Document ID: 06_05_99_1
Date Received: 1999-06-05 **Date Revised**: 1999-08-01 **Date Accepted**: 1999-08-08
Curriculum Topic Benchmarks: M1.4.1, M.1.4.3, M1.4.6, M1.4.7, M2.4.1, M2.4.3, M2.4.5, M4.4.6, M4.4.8
Grade Level: [9-12] High School
Subject Keywords: slide rule, analog computer, isomorphism, logarithm
Rating: advanced

Learning from Slide Rules

By: Martin P Cohen, Environmental Standards, Inc, 52 Hollybrook Drive , Langhorne PA 19047
e-mail: mpcohen@mindspring.com

From: The PUMAS Collection http://pumas.jpl.nasa.gov

Introduction -- In the days before calculators and personal computers an engineer always had a slide rule nearby. These days it is difficult to locate a slide rule outside of a museum.

I don't know how many of those who have used a slide rule ever thought of it as an analog computer, but that is really what it is. As such, the slide rule can be used to teach the modern view of the relationship between nature and mathematics and about the formalization of this concept known as isomorphism, which is one of the most pervasive and important concepts in mathematics.

In order to introduce the concepts that will be used, we will start with the simplest type of slide rule – one that is made up of two ordinary rulers.

Using Rulers for Addition -- Two rulers can be lined up as in the figure below to show that 5 + 3 = 8. Note that in order to do this it is not necessary to know how to add or even to be able to count. All that is required is to line up the start of the top ruler with the symbol "5" on the bottom ruler; locate the symbol "3" on the top ruler and the symbol "8" below it on the bottom ruler.

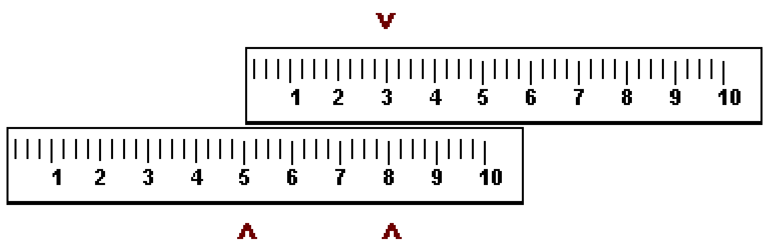

Figure 1 - Use of two rulers to add 5 and 3

For the purposes here I am going to define a slide rule as using two copies of the same scale, with one scale moving relative to the other. Actual slide rules may use two different scales, which could either be stationary or mobile with respect to the other.

Analog Computers -- An analog computer performs a calculation by transforming numbers into physical quantities, combining these physical quantities and then converting the result into the result of the calculation. The way in which a slide rule is an analog computer should be clear - a numerical computation (in this case addition) is performed by changing numbers into a physical representation (distances); the distances are added and the resultant distance is reverse transformed into the result of the numerical calculation.

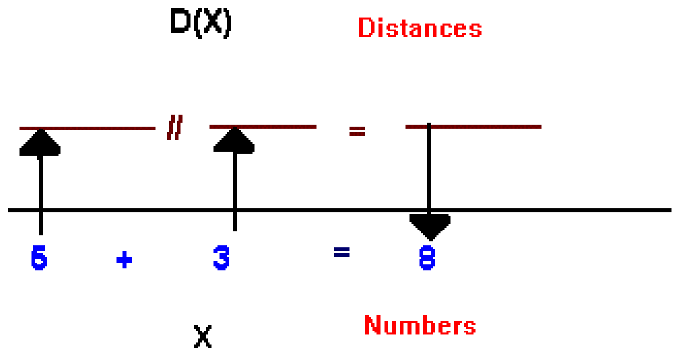

Figure 2 - Converting numbers to distances

Figure 2 above shows diagrammatically what is happening. Imagine two worlds, a number world X below the black horizontal line and a distance world above it. In the number world 3 is added to 5 to get 8. The arrows above "5" and "3" represent the transform of these numbers into distances. The symbol "//" represents the operation of adding the two distances to produce the distance to the right of the "//". The arrow above the "8" represents the transform of that distance into the number 8.

There are thus two ways to get to the "8". Symbolically, $5 + 3 = D^{-1}$ (D(5) // D(3)). That is, adding 5 and 3 is the same as adding their distances and then taking the inverse transform to convert the distance 8 into the number 8.

Isomorphisms -- Our intuition tells us that adding numbers is in some sense the same as adding distances. Let us look a little more closely at what is going on. We want to be able to say that 5 and 3 with respect to + are equivalent to D(5) and D(3) with respect to //. We can think of 5 and 3 as inputs to the + operator, which produces the output **(5 + 3).** We can think of D(5) and D(3) as inputs to the // operator, which produces the output **(D(5) // D(3)).**

In order to be able to say that the transformation D results in equivalence we must be able to say that equivalent inputs result in equivalent outputs or, D**(5 + 3)** = **(D(5) // D(3))**, in agreement with what we showed above. The relationship D**(5 + 3)** = **(D(5) // D(3))** is described by saying that adding numbers is *isomorphic* to adding distances. The word isomorphic is derived from Greek and means "same form". To generalize, if T is a transformation and & and % are operators then & and % are isomorphic if T(x & y) = T(x) % T(y). In the above example
T = D, & = + and % = //.

An Ancient Manuscript -- Consider the following story: An explorer comes upon the first known sample of writing from an ancient civilization. Based on translations from other civilizations that existed at the same time, it is believed that a snake stands for 5, a frog stands for 3, a cow stands for 8 and ~ stands for +. Without knowing anything about the writing, how can the explorer test this hypothesis? If the manuscript contained "snake ~ frog ... cow" then this would clearly be supporting evidence.

The point to be taken from this isomorphism is that an isomorphism is always a direct translation from one language to another. The two languages are structurally identical. Any statement in one can be turned into an equivalent statement in the other by swapping one set of symbols for the other. If the statement using one set of symbols is true then the transformed statement is true. This holds even if the objects being manipulated, for example numbers and distances, are completely different.

The Relationship between Science and Mathematics -- To the ancient Greeks there was no difference between mathematics and science. They believed that statements about mathematics directly described nature. They did not say that Euclidean geometry modeled the real world. There was no such concept. The Greeks believed that the study of geometry was a type of scientific exploration.

The modern view is much different. Mathematics is based on axioms and undefined terms. Mathematical truths are built on pure logic. Science is based on experimentation and observation. Its truths are empirical.

We now think of mathematics as the language of science. If we take a basic scientific law, I*R=V, or Current * Resistance = Voltage, what we have is an isomorphism between the physical quantities and their mathematical measurements.

Logarithms, Multiplication and Composition -- Isomorphisms are not just a way of relating mathematical and physical operations. They play an important part in mathematics because they relate different mathematical operations to each other.

The logarithmic function satisfies the following relationship between the multiplication of positive numbers and the addition of all real numbers: log (x * y) = log(x) + log(y).

This is an isomorphic relationship that says that that multiplication of positive numbers is structurally equivalent to addition. We have shown above that the D function establishes an isomorphism between addition and the combination of distances. By combining the log and D functions we establish an isomorphism between multiplication and the combination of distances:

D(log(x*y)) = D(log(x) + log(y)) = D(log(x)) // D(log(y))
D(log(x*y)) = D(log(x)) // D(log(y))

It follows that the transform (D log) formed by composing the D and log transforms is an isomorphism. The same process could be used to show that in general the composition of two isomorphic transforms is an isomorphic transform.

The construction of a slide rule for multiplication follows from the above equation. If the distance that a number is placed is equal to the log of the number then the result is the standard slide rule. Figure 3 shows how the slide rule is used to multiply 10 and 1000.

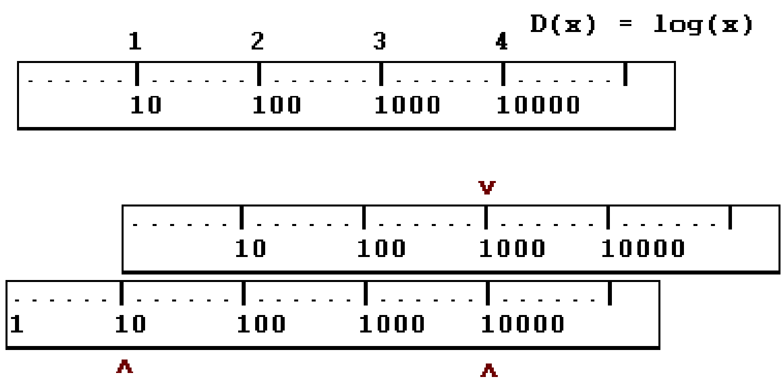

Figure 3 - Multiplication of 10 * 1000 = 10000

Exploring the Structural Equivalence of Multiplication and Addition -- I stated above that the logarithmic relationship shows that multiplication of positive numbers is structurally equivalent to addition of all numbers. Let us explore the nature of this equivalence. Doing so should help to make working with logarithms a little easier for the student. Addition is commutative and associative, that is a + b = b + a and (a + b) + c = a + (b + c). For multiplication to be equivalent it too must be both commutative and associative, which of course it is.

The number 1 satisfies the relationship 1 * x = x for any x. We say that 1 is the multiplicative identity. We have log(x) = log(1 * x) = log(1) + log(x) or
log(x) + log(1) = log(x). log(1) is the additive identity, which is zero.
1/x is the multiplicative inverse of x. x * 1/x = 1. log(x * 1/x) = log(1) or log(x) + log(1/x) = 0, so log(1/x) = -log(x). The log of the mulitiplicative inverse is the additive inverse.

Related Web Site: The document at http://www.mathed.org/slide.html covers some of the above information and also describes the Pythagorean relationship and the equation for relativistic addition of velocities as examples of isomorphisms. There are instructions for building slide rules for both of these relationships. The site also includes virtual slide rules for both right triangles and relativistic velocity addition.