case with $x = 0$, the median is infinitely rich compared to the poorest person and then becomes restricted to using a 100% tax rate, because no other tax rate leaves the poorest person with non-negative consumption.[9] Now the issue of existence of solutions with $h_2 \in (0, 1)$ is harder to explore. For example, whether $RHS(h_2)$ is above or below $LHS(h_2)$ in the neighborhood of $h_2 = 0$ depends on what $\psi_2(x_1)$ looks like there, and it may be that whether one curve is above the other depends on parameters.

We can say something about how $RHS(h_2)$ depends on primitives, however. Suppose that τ_1 falls. Then $1 + \frac{B}{h_2}$ falls for any given h_2 (the median-mean gap increases), and $\psi(1 + \frac{B}{h_2})$ will then increase. This leads $RHS(h_2)$ to fall. We can thus imagine a picture where $RHS(h_2)$ is concave and has a unique intersection with $LHS(h_2)$, but there will be a point where τ_1 is low enough that $RHS(h_2)$ falls sufficiently that it is everywhere below the 45-degree line for $h_2 > 0$. As τ_1 falls continuously in this case, the intersection point h_2 moves toward zero and eventually zero savings is the only "solution". Figure 3 illustrates this possibility.

The intuitive reasoning behind non-existence here is as follows. As we have seen above, the median will save less than the mean, and the lower is τ_1, the larger is the mean-median difference. This implies that next period, there is still significant inequality, which (as we know) will lead to

[9] The two cases are, in some sense the same: in both cases, the tax rate has to be 100% because of the poorest person, and the only difference is that when $x_1 < 1$, the median person is the poorest person, whereas when $x_1 > 1$, someone else is.

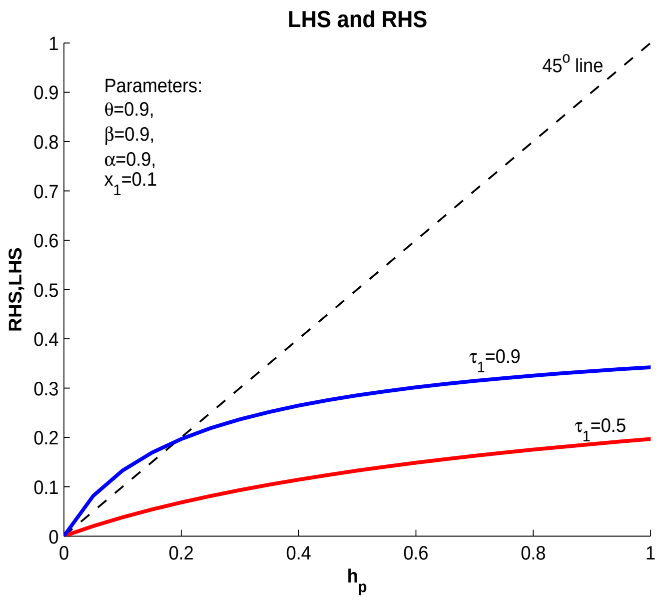

Figure 3: Existence and τ_1

a high tax rate τ_2. This high tax will make the return on saving lower, the lower is τ_1, and hence saving will fall as τ_1 is lowered. Ordinarily there would be a limit to this mechanism, because as savings fall, the return to capital rises in period 2, thus inducing more savings, so that savings are bounded away from zero. The return does increase here as well, but a rise in the returns to capital can also have another effect: it can make the median agents poorer in period 2. The reason is that when mean savings are close enough to zero, median savings have to be negative—see the formula for the difference above—and then a high interest rate means that the debt service is more costly for the median agent, thus lowering his before-tax income. The lower is τ_1, the more savings fall, and for a low enough τ_1, there is no period-2 tax rate that can maintain positive consumption for the median agent. As τ_1 falls, thus, we see mean savings fall toward zero, along with a rise in τ_2 toward one, consumption of both consumers going to zero in period 2, along with work effort.

All of this is a political-economy effect: wealth inequality in the second period rises as the before-tax return to capital rises, translating to a higher tax rate on savings via popular vote. In fact, as mean savings go to zero, period-2 wealth inequality, as measured by the ratio of mean to median asset income, becomes infinite, since the interest rate the median agents have to pay goes to infinity, and their borrowing is strictly positive and bounded away from zero. The rise in taxes overtakes the rise in the before-tax return to capital leading to a net return to savings equal to zero. The reason for this effect is that the tax going to one also makes the labor input go to zero, independently of what the level of capital is.

Fundamentally, the mechanism that leads to non-existence of a competitive equilibrium has to do with the period-2 tax rate being endogenous and not committed to in advance; in particular, we think of a situation without commitment where the endogeneity of the tax rate is given by what the government will choose in period 2.

A final interesting issue is whether the solution is always unique here, which it is in the case with $x_1 > 1$. What ensures, for example, that $RHS(h_2)$ is concave? The simple parametric model we are exploring here turns out to be well-behaved for all the parameter configurations we have explored. But more generally it seems hard to rule out cases with two solutions for h_2 in (0,1). The idea is easy to express intuitively: if there is an expectation of higher taxes tomorrow, all savings fall. But because savings of all consumers fall, and by a similar amount (because the *difference* between savings of different groups do not depend on the tax tomorrow), wealth inequality measured as the ratio of median to mean wealth will increase, justifying the tax tomorrow being higher. Or, if we think taxes will be very low, savings are high, leading to lower wealth inequality in the future, consistently with expecting a lower tax rate, i.e., we would have "self-fulfilling expectations".

3.4.3 The median voter's choice in the first period: decreasing tax rates

Finally, we report what the median voter would choose among all feasible values for τ_1, i.e., for values for τ_1 for which there is an equilibrium in the 2-period economy, taking into account the endogeneity of taxes in the second period. Of course, the choice set here is smaller than in the case with commitment, not just because τ_2 is not subject to choice, but because a narrower range of values for τ_1 is available: there are values for τ_1 that, together with specific choices of τ_2, would lead to equilibria, but would not be consistent with an equilibrium when τ_2 has to be taken as given through ψ_2. We find an equilibrium using both first-order conditions and with global search, and the following graphs describe our findings for the parameter configuration $\alpha = 0.3$, $\beta = 0.9$, and θ either equal to 0.4 or to 0.05.

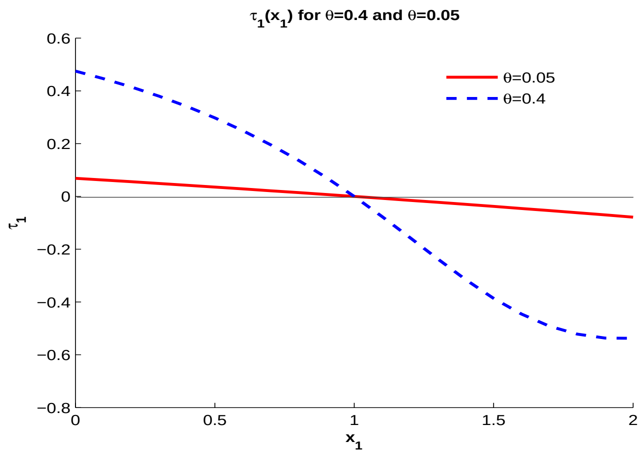

Figure 4: Taxes as a function of the median to mean wealth

We see from Figure 4 that, as in the 1-period model, the tax rate is globally decreasing in the median-mean wealth ratio. We also see that the (absolute) rate of taxation is higher the more important capital is in production: $\theta = 0.4$ leads to much larger tax rates than $\theta = 0.05$. Of course, as θ is increased, the share of labor, and the costs of distortions, also decrease, making it less costly to tax.

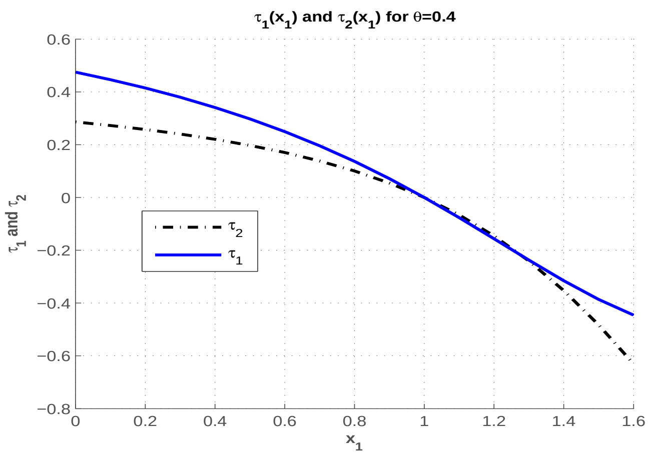

Figure 5: Tax rates in the two periods

Figure 5 shows the tax rates in the two periods as a function of the initial median-to-mean asset ratio. It shows that taxation (assuming that median income is below mean income) is lower in period 2 than in period 1. The source of this finding is not just the finite time horizon, but also the fact that capital income is inelastic. Under commitment, clearly, the rate of taxation should optimally—from the perspective of "optimal redistribution to the median voter"—decrease over time, since it is costlier to tax in the future than in the present. Here, the median voter cannot commit to the future tax rate, but on the other hand an increase in the current tax rate would automatically, by reducing inequality, lead to a lower future tax rate. Thus, the median voter can still enact a decreasing tax path. Thus the equilibrium features heavy initial taxation and subsequent lower inequality and less taxation.

Figure 6 illustrates the resulting path of inequality. The solid line in the figure depicts the mean-median ratio in the second period as a function of the initial ratio, while the dotted one is just the 45 degree line. Due to the decrease in inequality, there will indeed be less taxation in the future: more current taxation leads to less taxation in the future. This result is an important

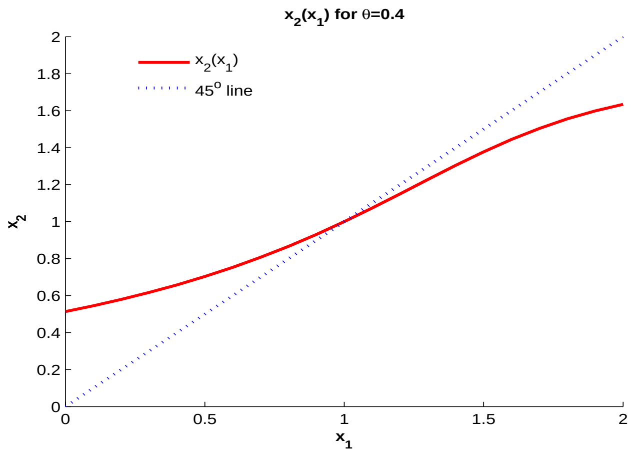

Figure 6: Median-mean wealth ratio tomorrow as a function of that today

insight that will have implications for the infinite-horizon setup: though in principle a range of different wealth distributions are possible as steady states in that model—each one associated with a different tax rate—this range is limited: with significant initial inequality, there will be high initial taxes and a decreasing path of taxes and inequality, leading to a limited set of long-run values for the median-mean wealth ratio. In Figure 6, for example, we see that even if the median agent is infinitely asset poor relative to the average agent, $A_{1m}/K_1 = 0$, the same period-2 ratio will be over 0.45.

4 The ∞-period model

The infinite-horizon version of the model is an important extension because it allows us to ask questions about the long-run level of taxes and inequality, i.e., just in the way the neoclassical growth model and its balanced growth path, or steady state, is used for describing, say, the post-war path of U.S. output and capital accumulation, so can we use the extension of the same model to discuss the forces behind taxation and, in turn, inequality. In particular, one can make quantitative assessments. In this paper, the focus is more on methods than on quantitative evaluation, though we do discuss some important implications of the present model. In particular, a more satisfactory quantitative model would involve inequality in labor income/productivity as well.[10]

The infinite-horizon model, as we shall show, inherits all the essential features of the 1- and 2-period models. Some new mechanisms appear—the dynamic components of the median voter's tradeoffs become richer—but the core of the political redistribution mechanism remains. Computationally, it raises some new issues, and we discuss these in some detail.

We first define recursive competitive equilibria for a given tax function Ψ. In the main text, we will presume that equilibria depend on the state vector (K, A_m) only, and not on other aspects of the asset distribution; in the Appendix, we make the formal statement to this effect: we show that the set of all equilibria contains the set of equilibria where aggregate outcomes depend only on (K, A_m).[11] Equilibria involve decision rules for savings and leisure of each of the types of agents as a function of the state vector. Moreover, for compactness, in the text we will not describe the competitive agent's dynamic programming problem, which would necessitate also including the individual state variable. Instead, we will only define aggregate decision rules.

In order to express aggregation as compactly as possible, we also define competitive equilibrium using another concepts of wealth, namely the present-value human and transfer wealth (net of taxes), E, and relative total net present-value wealth of the mean- and median-asset agents, λ. The latter variable is very convenient: it will dictate, and equal, the relative consumption and leisure levels of the mean- and median-asset agents.

Definition 8 ***A recursive competitive equilibrium with aggregation for a given tax function***

[10] This extension will be considered in future work.

[11] As pointed out before, this still leaves open the possibility that there are self-fulfilling equilibria where other moments of the asset distribution matter for aggregate outcomes.

$\Psi(A_m,K)$ is a set of functions $h_m(A_m,K)$, $n_m(A_m,K)$, $h(A_m,K)$, $n(A_m,K)$, $R(K,N)$, $W(K,N)$, $\lambda(A_m,K)$, $E(A_m,K)$, and $T(A_m,K)$ with the following properties.

1. *h_m and n_m solve, for all (A_m,K),*

   $$u_c(c_m(A_m,K),1-n_m(A_m,K)) = \beta R(K',n(A'_m,K'))(1-\Psi(A'_m,K'))u_c(c_m(A'_m,K'),1-n_m(A'_m,K')$$

   and

   $$u_c(c_m(A_m,K),1-n_m(A_m,K))W(K,n(A_m,K))(1-\Psi(A_m,K)) = u_l(c_m(A_m,K),1-n_m(A_m,K)),$$

   where

   $$c_m(A_m,K) \equiv A_m+[A_mR(K,n(A_m,K))+n_m(A_m,K)W(K,n(A_m,K))](1-\Psi(A_m,K)))+T(A_m,K)$$

   $$A'_m \equiv h_m(A_m,K),$$

   and

   $$K' \equiv h(A_m,K).$$

2. *$h(A_m,K)$ and $n(A_m,K)$ satisfy, for all (A_m,K),*

   $$h(A_m,K) \equiv R(K,n(A_m,K))(1-\Psi(A_m,K)))(K-\lambda(A_m,K)A_m)+$$

   $$+[W(K,n(A_m,K))(1-\Psi(A_m,K))+T(A_m,K)](1-\lambda(A_m,K))+\lambda(A_m,K)h_m(A_m,K)$$

and

$$1 - n(A_m, K) = \lambda(A_m, K)(1 - n_m(A_m, K)).$$

3. *W and R satisfy $W(K, N) = F_1(K, N)$ and $R = F_2(K, N)$ for all (K, N).*

4. *T satisfies $T(A_m, K) = \Psi(A_m, K)(R(K, n(A_m, K))K + W(K, n(A_m, K))n(A_m, K))$ for all (K, A_m).*

5. *$\lambda(A_m, K)$ satisfies, for all (A_m, K),*

$$\lambda(A_m, K) = \frac{K[R(K, n(A_m, K))(1 - \Psi(A_m, K))] + E(A_m, K)}{A_m[R(K, n(A_m, K))(1 - \Psi(A_m, K))] + E(A_m, K)}.$$

6. *$E(A_m, K)$ satisfies, for all (A_m, K),*

$$E(A'_m, K') =$$

$$[E(A_m, K) - W(K, n(A_m, K))(1 - \Psi(A_m, K)) - T(A_m, K)]R(K', n(A'_m, K'))(1 - \Psi(A'_m, K')),$$

where

$$A'_m \equiv h_m(A_m, K)$$

and

$$K' \equiv h(A_m, K).$$

Thus, E adds up the present value of the stream of total labor endowments, net of taxes, and the stream of transfers; these are the same for all agents, so E is a part of wealth that is common to all. The differences between agents are then captured with λ, which expresses net-present-value wealth ratios when current assets, net of taxes, are included.

Notice that this equilibrium definition is stated entirely in terms of functions: each condition in the equilibrium definition is required to hold for all values of the arguments of the functions, thus defining a set of *functional equations*.

Also, note that in this setup the ranking of asset between types stays the same over time. This follows rather straightforwardly from assuming normal goods—which our typical assumptions on additive time separability and the concavity of u imply—so that a higher asset holding in a given period translates into higher asset holdings (and thus more consumption) for the future as well. If one considers heterogeneity in preferences or in labor productivity, this result will no longer necessarily hold, and the asset ranking can change over time.

4.1 Endogenous policy: recursive majority-voting equilibrium

We now define a Markov-perfect equilibrium by requiring that the government policy function Ψ be the preferred choice by the agent whose asset holdings are median in the distribution. The function Ψ is stationary—due to future being infinite at any point in time, it does not have a time subscript—and we think of it as a limit of the sequence of first-period outcome functions Ψ_{0T} where T represents the time horizon. As in the 1- and 2-period models, and as in Krusell and Ríos-Rull (1999), we first define equilibrium by characterizing one-period deviations from the policy rule $\Psi(A_m, K)$: in the current period, the tax rate is τ, whereas all future tax rates are given by the rule Ψ, evaluated at the asset distributions that will result from current taxation at τ. We do this in order to be able to state the median voter's problem: the median voter needs to consider all possible current τ values, and their associated competitive equilibria, in order to see which one is best. The political equilibrium will then require that $\Psi(A_m, K)$ is the best among all policies τ, for all (A_m, K). To define behavior in the one-period deviations we add the argument τ and tildes to all associated functions. The resulting equilibrium is defined as follows. The one-period deviation will require functions that depend on τ and these will also satisfy the aggregation property; the proof, moreover, is parallel. A key element of this definition will be the value function of the median voter for a one-period deviation, $\tilde{V}(A_m, K, \tau)$, along with $\tilde{h}_m(A_m, \bar{A}, \tau)$, $\tilde{h}(A_m, \bar{A}, \tau)$, $\tilde{n}_m(A_m, \bar{A}, \tau)$ and $\tilde{n}(A_m, \bar{A}, \tau)$. A more formal definition follows.

Definition 9 *The **recursive competitive equilibrium with aggregation and a one-period tax deviation** is a set of functions $V(A_m, K)$, $\tilde{V}(A_m, K, \tau)$, $h_m(A_m, K)$, $\tilde{h}_m(A_m, K, \tau)$, $n_m(A_m, K)$, $\tilde{n}_m(A_m, K, \tau)$, $h(A_m, K)$, $\tilde{h}(A_m, K, \tau)$, $n(A_m, K)$, $\tilde{n}(A_m, K, \tau)$, $R(K, N)$, $W(K, N)$, $\lambda(A_m, K)$, $\tilde{\lambda}(A_m, K, \tau)$, $E(A_m, K)$, $\tilde{E}(A_m, K, \tau)$, $T(A_m, K)$ and $\tilde{T}(A_m, K, \tau)$ with the following properties:*

1. *$\tilde{V}(A_m, K, \tau)$ solves:*

$$\tilde{V}(A_m, K, \tau) = \max_{n_m, A'_m} u_c(c_m, 1 - n_m) + \beta V(A'_m, K')$$

subject to

$$c_m \equiv A_m + [A_m \tilde{R}(K, \tilde{n}(A_m, K, \tau)) + n_m \tilde{W}(K, \tilde{n}(A_m, K, \tau))](1 - \tau) +$$

$$\tilde{T}(A_m, K, \tau) - A'_m.$$

2. $\tilde{h}_m$ *and* $\tilde{n}_m$ *attain the argmax above.*

3. $\tilde{h}(A_m, K, \tau)$ *and* $\tilde{n}(A_m, K, \tau)$ *satisfy, for all* (A_m, K, τ)*,*

$$\tilde{h}(A_m, K, \tau) \equiv (1 + \tilde{R}(K, \tilde{n}(A_m, K, \tau)))(1 - \tau))(K - \tilde{\lambda}(A_m, K, \tau)A_m) +$$

$$[\tilde{W}(K, \tilde{n}(A_m, K, \tau))(1-\tau) + \tilde{T}(A_m, K, \tau)](1-\tilde{\lambda}(A_m, K, \tau)) + \tilde{\lambda}(A_m, K, \tau)h_m(A_m, K, \tau)$$

and

$$(1 - \tilde{n}(A_m, K, \tau)) = \tilde{\lambda}(A_m, K, \tau)(1 - \tilde{n}_m(A_m, K, \tau)).$$

4. $\tilde{T}$ *satisfies* $\tilde{T}(A_m, K, \tau) = \tau(R(K, \tilde{n}(A_m, K, \tau))K + W(K, \tilde{n}(A_m, K, \tau))\tilde{n}(A_m, K, \tau))$ *for all* (K, A_m, τ)*.*

5. $\tilde{\lambda}(A_m, K, \tau)$ *satisfies, for all* (A_m, K, τ)*,*

$$\tilde{\lambda}(A_m, K, \tau) = \frac{K[R(K, \tilde{n}(A_m, K, \tau))(1 - \tau)] + \tilde{E}(A_m, K, \tau)}{A_m[1 + R(K, n(A_m, K, \tau))(1 - \tau)] + \tilde{E}(A_m, K, \tau)}.$$

6. $\tilde{E}(A_m, K, \tau)$ *satisfies, for all* (A_m, K, τ)*,*

$$E(A'_m, K') = [\tilde{E}(A_m, K, \tau) - W(K, n(A_m, K, \tau))(1 - \tau) -$$

$$\tilde{T}(A_m, K, \tau)]R(K', n(A'_m, K'))(1 - \psi(A'_m/K')),$$

where

$$A'_m \equiv h_m(A_m, K)$$

and

$$K' \equiv h(A_m, K).$$

7. $V(A_m, K)$, $h_m(A_m, K)$, $n_m(A_m, K)$, $h(A_m, K)$, $n(A_m, K)$, $R(K, N)$, $W(K, N)$, $\lambda(A_m, K)$, $E(A_m, K)$ *and* $T(A_m, K)$ *constitute a recursive competitive equilibrium with aggregation.*

Finally, given a recursive competitive equilibrium with aggregation with a one-period tax deviation, it is straightforward to define a **Markov-perfect median voter equilibrium with aggregation**. Such an equilibrium thus has as its key requirement that

$$\psi(A_m, K) = \arg\max_{\tau} \tilde{V}(A_m, K, \tau)$$

for all (A_m, K).

4.2 The problem of the median voter: the Generalized Euler Equation

We saw in the 1- and 2-period model that the first-order necessary condition of the median voter amounted to an equation setting a weighted sum of "gaps" to zero. In the 1-period model, these gaps were the labor-leisure gap and a redistribution gap; in the 2-period model, these gaps reappeared, one for each time period, and in addition an intertemporal, or savings, gap appeared. In a 3-period model, the first-order condition for the median voter in the first of the three periods would involve three static gaps: one for each of the three periods. Models with 4 and more periods, however, also involve only gaps in three periods: the current one and two consecutive ones.

Thus, as we shall see here, the infinite-horizon model delivers a first-order condition of the median voter that consists of three consecutive periods of static gaps, together with two consecutive periods of savings gaps, but no more. The intuition for this finding will be discussed below. The detailed derivation can be found in Azzimonti, de Francisco, Krusell, and Ríos-Rull (2005).

The GEE thus reads

$$\underbrace{GAP_{a'_m}\frac{d\tilde{h}_m}{d\tau} + GAP_{l_m}\frac{d\tilde{n}_m}{d\tau} + GAP_{red}}_{\text{t=1}} + \beta\underbrace{\left[GAP_{a''_m}\frac{d\tilde{h}'_m}{d\tau} + GAP_{l'_m}\frac{d\tilde{n}'_m}{d\tau} + GAP_{red'}\right]}_{\text{t=2}} +$$

$$\beta^2\underbrace{\left[GAP_{l''_m}\frac{d\tilde{n}''_m}{d\tau} + GAP_{red''}\right]}_{\text{t=3}} = 0. \tag{1}$$

There are no conceptual news in this condition—the definitions of gaps are inherited from the 2-period model—so the only remaining issue is the one of why three and only three periods appear. This can be understood by thinking of the GEE as resulting from a variational experiment. The key insight in this regard is that there are two state variables and only one control in the median voter's maximization problem. Suppose the median agent kept (A_m, K) and (A''_m, K'') fixed and optimally varied the controls in between, as in a parallel of what occurs in a standard dynamic optimization problem. The controls would be τ and τ', or, alternatively, the vector (A'_m, K'). The problem with this experiments is that there are not enough degrees of freedom for a variational experiment: the two controls are completely pinned down by the end conditions, (A''_m, K''), and cannot be varied beyond that! This is why a variational experiment here has to involve keeping (A_m, K) and (A'''_m, K''') fixed and optimally varying τ, τ', and τ'', where there now is one degree of freedom and utility can be maximized. As a consequence, the GEE must contain terms also dated two periods from the current period.

4.3 The infinite-horizon example economy

Using the same functional forms as in the 2-period case, we will again be able to use the result that levels do not matter: the endogenous variables that are nontrivially determined in equilibrium depend only on the ratio of median to mean assets. This is very convenient since we only need to keep track of the evolution of one state variable.

To proceed, we will first solve for the recursive competitive equilibrium. Then we will characterize the competitive equilibrium under a one-period deviation. Finally, we will use the GEE derived before to solve for the tax rule using numerical methods.

4.3.1 Equilibrium for a given tax rule

Given a tax rule, $\psi(A_m/K)$, it is possible to find a closed-form solution to the decision rules in the recursive competitive equilibrium. The results are summarized in the following proposition.

Proposition 10 *The recursive competitive equilibrium for the parameterized economy is characterized as follows.*

1. *The value function satisfies:*

$$V(A_m, K) = S(A_m/K) + B\log(K)$$

where

$$S(A_m/K) = \big\{\alpha \ln \big[n^{1-\theta}[(1-\psi(A_m/K))(\alpha A_m/K + (1-\alpha))\theta(1-\beta) + 1 - \theta) + \psi(A_m/K)]\big]$$

$$+(1-\alpha)\ln[1 - n_m(A_m/K)] + \frac{\beta\theta\alpha}{1-\theta\beta} \ln \big[\beta\theta(1-\psi(A_m/K)n(A_m/K)^{1-\theta}\big]\Big\} \frac{1}{1-\beta}$$

and

$$B = \frac{\theta\alpha}{1-\theta\beta}.$$

2. *The median agent's savings and working decisions are*

$$h_m(A_m, K) = \beta(1-\psi(A_m/K))R(A_m, K)A_m$$

and

$$n_m(A_m, K) = \frac{(1-\theta) + (1-\alpha)\theta(1 - A_m/K)(1-\beta)}{1-\theta} n(A_m, K).$$

3. *The savings and working decisions of the mean-asset agent are*

$$h(A_m, K) = \beta\theta(1-\psi(A_m/K))n(A_m, K)^{1-\theta}K^{\theta}$$

and

$$n(A_m, K) = \frac{\alpha(1-\psi(A_m/K))(1-\theta)}{(1-\psi(A_m/K))(1-\alpha\theta) + (1-\alpha)\left[\psi(A_m/K) - \beta\theta(1-\psi(A_m/K))\right]}.$$

4. *The functions W and R satisfy $W(A_m, K) = (1-\theta)n(A_m, K)^{-\theta}K^{\theta}$ and $R(A_m, K) =$*

$\theta n(A_m, K)^{1-\theta} K^{\theta-1}$.

5. *The function* T *satisfies* $T(A_m, K) = \psi(A_m/K)[R(A_m, K)K + W(A_m, K)n(A_m, K)]$.

6. *The function* λ *satisfies*

$$\lambda(A_m, K) = \frac{1 - \beta\theta(1 - \psi(A_m/K))}{1 - \beta\theta(1 - \psi(A_m/K)) - (1-\beta)\alpha\theta(1 - \psi(A_m/K))\left(1 - \frac{A_m}{K}\right)}.$$

7. *The function* $E(A_m, K)$ *satisfies, for all* (A_m, K)*,*

$$E(A_m, K) = R(A_m, K)K(1 - \psi(A_m/K))\frac{1 - \lambda(A_m, K)A_m/K}{\lambda(A_m, K) - 1}.$$

This equilibrium has several interesting properties. First, notice that the ratio between the mean and the median capital holdings is constant over time, i.e., $A'_m/K' = h_m(A_m, K)/h(A_m, K) = A_m/K$. In this economy, all agents' assets will grow at the same rate until reaching the steady state. Hence, the distribution of income remains unchanged along the transition path. More importantly, a constant ratio in equilibrium implies that the tax rate does not vary over time. In the 2-period model, in contrast, we saw that equal tax rates over time implied that asset positions moved closer together. However, in the 2-period model, a measure of wealth that includes all sources of income would have remained constant across the two periods. The reason for assets moving closer together, thus, was that asset wealth was a larger fraction of total wealth in the final period than in the first period, when human wealth derives from two periods of income. That is, a given total equality requires larger asset inequality in the first period, since human wealth is larger then and equal across agents.

Second, mean savings are a constant proportion of total output: the rate given by $\beta\theta(1 - \psi(A_m/K))$. Given the aggregate paths and the constant tax rule, the median agent chooses to save a constant fraction of after-tax capital income, the function being linear in his capital income RA_m[12]. The fact that the level of transfers and the wage income do not affect savings is a result of assuming logarithmic utility and Cobb-Douglas production with full depreciation, where income and substitution effects offset each other.

Third, aggregate labor is constant and only depends on the ratio of mean to median capital via the tax rate. This is also a result of the above functional-form assumptions.

Finally, if the median agent is currently poorer than the mean (i.e., $A_m < K$), the relative present value wealth is greater than one: $\lambda(A_m, K) > 1$. Hence, he will remain poorer in the future despite of the existence of lump-sum transfers. We can also see that the median voter will consume less and work more than the average agent in equilibrium.

4.3.2 One-period deviations

In order to find the equilibrium tax rate, we need to characterize the decision rules in a competitive equilibrium after a one-period deviation. It is important to find out how agents will react to a deviation of τ from the rule $\psi(A_m/K)$ by the current government. Since we are studying a one-period deviation only, future governments are assumed to follow $\psi(A_m/K)$. Therefore, agents' decision rules depend on both τ and $\psi(A_m, K)$.

Proposition 11 *The recursive competitive equilibrium with aggregation and a one-period tax*

[12]Krusell, Kuruşçu, and Smith (1999) find a similar result in the context of hyperbolic discounting.

deviation for the parameterized economy is characterized as follows.

1. *The median and mean agents' savings and working decisions are given implicitly by*

$$\tilde{h}_m(A_m,K,\tau)=\beta(1-\tau)\tilde{R}(A_m,K)A_m+\beta\theta\left[\tau-\psi[\tilde{h}_m(A_m,K,\tau)/\tilde{h}(A_m,K,\tau)]\right]K^{\theta}\tilde{n}(A_m,K,\tau)^{1-\theta},$$

$$\tilde{n}_m(A_m,K,\tau)=\frac{(1-\theta)+(1-\alpha)\theta(1-A_m/K)(1-\beta)}{1-\theta}\tilde{n}(A_m,K,\tau),$$

$$\tilde{h}(A_m,K,\tau)=\beta\theta(1-\psi[\tilde{h}_m(A_m,K,\tau)/\tilde{h}(A_m,K,\tau)]K^{\theta}\tilde{n}(A_m,K,\tau)^{1-\theta}$$

and

$$\tilde{n}(A_m,K,\tau)=\frac{(1-\theta)(1-\tau)\alpha}{(1-\tau)(1-\alpha\theta)+(1-\alpha)[\tau-\beta\theta(1-\psi[\tilde{h}_m(A_m,K,\tau)/\tilde{h}(A_m,K,\tau)]]}.$$

2. *The functions* $\tilde{W}$ *and* $\tilde{R}$ *satisfy* $\tilde{W}(A_m,K,\tau)=(1-\theta)\tilde{n}(A_m,K,\tau)^{-\theta}K^{\theta}$ *and* $\tilde{R}(A_m,K,\tau)=\theta\tilde{n}(A_m,K,\tau)^{1-\theta}K^{\theta-1}$.

3. *The function* $\tilde{T}$ *satisfies* $\tilde{T}(A_m,K,\tau)=\tau[\tilde{R}(A_m,K,\tau)K+\tilde{W}(A_m,K,\tau)\tilde{n}(A_m,K,\tau)]$.

4. *The function* $\tilde{\lambda}$ *satisfies*

$$\tilde{\lambda}(A_m,K,\tau)=\frac{1-\beta\theta(1-\psi[\tilde{h}_m(A_m,K,\tau)/\tilde{h}(A_m,K,\tau)]}{1-\beta\theta(1-\psi[\tilde{h}_m(A_m,K,\tau)/\tilde{h}(A_m,K,\tau)]-(1-\tau)\alpha\theta(1-A_m/K)(1-\beta)}.$$

5. The function $\tilde{E}$ satisfies

$$\tilde{E}(A_m, K, \tau) = \tilde{R}(A_m, K, \tau)K(1-\tau)\frac{1-\tilde{\lambda}(A_m, K, \tau)A_m/K}{\tilde{\lambda}(A_m, K, \tau)-1}.$$

It is straightforward to show that if $\tau = \psi(A_m/K)$, the equations above collapse to those in Proposition 8: if there were no deviation, we would be back in a recursive competitive equilibrium for the given tax function.

When the government deviates, agents react by modifying their savings and working decisions in the current period. For example, the median savings' rule is adjusted by the difference between current and future taxes. The first term of $h_m(A_m, K, \tau)$ takes the same form as in the equilibrium with no deviation: savings are linear in capital income net of taxes. The second term increases or decreases savings depending on the spread between taxes today and tomorrow. While the ratio of mean to median hours worked remains unchanged (not only over time but also compared to the previous case), the levels are modified when there is a deviation. Since aggregate labor supply changes, so do wages and interest rates. Moreover, individual savings decisions are affected as well, so aggregate capital in the future will change via that indirect effect.

In contrast to the case of labor decisions, the ratio of median to mean capital is no longer constant over time. In particular,

$$\frac{A'_m}{K'} = \frac{(1-\tau)A_m/K + \tau - \psi(A'_m/K')}{1-\psi(A'_m/K')} \neq \frac{A_m}{K}. \tag{2}$$

The deviation in the tax rule results in a deviation in relative capital holdings, which in turn affects future taxes. So even if future governments decide not to deviate from the equilibrium path, the change in their inherited state of the world causes them to adjust the tax rate. We see from the formula that, as in the 2-period model, if the current tax rate is higher than the future tax rate, inequality decreases. Note that in all periods after tomorrow, the capital ratio remains unchanged, that is, $\frac{A'_m}{K'} = \frac{A''_m}{K''} = \frac{A'''_m}{K'''}$ and so on. Thus, if a deviation from ψ is considered and the current tax is raised, it will lead to a higher tax today than tomorrow: not only is the current tax higher, but since inequality will fall, the tax in the next period, and all periods hence, will be lower as well.

4.3.3 The median voter's problem

Now we will characterize the tax rule chosen by the median voter for this economy. The median voter's decision problem is thus to choose τ to maximize

$$\alpha \log([A_m R(K, \tilde{n}(A_m, K, \tau)) + \tilde{n}_m(A_m, K, \tau) W(K, \tilde{n}(A_m, K, \tau))](1-\tau)) + \tilde{T}(A_m, K, \tau) - \tilde{h}_m(A_m, K, \tau))$$

$$+(1-\alpha)\ \log(1 - \tilde{n}_m(K, \tilde{n}(A_m, K, \tau)) + \beta V(\tilde{h}_m(A_m, K, \tau), \tilde{h}(A_m, K, \tau)),$$

with $V(A_m, K)$ defined above. If, for every (A_m, K), $\psi(A_m, K)$ solves this problem, we have a Markov-perfect equilibrium. Of course, since decision rules, as well as V, depend on ψ here, we are looking at a nontrivial fixed-point problem in the function ψ. Unfortunately, no closed-form solution is available for the function $\psi(A_m/K)$. It is a function of one variable only, so it is in that sense not a higher-dimensional problem than that of solving the neoclassical growth model for the optimal savings function (in a case where no analytical solution is available). On the other hand, there is not, as far as we are aware, any general guarantee of existence or uniqueness unlike in the case of the standard growth model.

4.3.4 Numerical solution method

Our main aim is to find the set of steady states. With the tax rate set at an arbitrary constant $\bar{\tau}$, it is well known that the present model has a unique steady state in terms of the level of average capital, and in addition that there is a continuum of associated steady-state asset distributions; see Krusell and Ríos-Rull (1999) for a discussion. Here, in principle, there is also the possibility of a continuum of steady states, though each one would be associated with its own tax rate (along with an average capital stock). The conjecture is thus that there is a $\psi(x)$, at least in the neighborhood of $x = 1$, such that the median voter chooses $\tau = \psi(x)$ at x and, thus, to remain at a constant tax rate and constant level of inequality x.

We proceed to find a candidate ψ function by analyzing the GEE. A problem here is that it depends on the derivatives of the rules that determine savings and hours worked, and these functions in turn depend on the equilibrium tax rule itself. So the GEE constitutes a functional equation in ψ that also contains the derivative of ψ, through the derivatives of the equilibrium decision rules. One implication is that standard linearization methods cannot be applied. Of course, the equilibrium functions can be arbitrarily well approximated by linear functions around the steady state. However, the steady state cannot be found without knowing (some of) the derivatives of these functions, as in standard dynamic problems. In particular, because the median voter's first-order condition involves derivatives of the ψ, one cannot solve for steady-state levels independently of solving for higher-order features of these functions.

We use an algorithm that can be viewed as an extension of linearization: a version of that is outlined in Krusell and Smith (2001), where it has proven useful both in terms of speed and accuracy.[13] This algorithm is feasible to implement for this problem, fast, and does not require

[13]The method was applied there to a consumption-savings problem under time-inconsistent preferences and has

significant work beyond deriving the equilibrium first-order conditions as functional equations, a task which was accomplished in the previous section. Judd (2005) discusses this method in some detail and points to a potential pitfall—multiplicity of candidate solutions—but also points to ways to discriminate between these. Here, we use the method to find a candidate rule ψ and then verify with global methods, in the neighborhood of x, that it indeed is (close to) an optimal choice for the median voter. The essential idea behind the method is to approximate the equilibrium function with a polynomial evaluated *at a single point*: the steady-state point. Thus, a 0th-order approximation would let all the derivatives be zero and the steady state could be found from the system of first-order conditions. A 1st-order (linear) approximation would involve more unknown parameters—first derivatives in addition to levels—of the five functions. The additional equations needed to pin down these unknown parameters are obtained by partial differentiation, with respect to each argument A_m and K, of each of the equilibrium functional equations. With this procedure, successively higher-order polynomial approximations are thus rather straightforward to derive, and convergence is obtained when the addition of higher orders does not alter the steady state more than by a very small amount. Differentiation of the functional equations is extremely tedious to implement with pencil and paper, but it can be automated using a symbolic math program (one of which is available as part of MATLAB).

In this paper, we will stop at a linear approximation of the key functions, which we later verify is a good approximation by global search. We will thus in particular use the approximation

$$\psi\left(A_m/K\right) = x_0 + x_1 \frac{A_m}{K}. \tag{3}$$

also been employed in an optimal-public-expenditure problem without commitment (Klein, Krusell, and Ríos-Rull (2003)).

As an intermediate step, we need to find the rule that determines the evolution of relative asset holdings under a one period deviation of τ from equilibrium taxes. We will approximate equation (2) using a linear function as well.[14] Thus, we let

$$\frac{\tilde{h}_m\left(A_m, K, \tau\right)}{\tilde{h}\left(A_m, K, \tau\right)} = x_2 + x_3 \frac{A_m}{K} + x_4 \tau. \tag{4}$$

We can replace our guess on ψ into the first-order conditions with respect to capital of the mean and the median agent and obtain a system (of two equations) determining savings per unit of output for the two types of voters:

$$\tilde{h}_{m0} = \beta\theta\left[(1-\tau)A_m/K + \tau - \psi(\tilde{h}_{mo}/\tilde{h}_o)\right],$$

$$\tilde{h}_o = \beta\theta\left[1 - \psi(\tilde{h}_{mo}/\tilde{h}_o)\right],$$

Since $\tilde{h}_{mo}/\tilde{h}_o = \tilde{h}_m/\tilde{h}$, we can solve the system above to obtain one equation governing the evolution of relative asset holdings as a function of the current ratio of median to mean assets, and the tax deviation τ. These equations together with the GEE will be used to solve for the unknown coefficients. The perturbation method is used to solve for the parameters of these polynomials. We have five unknowns, x_0, x_1, x_2, x_3 and x_4. The first derivative of the GEE determines the change in the tax rule when the capital ratio changes (using the implicit function theorem). We can obtain the remaining two equations by taking the derivative of the ratio of asset holdings with respect to the current ratio and its derivative with respect to taxes. The system can then

[14] Given ψ, the linearization of the rule governing the evolution of relative asset holdings is straightforward and does not involve derivatives.

be solved using a standard non-linear equation solver. Thus, these equations deliver values for the parameters around the chosen A_m/K ratio. We will now look at the numerical solution to a specific example to further characterize the tax rule.

4.3.5 Numerical results

We again use $\alpha = 0.3$ and $\beta = 0.9$ to compute the solution in the infinite-horizon economy. As a benchmark, we set $\theta = 0.05$, and then consider the case where $\theta = 0.4$.

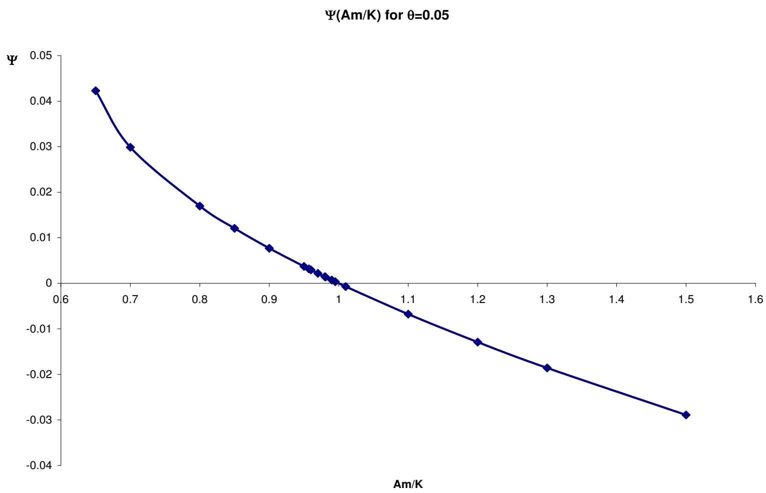

Figure 7: Equilibrium tax function ($\theta = 0.05$)

Figure 7 shows that the tax function, as in the finite-horizon case, is decreasing in the median-mean ratio. A relatively poor pivotal voter would push forward high taxes so as to redistribute resources in his favor. We can also observe that even when wealth inequality is large, taxes are not too distortionary; for example, when the ratio is 0.7 the optimal tax rate does not exceed 5%. This is clearly due to the low productivity of capital assumed for this economy (recall that

$\theta = 0.05$). It is useful then to analyze the behavior of taxes for the case with $\theta = 0.4$. The resulting tax rule is depicted in Figure 8.

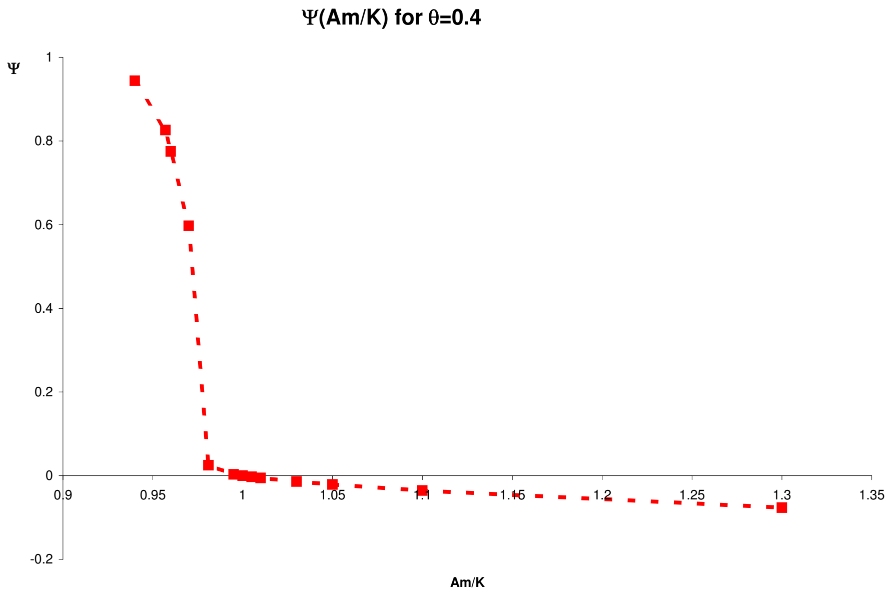

Figure 8: Equilibrium tax function ($\theta = 0.4$)

As in the one and two-period examples, as the economy becomes more productive, the benefits of taxing the current stock of capital increase. It is interesting to notice as soon as we move away from the representative-agent case to one where the median agent is poorer than the mean, the tax rate increases very steeply. When the ratio is only 0.95, taxes are almost 100%. Intuitively, steady states, which we are studying here, are long-run outcomes, and we saw that even in a two-period model taxes fall over time, and inequality declines. Thus, with many time periods, only a small set of values for median-mean wealth are possible: too much initial inequality would be taxed away over time. And the small set of values for A_m/K that are feasible steady states are then associated to high rates of taxation: inequality is “almost” taxed away. Thus, with a reasonable value for θ—similar to that used in most macroeconomic calibrations, the model predicts taxes that are too high, and a range of feasible values for asset inequality that is nowhere near the level of asset inequality observed in data.

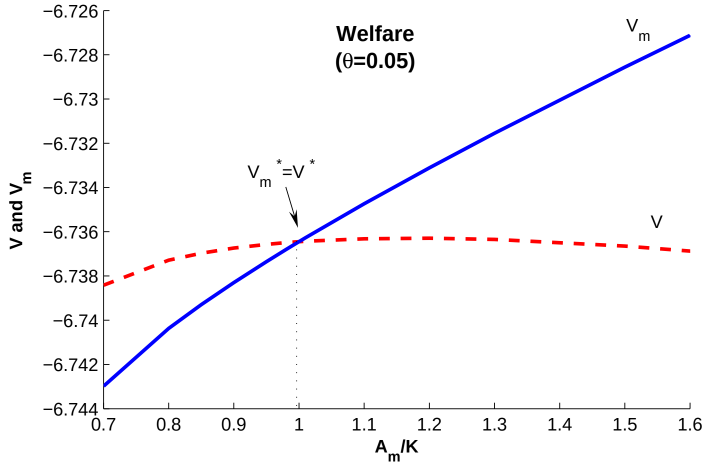

Figure 9: Welfare ($\theta = 0.05$)

In Figure 9, finally, we have graphed the welfare function for the average agent and the median voter (assuming that the asset holdings of the average agent are always one ($\overline{A} = 1$)), as a function of the asset holdings of the median voter. The welfare of the median voter is monotonically increasing in A_m. However, the welfare of the average agent with $\overline{A} = 1$ is increasing in A_m for $A_m < \overline{A}$, but decreasing in A_m for $A_m > \overline{A}$, so that a maximum is reached when there is no inequality ($A_m = \overline{A}$). Production is always increasing in A_m since taxes decrease and they become subsidies when $A_m > \overline{A}$ so there is overaccumulation of capital in this case. However, subsidies are also distortionary and they decrease the welfare of the average agent.

5 Concluding comments

In this paper, we have developed finite-horizon models of endogenous redistribution using the median-voter construct, and we have explored the infinite-horizon version of the setup as well. The analysis demonstrates first that, under assumptions about the utility function that are common in the applied macroeconomic literature, an aggregation result applies: the aggregate politico-economic equilibrium outcomes, i.e., taxes, output, prices, etc., depend on the mean level of assets and on the median asset holding, and on no other aspect of the asset distribution. This result facilitates tractability considerably; dynamic models with forward-looking, rational agents rapidly become more complex as the number of state variables grows. Thus, it would for example be feasible to study the economy considered here with aggregate productivity shocks and thereby analyze any "political business cycles" arising from median-voter tax determination in a quantitative context.[15]

The aggregation result requires complete markets, and in the present context—which does not have uncertainty—this just means that all agents can borrow and lend at the same rate. Under uncertainty, aggregation would require complete insurance markets. We know, however, from Krusell and Smith (1998), that a setting with idiosyncratic shocks and no insurance markets but precautionary savings using one asset leads to "approximate aggregation". Thus we conjecture that politico-economic equilibria in such a model would approximately depend only on an aggregate state vector (K, A_m) and not (almost at all) on any other moments of the asset distribution. We hope to explore this kind of setup in future work.

Second, we used first-order conditions of the median voter to interpret how taxes are chosen.

[15]The addition of an exogenous state variable—aggregate productivity—does make the analysis more difficult but would be entirely feasible.

We thus showed that the tax choice can be viewed as a tradeoff between direct redistribution effects and distortions to the labor-leisure and the consumption-savings choices in three consecutive time periods. One noteworthy point is that the consumption-savings distortion is a consideration for the median voter, even though capital income is inelastic ex post; recall that there is no commitment in advance to the tax choice. The reason is that the current tax influences savings from the present to the future, since it influences total resources available.

Third and finally, we used numerical methods to find a set of steady states for the infinite-horizon model. We compared the same parametric setup in a 1-period model, a 2-period model, and an infinite-horizon model. We found for the 2-period model that inequality, as measured by the median-to-mean wealth ratio, falls substantively between periods 1 and 2, as do tax rates. As an implication, the set of steady states is quite narrow, at least for parameter values that are close to those used in the macroeconomic literature. More specifically, we found that only very modest levels of inequality could be supported as long-run outcomes of the model. This indicates that models that have a chance of generating inequality/tax combinations that resemble those we observe in most developed countries would need different ingredients. One possibility is that explored in Krusell and Ríos-Rull (1999), namely, that there is an implementation lag for taxes, so that taxes are perceived as more distortionary when they are chosen. Another possibility is that inequality in labor productivity/wages, which is abstracted from here, would improve the quantitative performance of the model. In general, features that make it more costly to tax, or less beneficial to redistribute, would be required to improve the quantitative performance of the model.

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Appendix

The state variable of the economy is the distribution of asset holdings, which we denote $\mathbf{A} \equiv (A_1, \ldots, A_I)$. Thus let $H(a, \mathbf{A})$ be a function specifying the law of motion of the asset holding of an individual agent with beginning-of-period holdings a: $a' = H(a, \mathbf{A})$ (primes denote next-period values). Similarly, we let the leisure choice of a given agent be $L(a, \mathbf{A})$, with the associated aggregate labor supply function $N(\mathbf{A})$. Let $\Psi(\mathbf{A})$ be the tax rate on income imposed by the government. The function for transfers, $T(\mathbf{A})$, is specified residually to obey government budget balance.

Definition 12 *A **recursive competitive equilibrium** is a set of functions* $V(a, \mathbf{A})$, $H(a, \mathbf{A})$, $N(\mathbf{A})$, $L(a, \mathbf{A})$, $R(K, N)$, $W(K, N)$, $\Psi(\mathbf{A})$, *and* $T(\mathbf{A})$ *with the following properties.*

1. V *solves*

$$V(a, \mathbf{A}) = \max_{l,a'} u(c, l) + \beta V(a', \mathbf{A}')$$

subject to

$$c + a' = W(K, N(\mathbf{A}))(1 - l)(1 - \Psi(\mathbf{A})) + R(K, N(\mathbf{A}))(1 - \Psi(\mathbf{A}))a + T(\mathbf{A})$$

and $\mathbf{A}' = (H(A_1, \mathbf{A}), \ldots, H(A_I, \mathbf{A}))$ *for all* $(a, \mathbf{A})$, *where* $K \equiv \sum_{i=1}^{I} \mu_i A_i$.

2. H *and* L *attain the argmax above.*

3. N *satisfies* $N(\mathbf{A}) = \sum_{i=1}^{I} \mu_i(1 - L(A_i, \mathbf{A}))$.

4. W *and* R *satisfy* $W(K, N) = F_N(K, N)$ *and* $R = F_K(K, N)$ *for all* (K, N).

5. T *satisfies* $T(\mathbf{A}) = \Psi(\mathbf{A})(R(K, N(\mathbf{A}))K + W(K, N(\mathbf{A}))N(\mathbf{A}))$ *for all* $\mathbf{A}$, *where* $K \equiv \sum_{i=1}^{I} \mu_i A_i$.

Next, we look at one-period deviations.

Definition 13 *A **recursive competitive equilibrium with a one-period tax deviation** is a set of functions* $V(a, \mathbf{A})$, $\tilde{V}(a, \mathbf{A}, \tau)$, $H(a, \mathbf{A})$, $\tilde{H}(a, \mathbf{A}, \tau)$, $N(\mathbf{A})$, $\tilde{N}(\mathbf{A}, \tau)$, $L(a, \mathbf{A})$, $\tilde{L}(a, \mathbf{A}, \tau)$, $R(K, N)$, $W(K, N)$, $\Psi(\mathbf{A})$, $T(\mathbf{A})$, *and* $\tilde{T}(\mathbf{A}, \tau)$ *with the following properties.*

1. $V(a, \mathbf{A})$, $H(a, \mathbf{A})$, $N(\mathbf{A})$, $L(a, \mathbf{A})$, $R(K, N)$, $W(K, N)$, $\Psi(\mathbf{A})$, *and* $T(\mathbf{A})$ *is a recursive competitive equilibrium.*

2. *For all* τ, $\tilde{V}$ *satisfies*

$$\tilde{V}(a, \mathbf{A}, \tau) = \max_{l,a'} u(c, l) + \beta V(a', \mathbf{A}')$$

subject to

$$c + a' = W(K, \tilde{N}(\mathbf{A}, \tau))(1 - l)(1 - \tau) + R(K, \tilde{N}(\mathbf{A}, \tau))(1 - \tau))a + \tilde{T}(\mathbf{A}, \tau)$$

and $\mathbf{A}' = (\tilde{H}(A_1, \mathbf{A}, \tau), \ldots, \tilde{H}(A_I, \mathbf{A}), \tau)$ *for all* $(a, \mathbf{A})$*, where* $K \equiv \sum_{i=1}^{I} \mu_i A_i$.

3. $\tilde{H}$ *and* $\tilde{L}$ *attain the argmax above.*

4. $\tilde{N}$ *satisfies* $\tilde{N}(\mathbf{A}, \tau) = \sum_{i=1}^{I} \mu_i (1 - \tilde{L}(A_i, \mathbf{A}, \tau))$.

5. $\tilde{T}$ *satisfies* $\tilde{T}(\mathbf{A}, \tau) = \tau(R(K, \tilde{N}(\mathbf{A}, \tau))K + W(K, \tilde{N}(\mathbf{A}, \tau))\tilde{N}(\mathbf{A}, \tau))$ *for all* $\mathbf{A}$*, where* $K \equiv \sum_{i=1}^{I} \mu_i A_i$.

Here, note that the deviation equilibrium will satisfy $H(a, \mathbf{A}) = \tilde{H}(a, \mathbf{A}, \Psi(\mathbf{A}))$, $N(\mathbf{A}) = \tilde{N}(\mathbf{A}, \Psi(\mathbf{A}))$, and $L(a, \mathbf{A}) = \tilde{L}(a, \mathbf{A}, \mathbf{\Psi}(\mathbf{A}))$.

We can now state a definition of a Markov-perfect median-voter equilibrium. Let m denote the median type; A_m is thus the median asset holding.

Definition 14 *A* ***Markov-perfect median-voter equilibrium*** *is a set of functions* $V(a, \mathbf{A})$*,* $\tilde{V}(a, \mathbf{A}, \tau)$*,* $H(a, \mathbf{A})$*,* $\tilde{H}(a, \mathbf{A}, \tau)$*,* $N(\mathbf{A})$*,* $\tilde{N}(\mathbf{A}, \tau)$*,* $L(a, \mathbf{A})$*,* $\tilde{L}(a, \mathbf{A}, \tau)$*,* $R(K, N)$*,* $W(K, N)$*,* $\Psi(\mathbf{A})$*,* $T(\mathbf{A})$*, and* $\tilde{T}(\mathbf{A}, \tau)$ *which is a recursive competitive equilibrium with a one-period tax deviation and* *which satisfies the following property:*

$$\Psi(\mathbf{A}) = \arg\max_{\tau} \tilde{V}(A_m, \mathbf{A}, \tau)$$

for all $\mathbf{A}$.

Suppose now that we have found a recursive equilibrium with aggregation as defined in the main text, i.e., suppose that we have functions that satisfy all the stated conditions. We can then use these functions in order to construct additional functions, specifying behavior and utility of agents with arbitrary asset holdings, that together with the given functions meet all the conditions of the earlier, general definition of a recursive competitive equilibrium. This constitutes our aggregation theorem, which thus reads as follows:

Proposition 15 *Suppose that $u(c, l)$ satisfies Assumption 6 and that in any median-voter equilibrium all agents' solutions are interior. Given a recursive competitive equilibrium with aggregation, thus satisfying Definition 8,*

1. *define*

$$\lambda(a, A_m, K) \equiv \frac{a[R(K, n(A_m, K))(1 - \Psi(A_m, K))] + E(A_m, K)}{A_m[R(K, n(A_m, K))(1 - \Psi(A_m, K))] + E(A_m, K)};$$

2. *define*

$$H(a, \mathbf{A}) \equiv R(K, n(A_m, K))(1 - \Psi(A_m, K)))(a - \lambda(a, A_m, K)A_m) +$$

$$+[W(K, n(A_m, K))(1 - \Psi(A_m, K)) + T(A_m, K)](1 - \lambda(a, A_m, K)) + \lambda(a, A_m, K)h_m(A_m, K);$$

3. *define*

$$L(a, \mathbf{A}) \equiv \lambda(a, A_m, K) n_m(A_m, K)$$

and

$$N(\mathbf{A}) \equiv n(A_m, K);$$

4. *Let* $K = \sum \mu_i A_i$ *where* $A_i \in \mathbf{A}$

5. *define* $\Psi(\mathbf{A}) \equiv \Psi(A_m, K)$ *and* $T(\mathbf{A}) \equiv T(A_m K)$*; and*

6. *solve, for all* $(a, \mathbf{A})$*, for*

$$V(a, \mathbf{A}) = u(c(a, \mathbf{A}), 1 - L(a, \mathbf{A})) + \beta V(H(a, \mathbf{A}), \mathbf{A}'),$$

where $A'_i = H(A_i, \mathbf{A})$ *for all* i *and*

$c(a, \mathbf{A}) \equiv a + [aR(K, n(A_m, K)) + W(K, n(A_m, K))\lambda(a, A_m, K) n_m(A_m, K)](1 - \Psi(A_m, K))) +$
$T(A_m, K) - H(a, \mathbf{A})$.

Then $V(a, \mathbf{A})$*,* $H(a, \mathbf{A})$*,* $N(\mathbf{A})$*,* $L(a, \mathbf{A})$*,* $R(K, N)$*,* $W(K, N)$*,* $\Psi(\mathbf{A})$*, and* $T(\mathbf{A})$ *constitute a recursive competitive equilibrium, i.e., they satisfy Definition 12.*

Proof. Note first that the solution for V is well-defined: it is the fixed point of a contraction mapping. Given concavity of the consumer's problem in Definition 12, the first-order conditions, which appear in Definition 8, are sufficient for maximization. The remainder of the proof uses the functional-form version of the first-order conditions, which imply that all consumption goods (consumption and leisure at all points in time) are a constant fraction of net-present value wealth (asset holdings plus the present value of non-asset wealth). This feature allows us to show that if the first-order conditions hold for the median agent—and they do by assumption—they also hold for agents with all other asset levels. The details of the manipulations required to demonstrate this are straightforward and only involve algebraic manipulations. ■