

Probability of a Two-Hop Connection in a Random Mobile Network

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Abstract – The probability is found for the event that, in an N -node randomly distributed (Gaussian) deployment of mobile radio terminals (nodes), any two nodes are connected by a two-hop path and are not directly connected. An approximate closed form expression, parametric in N and a function of the relative size of the deployment area, is given based on fitting to the results obtained by numerical integration. An upper bound is also obtained for the probability of an m -hop connection, and this expression is applied to the calculation of an upper bound for the average hop distance in the network.

KEY WORDS: Mobile networks, connectivity, probability

I. INTRODUCTION

Suppose that a number of mobile radio terminals (nodes) are randomly deployed over a certain area, as depicted in Figure 1. The probability that a link between two nodes has sufficient signal-to-noise ratio for acceptable transmission quality or reliability is, other factors being equal, the probability that the link distance r is less than some value R , where R is termed the transmission range:

$$\Pr\{\text{Link is good}\} = \Pr\{r \leq R\} = F_r(R) \quad (1.1)$$

The function $F_r(\cdot)$ in (1.1) is the cumulative probability distribution function (cdf) for the link distance. Assuming that different links are independent, the quantity $F_r(R)$ can be taken as the probability of success (acceptable transmission quality) in a binomial trial in which two link endpoints are selected; if the trial is repeated K times, then an estimate of the number of good links is $K F_r(R)$. For this reason, the cdf for the link distances in a mobile radio system is an important quantity ([1], [2]). General expressions for this cdf are given in [2] for a uniform distribution of mobiles positioned over a rectangular area and for a bivariate Gaussian distribution of positions.

In general, an ad hoc mobile network is a multihop network in which the distance of a particular node A from some other nodes may be greater than R , and A is then said to be "hidden" from those other nodes ([3], [4]). Under a carrier sense multiple access procedure, the "hidden terminal problem" may arise, in which two nodes that are more than distance R apart transmit at the same time and their transmissions "collide" at a third node that is within distance R of both of the other nodes. Thus, all node pairs that are connected by a single relay node (are two hops away from each other) have the potential for creating a hidden terminal problem. In this paper we calculate

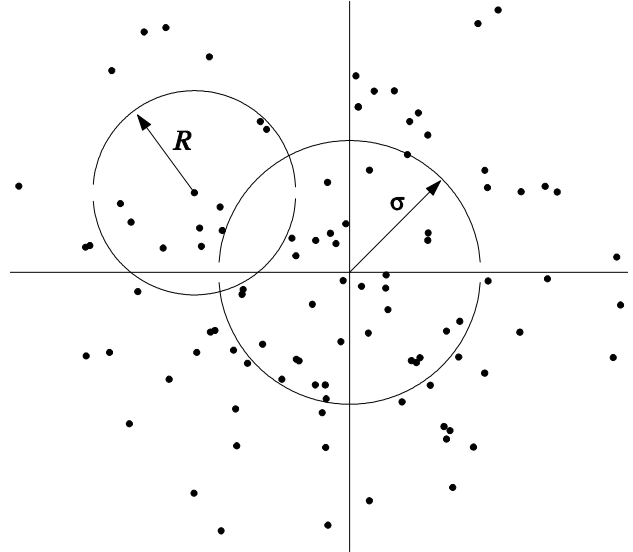


Figure 1. Random network with dispersion parameter σ , transmission range R .

the probability that two nodes are connected by a two-hop path for a network deployment in which the x and y coordinates of the mobile locations have Gaussian distributions.

II. DISTRIBUTIONS OF NODE LOCATIONS AND LINKS

We assume that the x and y coordinates of the mobile locations all have independent zero-mean Gaussian distributions, where σ_1 and σ_2 are, respectively, the standard deviations of the x and y coordinates. Thus the x and y components $d_x = x_2 - x_1$ and $d_y = y_2 - y_1$ of the link distance between two nodes are also independent Gaussian random variables (RVs) with standard deviations $\sigma_1\sqrt{2}$ and $\sigma_2\sqrt{2}$, respectively, and the probability density function (pdf) of the link distance $r = \sqrt{d_x^2 + d_y^2}$ is found to be [2]

$$p_r(r) = \frac{r}{2\sigma_1\sigma_2} \exp\left\{-\frac{r^2}{8}(\sigma_1^{-2} + \sigma_2^{-2})\right\} \times I_0\left[\frac{r^2}{8}(\sigma_1^{-2} - \sigma_2^{-2})\right] \quad (2.1)$$

in which $I_0(\cdot)$ is the modified Bessel function of the first kind and order zero. For $\sigma_1 = \sigma_2 = \sigma$, the link distance is a scaled Rayleigh RV; (2.1) then simplifies to the pdf expression

$$p_r(r) = \frac{r}{2\sigma^2} e^{-r^2/4\sigma^2} \quad (2.2)$$

and P_1 , the probability of a one-hop connection, is easily determined as a function of the ratio of the dispersion parameter σ to R (quantities illustrated in Figure 1):

$$\begin{aligned} \Pr\{\text{1-hop connection}\} &= \Pr\{r \leq R\} \triangleq P_1 \\ &= \int_0^R dr p_r(r) = 1 - e^{-R^2/4\sigma^2} \end{aligned} \quad (2.3)$$

Calculations of P_1 were made for the general case of $\sigma_1 \neq \sigma_2$ and are shown in Figure 2 as a function of σ_1/R . The figure shows that, as the variances of the x and y coordinates increase, P_1 decreases because the nodes are typically farther apart.

III. ANALYSIS OF TWO-HOP CONNECTIONS

A two-hop connection between nodes 1 and 2 exists if two conditions are met: (1) the positions (x_1, y_1) for node 1 and (x_2, y_2) for node 2 are such that the distance between the nodes is greater than the transmission range R but less than $2R$; and (2) the position (x_3, y_3) for at least one other node is within the distance R of both nodes 1 and 2.

The geometry of a two-hop connection is illustrated in Figure 3. Given the positions (x_1, y_1) and (x_2, y_2) such that the two circles with radius R intersect but their centers are greater than R apart, the position of a relay node (x_3, y_3) must lie within the area of the intersection of the two circles. Note that the center of the area of intersection is the midpoint of the line between (x_1, y_1) and (x_2, y_2) , which has the coordinates $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$. Note also that, in general, the orientation of the area of intersection with respect to the origin of the coordinate system depends on the relation between positions (x_1, y_1) and (x_2, y_2) , so that we may denote the area of intersection as $A(x_1, y_1, x_2, y_2)$. The probability of a two-hop connection for the case of $N \geq 3$ nodes, denoted P_2 , can be formulated as follows:

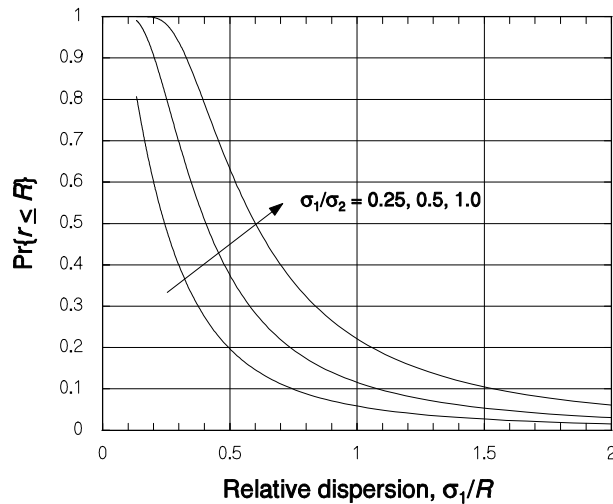


Figure 2. One-hop connection probability.

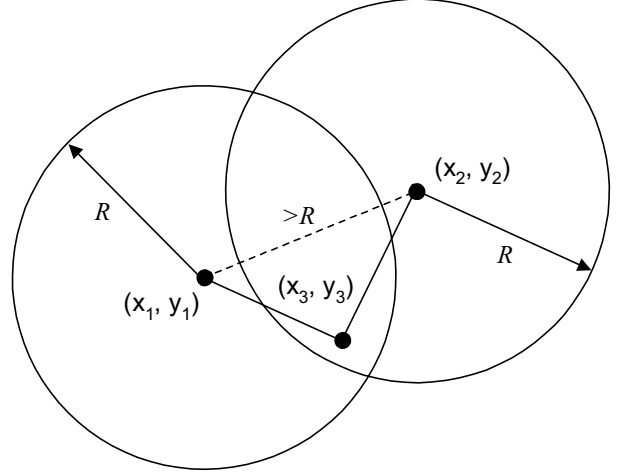


Figure 3. Geometry of a two-hop connection.

$$\begin{aligned} \Pr\{1 \rightarrow 2 \text{ in 2 hops}\} &\triangleq P_2 \\ &= \Pr\{R < r < 2R \text{ and} \\ &\quad \text{at least 1 other node in the area of intersection}\} \\ &= \Pr\{R < r < 2R \text{ and} \\ &\quad \text{NOT (no other node in the area of intersection)}\} \\ &= \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2 p_{x,y}(x_1, y_1, x_2, y_2)}_{R < r < 2R} \\ &\quad \times \left[1 - \left[1 - \underbrace{\int dx_3 \int dy_3 p_{x,y}(x_3, y_3)}_{A(x_1, y_1, x_2, y_2)} \right]^{N-2} \right] \end{aligned} \quad (3.1)$$

where r is the distance between nodes 1 and 2. Note that the inner integration of $p_{x,y}(x_3, y_3)$ in (3.1) depends on the position and orientation of the area through the weighting of the pdf. As an approximation, we may write the inner double integral as

$$\begin{aligned} &\underbrace{\int dx_3 \int dy_3 p_{x,y}(x_3, y_3)}_{A(x_1, y_1, x_2, y_2)} \\ &\approx p_{x,y}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \underbrace{\int dx_3 \int dy_3}_{A(x_1, y_1, x_2, y_2)} \end{aligned} \quad (3.2)$$

where the area of intersection, denoted $B(r)$, is given by [5]

$$\begin{aligned} &\underbrace{\int dx_3 \int dy_3}_{A(x_1, y_1, x_2, y_2)} \triangleq B(r) \\ &= R^2 \left(2 \cos^{-1}\left(\frac{r}{2R}\right) - \sin\left[2 \cos^{-1}\left(\frac{r}{2R}\right)\right] \right) \end{aligned} \quad (3.3)$$

Note that the area of intersection depends only on the distance between nodes 1 and 2, relative to R . Using the approximation approach of (3.2), the expression for P_2 becomes

$$P_2 \approx \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R < r < 2R} p_{x,y}(x_1, y_1, x_2, y_2) \times \left[1 - \left[1 - p_{x,y}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) B(r) \right]^{N-2} \right] \quad (3.4)$$

For convenience, we restrict our attention to the case of $\sigma_1 = \sigma_2 = \sigma$. For this case, the expression in (3.4) can be manipulated into the following form:

$$P_2 \approx \int_R^{2R} dr \int_0^\infty d\rho \frac{r\rho}{\sigma^4} \exp\left\{-\frac{r^2}{4\sigma^2} - \frac{\rho^2}{\sigma^2}\right\} \times \left[1 - \left[1 - \frac{B(r)}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2} \right]^{N-2} \right] \quad (3.5)$$

Let us define $a \equiv a(r) \triangleq B(r)/2\pi\sigma^2$. The integration over ρ is carried out by making a sequence of changes of variable:

$$\begin{aligned} & \int_0^\infty d\rho \frac{\rho}{\sigma^2} e^{-\rho^2/\sigma^2} \left[1 - \left[1 - a e^{-\rho^2/2\sigma^2} \right]^{N-2} \right] \\ &= \frac{1}{2} - \int_0^1 dv v (1 - av)^{N-2} \text{ using } v = e^{-\rho^2/2\sigma^2} \\ &= \frac{1}{2} - a^{-2} \int_{1-a}^1 dw w^{N-2} (1-w) \text{ using } w = 1 - av \\ &= \frac{1}{2} - \frac{1 - (1-a)^N - Na(1-a)^{N-1}}{a^2 N(N-1)} \end{aligned} \quad (3.6)$$

Using (3.6) the total integral (3.5) becomes

$$\begin{aligned} P_2 &\approx \int_R^{2R} dr \frac{r}{2\sigma^2} e^{-r^2/4\sigma^2} \\ &\times \left\{ 1 - \frac{2[1 - (1-a)^N - Na(1-a)^{N-1}]}{a^2 N(N-1)} \right\} \\ &= \int_{R^2/4\sigma^2}^{R^2/\sigma^2} d\nu e^{-\nu} \\ &\times \left\{ 1 - \frac{2[1 - (1-b)^N - Nb(1-b)^{N-1}]}{b^2 N(N-1)} \right\} \end{aligned} \quad (3.7)$$

where in keeping with the change of variable, we define $b \equiv b(\nu) \triangleq B(2\sigma\sqrt{\nu})/2\pi\sigma^2$.

An alternative solution to the integration over ρ is to expand the binomial factor in the integrand of (3.5) to obtain

$$\begin{aligned} & \int_0^\infty d\rho \frac{\rho}{\sigma^2} e^{-\rho^2/\sigma^2} \left[1 - \left[1 - a e^{-\rho^2/2\sigma^2} \right]^{N-2} \right] \\ &= \frac{1}{2} - \frac{1}{2} \int_0^\infty du e^{-u} \left[1 - a e^{-u/2} \right]^{N-2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \sum_{n=1}^{N-2} \binom{N-2}{n} (-1)^{n+1} a^n \int_0^\infty du e^{-u(1+n/2)} \\ &= \frac{1}{2} \sum_{n=1}^{N-2} \binom{N-2}{n} \frac{(-1)^{n+1} a^n}{1+n/2} \end{aligned} \quad (3.8)$$

Substituting (3.8) into (3.5) yields the following expression for P_2 :

$$\begin{aligned} P_2 &\approx \sum_{n=1}^{N-2} \binom{N-2}{n} \frac{(-1)^{n+1}}{1+n/2} \int_R^{2R} dr \frac{r}{2\sigma^2} e^{-r^2/4\sigma^2} \left[\frac{B(r)}{2\pi\sigma^2} \right]^n \\ &= \sum_{n=1}^{N-2} \binom{N-2}{n} \frac{(-1)^{n+1}}{1+n/2} \frac{1}{(2\pi)^n \gamma^{2n}} \\ &\times \int_{1/4\gamma^2}^{1/\gamma^2} d\nu e^{-\nu} [2 \cos^{-1} \gamma \sqrt{\nu} - \sin(2 \cos^{-1} \gamma \sqrt{\nu})]^n \end{aligned} \quad (3.9)$$

in which we use $\gamma \triangleq \sigma/R$ to index the dispersion of the nodes relative to their transmission range.

An upper bound on P_2 can be derived by considering the effect on the formulation of the probability in (3.7) as the number of nodes approaches infinity. As $N \rightarrow \infty$, there is an upper bound given by

$$\begin{aligned} P_2 &< \int_{R^2/4\sigma^2}^{R^2/\sigma^2} d\nu e^{-\nu} \{1 - 0\} \\ &= e^{-R^2/4\sigma^2} - e^{-R^2/\sigma^2} = P_{2\infty} \end{aligned} \quad (3.10)$$

IV. NUMERICAL RESULTS AND APPROXIMATIONS

Calculations of the two-hop connectivity probability that were performed using the integral form in (3.9) are shown in Figure 4 as a plot of P_2 vs. $\gamma = \sigma/R$, for $N = 3, 4, 5, 6, 7, 8, 12, 17, 22, 50$. Also plotted are the one-hop connectivity probability given in (2.3) and the asymptotic two-hop connectivity expression given in (3.10). We observe in Figure 4 that P_2 is practically zero for $\gamma < 0.25$ (i.e., $\sigma < \frac{1}{4}R$). For this small a dispersion, all the mobile terminals can hear each other, as evidenced in Figure 4 by the fact that $P_1 \doteq 1$ for this range of γ . As γ increases, the locations of the terminals are dispersed over a wider area, and in Figure 4 we see that P_2 at first increases for $\gamma > 0.25$. The two-hop probability then peaks at a value of about 10% for $N = 3$ nodes to about 45% for an infinite number of nodes, for γ from $\gamma \approx 0.4$ to $\gamma \approx 0.7$, respectively. Thereafter, as γ increases, the probability decreases because the nodes are more likely to be too far apart for a two-hop connection to succeed.

Using the CoPlot graphing and analysis software [6], a nonlinear regression was used to fit the two-hop connection probability results in Figure 4 to curves of the form given by the following equation, parametric in α and β :

$$f(\gamma; \alpha, \beta) = \left(1 - \alpha e^{-\beta/\gamma^2} \right) P_{2\infty}, \text{ using } \frac{\sigma}{R} \triangleq \gamma \quad (4.1)$$

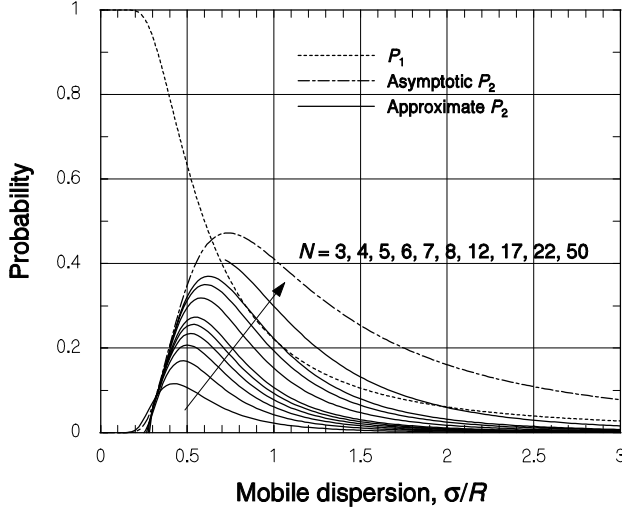


Figure 4. Plot of P_2 and P_1 vs. $\gamma = \sigma/R$.

As demonstrated in Figure 5, except for the case of $N = 3$, these parameters are well approximated by the empirical expressions

$$\alpha \approx 1.06 e^{-0.0055(N-1)} \quad \text{and} \quad \beta \approx 0.157 \sqrt{N-3} \quad (4.2)$$

Using the results of the regressions and curve fitting, an empirical expression for P_2 as a function of N is given by

$$P_2 \approx \left(1 - 1.0383 e^{-0.0988/\gamma^2}\right) P_{2\infty}, \quad N = 3 \quad (4.3)$$

and

$$P_2 \approx \left(1 - 1.06 e^{-0.0055(N-1) - 0.157 \sqrt{N-3}/\gamma^2}\right) P_{2\infty}, \quad N > 3 \quad (4.4)$$

These expressions are plotted in Figure 6, which compares well with Figure 4, indicating that the approximations should be useful for studies of two-hop connectivity with different numbers of nodes.

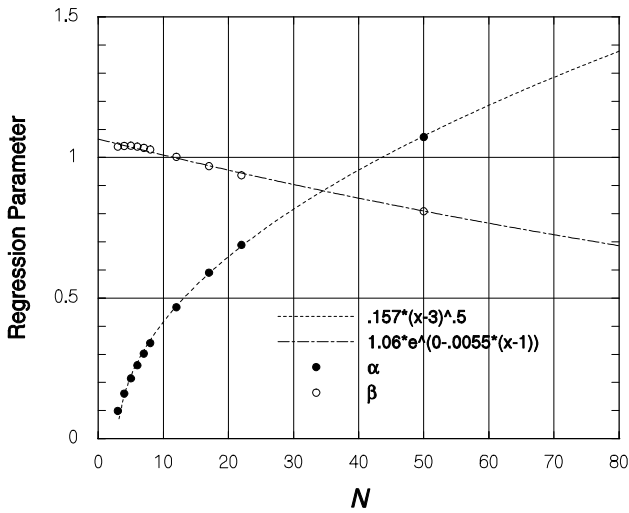


Figure 5. Curve fit for regression parameters α and β .

V. ASYMPTOTIC m -HOP CONNECTION PROBABILITY AND APPLICATIONS

The form of the asymptotic probability $P_{2\infty}$ for a two-hop connection suggests the following form for the asymptotic probability of an m -hop connection:

$$\begin{aligned} P_m &< \Pr\{(m-1)R < r < mR\} \\ &= e^{-(m-1)^2 R^2 / 4\sigma^2} - e^{-m^2 R^2 / 4\sigma^2} \triangleq P_{m\infty} \end{aligned} \quad (5.1)$$

Using $\gamma = \sigma/R$, the m -hop asymptotic connection probabilities are plotted in Figure 7. Note that the sum of the probabilities equals 1. The most likely number of hops, given the value of γ , the relative dispersion of the nodes, is determined from (5.1) as

$$\text{Most likely \# hops} = \begin{cases} 1, & \sigma/R < 0.64 \\ 2, & 0.64 < \sigma/R < 1.39 \\ 3, & 1.39 < \sigma/R < 2.10 \\ 4, & 2.10 < \sigma/R < 2.81 \\ 5, & 2.81 < \sigma/R < 3.52 \end{cases} \quad (5.2)$$

The asymptotic probability $P_{m\infty}$ is an upper bound to the actual probability of an m -hop connection, P_m . Therefore, an upper bound on the average number of hops between node pairs is given by the expectation

$$E\{h\} = \sum_{m=1}^{N-1} m P_m < \sum_{m=1}^{\infty} m P_{m\infty} \quad (5.3)$$

Substituting (5.1) in the right-hand side of (5.3) yields the following upper bound on the average number of hops:

$$\begin{aligned} E\{h\} &< \sum_{m=1}^{\infty} m \left(e^{-(m-1)^2 / 4\gamma^2} - e^{-m^2 / 4\gamma^2} \right) \\ &= \sum_{m=0}^{\infty} (m+1) e^{-m^2 / 4\gamma^2} - \sum_{m=0}^{\infty} m e^{-m^2 / 4\gamma^2} \\ &= \sum_{m=0}^{\infty} e^{-m^2 / 4\gamma^2} \triangleq \bar{h}_+(\gamma) \end{aligned} \quad (5.4)$$

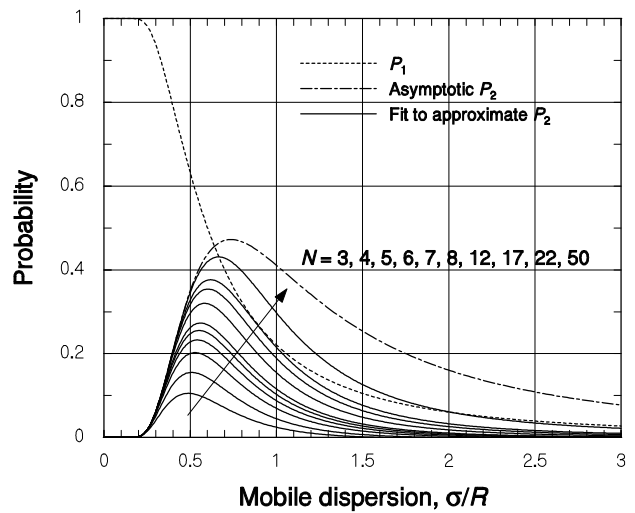


Figure 6. Approximations for P_2 .

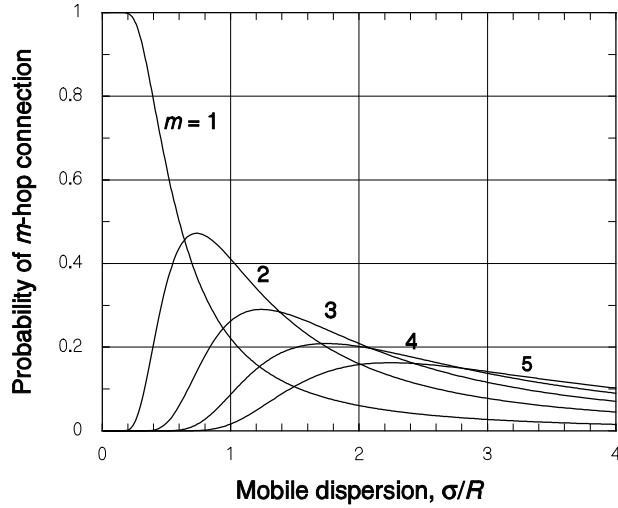


Figure 7. Asymptotic m -hop connection probabilities.

This bound is plotted in Figure 8, which suggests that the summation in (5.4) converges to a nearly linear function of the ratio of node dispersion parameter to transmission range, γ , for $\gamma > 0.25$. That linear function can be found from the Taylor's series expansion of (5.4) about some value of γ , say, $\gamma = 0.5$, to derive the piecewise linear function given by $\bar{h}_+(\gamma) \approx 1$ for $\gamma < 0.25$, and

$$\begin{aligned} \bar{h}_+(\gamma) &\approx 1 + e^{-1} + e^{-4} + e^{-9} + \dots \\ &\quad + (\gamma - 0.5) \times 4(e^{-1} + 4e^{-4} + 9e^{-9} + \dots) \\ &\doteq 0.501 + 1.769\gamma, \quad \gamma \geq 0.25 \end{aligned} \quad (5.5)$$

The linear approximation also can be obtained by the following derivation:

$$\sum_{m=0}^{\infty} e^{-m^2/4\gamma^2} = \frac{1}{2} + \frac{1}{2} \int_{-\infty}^{\infty} dx e^{-x^2/4\gamma^2} g(x) \quad (5.6)$$

where $g(x)$ is a periodic train of impulses, with the Fourier series representation (using $i = \sqrt{-1}$) [7, §6.4.3]

$$g(x) = \sum_{m=-\infty}^{\infty} \delta(x - m) = \sum_{k=-\infty}^{\infty} e^{2\pi k i x} \quad (5.7)$$

Substituting (5.7) in (5.6), then exchanging the order of summation and integration, yields an expression involving the characteristic function of a Gaussian random variable that reduces to

$$\bar{h}_+(\gamma) = \frac{1}{2} + \gamma \sqrt{\pi} \sum_{k=-\infty}^{\infty} e^{-(2\pi k \gamma)^2} \quad (5.8a)$$

$$\approx 0.5 + 1.772\gamma, \quad \gamma > 0.5 \quad (5.8b)$$

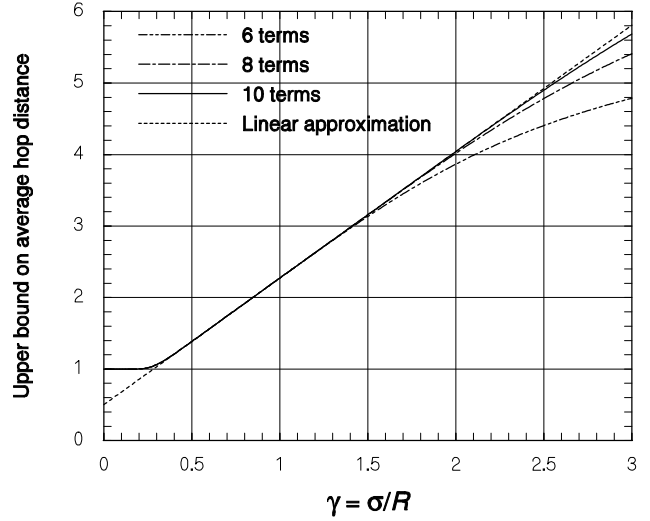


Figure 8. Upper bound on average hop distance.

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