Joint Distribution of Link Distances

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Abstract—**The calculation of two-hop connectivity between two terminals for randomly deployed wireless networks requires the joint probability distribution of the distances between these terminals and the terminal that is acting as a relay. In general the distances are not independent since a common terminal is involved. The marginal distributions for link distances are known for various random deployment models. However, the joint distribution of two or more link distances is not known. In this paper, the derivation of the joint distribution is given in general form and in a new form suitable for computation for a network of terminals randomly deployed in a square area**.

I. Introduction

The connectivity between pairs of terminals in a wireless network is a function of the distances between the terminals. The distribution of the distance d between one pair of terminals in a randomly deployed wireless network has been derived for various deployment models [1-3]. For example, for a uniform distribution of terminals in a $D \times D$ square area, the cumulative probability distribution function (CDF) for the distance is given by [1]

$$\mathrm{F}_d(\xi D) = \Pr\{d \le \xi D\}$$
$$= \begin{cases} 0, & \xi < 0 \\ \xi^2\left(\frac{1}{2}\xi^2 - \frac{8}{3}\xi + \pi\right), & 0 \le \xi < 1 \\ \frac{4}{3}\sqrt{\xi^2 - 1}(2\xi^2 + 1) \\ \qquad - \left(\frac{1}{2}\xi^4 + 2\xi^2 - \frac{1}{3}\right) \\ \quad + 2\xi^2[\sin^{-1}(1/\xi) - \cos^{-1}(1/\xi)], & 1 \le \xi < \sqrt{2} \\ 1, & \xi \ge 1 \end{cases} \tag{1}$$

For calculation of single-hop connectivity, the distribution in (1) is sufficient, assuming that the terminals are connected when d is less than a certain value (transmission radius). For example, since $\mathrm{F}_d(0.1D) = 0.0288$, if the transmission radius equals $R = 0.1D$ then a particular terminal pair is so positioned as to be connected with probability $p = 0.0288$, and the expected number of neighbors for a terminal in an N-terminal network equals $\nu = (N-1)p$.

For calculation of two-hop connectivity, the distribution in (1) is not sufficient because the two link distances in the connection are not independent—the second-order (joint) distribution for link distances is needed. In what follows, a derivation of this joint distribution is given and its computation and application are discussed.

II. General Form of Joint Distribution

For a network of N wireless terminals randomly deployed in some area, let the position of a reference terminal be given by $(x,\, y)$. The joint CDF for the distances between the reference terminal and the other $N-1$ terminals can be expressed as follows:

$$\mathrm{F}_{\boldsymbol{d}}(\alpha_1, \ldots, \alpha_{N-1}) = \mathrm{E}_{x,y}\left\{\prod_{n=1}^{N-1} \Pr\{d_n \le \alpha_n |\, x, y\}\right\}$$
$$= \mathrm{E}_{x,y}\left\{\prod_{n=1}^{N-1} \mathrm{F}_{dc}(\alpha_n |\, x, y)\right\} \tag{2}$$

where $\mathrm{F}_{dc}(\alpha_n |\, x, y)$ denotes the conditional CDF of a single terminal's distance from the reference terminal, given the position of the reference terminal.

III. Joint Distribution for Square Area

Consider a network of N wireless terminals randomly deployed in a $D \times D$ square area with independent x- and y-position coordinates between $-D/2$ and $D/2$. Making maximum use of symmetry, the joint CDF for the distances between the reference terminal and the other $N-1$ terminals for this case can be expressed as follows:

$$\mathrm{F}_{\boldsymbol{d}}(\alpha_1, \ldots, \alpha_{N-1}) = \frac{8}{D^2}\int_0^{D/2} dx \int_0^x dy \prod_{n=1}^{N-1} \mathrm{F}_{dc}(\alpha_n |\, x, y) \tag{3}$$

For the two-hop connectivity calculation, the reference terminal is the relay and the other $N-1$ terminals need to be considered in pairs. Regardless of how many other terminals are involved in the calculation, the conditional distribution is required. Based on (3), we assume that $0 \le y \le x \le D/2$ and use the diagram in Figure 1 to formulate the conditional CDF as follows:

$$\begin{aligned} \mathrm{F}_{dc}(\alpha |\, x, y) = \frac{1}{2D^2}\Bigg\{ &\int_{-\theta_{12}}^{\theta_{11}} d\theta \left[\min\left\{\alpha,\, \frac{D/2 - x}{\cos\theta}\right\}\right]^2 \\ &+ \int_{-\theta_{22}}^{\theta_{21}} d\theta \left[\min\left\{\alpha,\, \frac{D/2 - y}{\cos\theta}\right\}\right]^2 \\ &+ \int_{-\theta_{42}}^{\theta_{41}} d\theta \left[\min\left\{\alpha,\, \frac{D/2 + y}{\cos\theta}\right\}\right]^2 \\ &+ \int_{-\theta_{32}}^{\theta_{31}} d\theta \left[\min\left\{\alpha,\, \frac{D/2 + x}{\cos\theta}\right\}\right]^2 \Bigg\} \end{aligned} \tag{4}$$

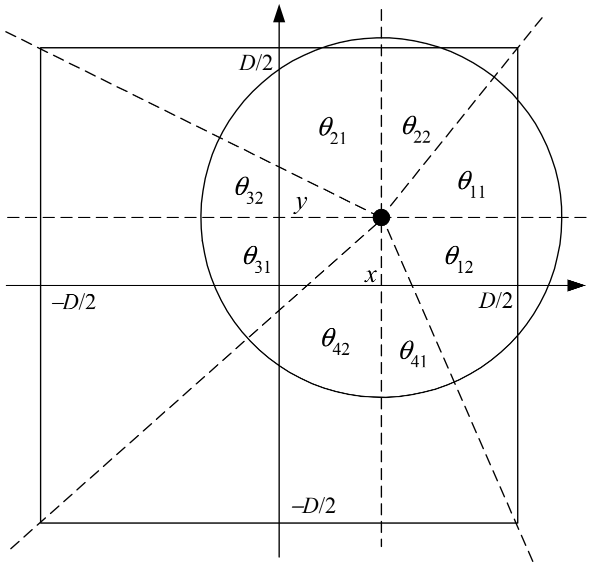

Figure 1. Key angles used in the derivation.

where

$$\theta_{11} = \tan^{-1}\left(\frac{D/2 - y}{D/2 - x}\right), \ \theta_{12} = \tan^{-1}\left(\frac{D/2 + y}{D/2 - x}\right) \quad (5a)$$

$$\theta_{31} = \tan^{-1}\left(\frac{D/2 + y}{D/2 + x}\right), \ \theta_{32} = \tan^{-1}\left(\frac{D/2 - y}{D/2 + x}\right) \quad (5b)$$

$$\theta_{21} = \frac{\pi}{2} - \theta_{32}, \quad \theta_{22} = \frac{\pi}{2} - \theta_{11} \quad (5c)$$

$$\theta_{41} = \frac{\pi}{2} - \theta_{12}, \quad \theta_{42} = \frac{\pi}{2} - \theta_{31} \quad (5d)$$

Note that the sum of the eight angles in (5) equals 2π and that, for the assumed reference terminal position,

$$D/2 - x \le D/2 - y \le D/2 + y \le D/2 + x \quad (6)$$

Therefore, for $\alpha \le D/2 - x$, the integrand in all of the integrals in (4) is α^2 and the conditional probability equals $\pi\alpha^2/D^2$. As α increases through the ordered values of distance in (6), the probability increases but is less than $\pi\alpha^2/D^2$ because the integrands of the integrals in (4) in order change from α^2 to an expression of the form $\beta^2/\cos^2\theta$, according to the rule given by

$$\min\left\{\alpha, \frac{\beta}{\cos\theta}\right\} = \begin{cases} \alpha, & \alpha < \beta \\ \alpha, & \alpha \ge \beta \text{ and } \cos\theta > \beta/\alpha \\ \dfrac{\beta}{\cos\theta}, & \alpha \ge \beta \text{ and } \cos\theta \le \beta/\alpha \end{cases} \quad (7)$$

The quantity β for each integral in (4) is the shortest distance to the edge of the square in the applicable angular region. When $\alpha > \sqrt{(D/2+x)^2 + (D/2+y)^2}$, the total area of the square is encompassed by the circle centered at (x, y).

IV. Discussion

The form of the joint CDF given here for the square area is designed for ease of computation. It is decomposed into four separate integrals involving angles less than $\pi/2$, each of which can be simplified further, as follows:

$$\frac{1}{2D^2}\int_{-\theta_2}^{\theta_1} d\theta \left[\min\left\{\alpha, \frac{\beta}{\cos\theta}\right\}\right]^2$$
$$= \frac{\alpha^2}{2D^2}[\theta_2 + \theta_1 - \min\{\theta_3, \theta_1\} - \min\{\theta_3, \theta_2\}]$$
$$+ \frac{\beta^2}{2D^2}[\tan(\min\{\theta_3, \theta_1\}) + \tan(\min\{\theta_3, \theta_2\})] \quad (8a)$$

where

$$\theta_3 = \cos^{-1}(\min\{\beta, \alpha\}/\alpha) \quad (8b)$$

For calculation of the joint probability density function (PDF), it is straightforward to show that it is the average over (x, y) of the product of the sums of the derivatives of the four integrals, where

$$\frac{\partial}{\partial\alpha}\left\{\frac{1}{2D^2}\int_{-\theta_2}^{\theta_1} d\theta \left[\min\left\{\alpha, \frac{\beta}{\cos\theta}\right\}\right]^2\right\}$$
$$= \frac{\alpha}{D^2}[\theta_2 + \theta_1 - \min\{\theta_3, \theta_1\} - \min\{\theta_3, \theta_2\}] \quad (9)$$

This derivation is part of an ongoing effort to develop analytical tools for the assessment of wireless ad hoc networks, with an emphasis on heuristics that simplify computation and analysis. Numerical evaluations of the expressions disclosed in this paper will be used to quantify the error that is involved in ignoring the dependence among the distances of links having a common terminal. Graphical comparison of the joint CDF with the case of independent distances is shown in Figure 2 for two distances with a common endpoint.

References

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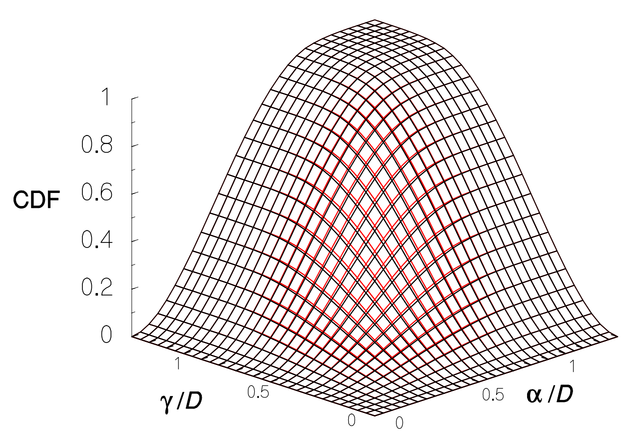

Fig. 2 Comparison of joint CDF with product of CDFs.