OPTIMAL SIGNAL SETS FOR NON-GAUSSIAN DETECTORS *

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Abstract. Identifying a maximally-separated set of signals is important in the design of modems. The notion of optimality is dependent on the model chosen to describe noise in the measurements; while some analytic results can be derived under the assumption of Gaussian noise, no such techniques are known for choosing signal sets in the non-Gaussian case. To obtain numerical solutions for non-Gaussian detectors, minimax problems are transformed into nonlinear programs, resulting in a novel formulation yielding problems with relatively few variables and many inequality constraints. Using sequential quadratic programming, optimal signal sets are obtained for a variety of noise distributions.

Key words. Optimal Design, Inequality Constraints, Sequential Quadratic Programming

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1. Introduction. The transmission of digital information requires signals (finite time series) that can be distinguished from one another in the presence of noise. These signals may be constrained by bounds on their energy or amplitude; the degree to which they can be distinguished depends on the distribution of noise.

We study the design of optimal signal sets under amplitude constraints and in the presence of non-Gaussian noise. We will call a signal set optimal when, roughly speaking, the *largest* probability of mistaking any one signal for any other is minimal. For this reason, as we show below, this problem is naturally formulated as a smooth and twice continuously differentiable minimax problem.

We shall assume that M signals

$$s_0, s_1, \dots, s_{M-1}$$

are to be constructed, where each signal is to be a linear combination of K given signals

$$\phi_0, \phi_1, \dots, \phi_{K-1}.$$

Moreover, it is assumed that each ϕ_k is a time series of length N, and that $\{\phi_k\}$ is an orthonormal set under the Euclidean inner product. We denote the components of ϕ_k by $\phi_k(n)$, $n = 0, 1, \dots, N-1$ and the components of s_m similarly. The unknowns to be determined are the weights $\{\alpha_{mk}\}$ defining the signals:

$$s_m = \sum_{k=0}^{K-1} \alpha_{mk}\phi_k, \; m = 0, 1, \dots, M-1.$$

A signal set is often referred to as a *constellation.*

In the special case of only two basis functions, ϕ_0, ϕ_1, the signals can be represented in the plane by the coefficients $(\alpha_{00}, \alpha_{01}), \dots, (\alpha_{M-1,0}, \alpha_{M-1,1})$. In the case of Gaussian noise, it turns out that the problem reduces to maximizing the minimum Euclidean distance between any two signals (subject to constraints on the energy or amplitudes of the signals). For this reason, heuristic methods have been used to design good signals sets.

Typically, these heuristics have taken the form of choosing the points lying on a lattice and somehow densely packing them within a fixed region of the plane. Identifying these lattice-based constellations associated with low average energies has been an active area of research (examples and pictures of these constellations can be found in [11], [13], and [14]).

Although we are not aware of previous attempts to find optimal constellations according to the criterion we describe below, related problems have been investigated. The most famous is the sphere-packing problem of communication theory (see [1]); this requires a constellation which maximizes the probability of detection under Gaussian noise. Modern research into this question has focused on the case in which the signals are chosen from a large dimensional space (in particular, an important research topic has been the Strong Simplex Conjecture, which deals with the case $M = K + 1$; see [12]). We are concerned with the case in which the signals are chosen from a small dimensional space ($K = 2$ or 3).

In the remainder of this paper, we show that this problem can be formulated as a smooth nonlinear programming problem with relatively few variables but many inequality constraints. This problem is solved using an SQP algorithm.

In the following second section, we explain the formulation of the optimization problem that describes optimal signal sets. In Section 3, we describe the SQP algorithm employed to solve these problems. The noise distributions used in our computations are described in Section 4, and in Section 5, numerical tests and results are presented. We conclude the paper with observations and comments on future work in Section 6.

2. Problem formulation. As mentioned above, we wish to find a signal set which minimizes the largest probability of mistaking any one for any other. This notion of optimality is therefore based on *hypothesis testing* (see, for example, [17]). We consider for the moment that one of two signals, s_0 and s_1, is to be transmitted, and that the received signal is denoted by y. We assume further that the transmitted signal is corrupted by independent, identically distributed (i.i.d.) additive noise drawn from some fixed distribution with probability density function (pdf) p_N. In other words,

$$y = s_m + \eta,\ m = 0 \text{ or } 1.$$

We assume that the *a priori* probabilities of s_0 and s_1 being transmitted are P_0 and P_1, respectively, and that there is a cost C_m associated with detecting signal s_0 when s_1 is actually present (a *miss*), and a cost C_f associated with detecting s_1 when s_0 is present (a *false alarm*). It is then easy to show (see [17]) that the expected cost, or *risk*, is minimized by detecting s_0 whenever

$$\frac{p(y|s_0)}{p(y|s_1)} > \frac{P_1 C_m}{P_0 C_f},$$

and otherwise detecting s_1. By taking the logarithm of both sides and using the fact that the noise is i.i.d., we obtain the following *optimal detector*: Detect s_0 whenever

$$\frac{1}{N}\sum_{n=0}^{N-1} \log \frac{p_N(y(n) - s_0(n))}{p_N(y(n) - s_1(n))} > \gamma, \tag{2.1}$$

(where γ is the threshold determined by the *a priori* probabilities and the costs of errors), and otherwise detect s_1.

Now assume that s_0 was actually transmitted (so that $y = s_0 + \eta$) and let $\Delta s = s_0 - s_1$; the optimal detector then computes

$$\frac{1}{N}\sum_{n=0}^{N-1} \log \frac{p_N(\eta(n))}{p_N(\eta(n) + \Delta s(n))}.$$

The expected value of the nth term is

$$K_N(\Delta s(n)) = \int \log\left[\frac{p_N(\tau)}{p_N(\tau + \Delta s(n))}\right] p_N(\tau) d\tau;$$

this quantity is the Kullback-Leibler distance between the noise density and the noise density shifted by $\Delta s(n)$ (see [18]).

Thus, if s_0 is actually transmitted, the expected value of the sum in (2.1) is

$$\frac{1}{N}\sum_{n=0}^{N-1} K_N(\Delta s(n)). \tag{2.2}$$

If one assumes that p_N is symmetric, as we will, it is easy to show that if s_1 is transmitted, the expected value of the sum in (2.1) is the negative of (2.2). Therefore, the probability of detecting the correct signal increases with (2.2).

From this discussion, we see that we wish to choose the signals $s_0, s_1, \ldots, s_m$ so that

$$\min_{m_1 \neq m_2} \sum_{n=0}^{N-1} K_N(s_{m_1}(n) - s_{m_2}(n))$$

is maximized. For physical reasons, either the average power (energy—L^2 norm) or the peak power (amplitude—L^∞ norm) of the signals must be constrained. In this paper, we are concerned with amplitude constraints.

We are thus faced with a constrained minimax problem. Due to the difficulty of solving such problems, we rewrite it as a smooth NLP by introducing an auxiliary variable t:

$$\min_{t,\alpha}\ -t^2 \tag{2.3}$$

$$s.t.\ \sum_{n=0}^{N-1} K_N(s_{m_1}(n) - s_{m_2}(n)) \geq t^2,\ m_1 < m_2 \tag{2.4}$$

$$s_m(n)^2 \leq C^2,\ n = 0, \ldots, N-1,\ m = 0, \ldots, M-1 \tag{2.5}$$

$$t \geq 0, \tag{2.6}$$

where $C > 0$ is the bound on the amplitudes of the signals. This problem has $MK + 1$ variables and $M(M+1)/2 + MN + 1$ inequality constraints. A typical problem would have $K = 2, M = 16, N = 50$, giving 33 variables and 937 constraints.

Difficulties arise when one tries to solve the above NLP (2.3) with standard algorithms. The fact that there are more variables than constraints results in three specific difficulties:

- There are many "almost" binding constraints at the solution.
- The linearized constraints are often inconsistent.
- The boundary of the feasible region is noticeably nonlinear.

The proximity of near-binding constraints to the solution suggests that correct identification of the active set becomes more difficult near the solution. In turn this suggests that the region of rapid local convergence of the iteration sequence $\{x_k, \lambda_k\}$ will be "small". The lack of consistent linear inequalities complicates the calculation of the SQP step (the solution of the quadratic subproblem). As observed in previous works, ([15] and others), nonlinear feasible regions can result in small acceptable steps if penalty parameters become small. Iterates will follow too closely a feasible region boundary that leads away from optimality. All three of these issues were relevant to the solution of the optimal signal set problem presented here.

3. Nonlinearly Constrained Optimization. Many algorithms have been developed for the solution of smooth, inequality-constrained optimization problems. Among the most popular methods is the Sequential Quadratic Programming (SQP) family of algorithms (see [4] for a review of these methods). Given an estimate of the solution, an SQP algorithm progresses by solving a quadratic program (QP), which is defined by a local quadratic model of the objective function and linearized constraints. The solution to the quadratic program is then used to construct an improved estimate of the solution. Many different SQP algorithms can be constructed by varying the algorithm for solving the QP, the Lagrange multiplier estimates, and the globalization strategy.

One version of the SQP algorithm, proposed by Boggs, Tolle, and Kearsley (see [7], [6], [9] and [8]), appears to be well suited for solving this class of NLPs. The algorithm employs a combined trust-region and merit function (see [5]) line search procedure for globalization. A modern interior point method called O3D, **O**ptimal **3**-**D**imensional subspace method (see Boggs, Domich, Rogers, and Witzgall [3]), is used to solve the quadratic programming subproblems. It appears that 03D is quite compatible with the globalization procedure (e.g. the steps produced by the O3D algorithm decrease the merit function and do not impede convergence).

A nonstandard feature of this algorithm is useful because of the need to solve NLP's with very small and highly nonlinear feasible regions. A perturbation is added to the right hand side of the system of linearized constraints to guarantee that this linear system is always consistent. Similar constraint relaxations have appeared in the literature before (see for instance the papers by Biggs [2], Tone [22] and Powell [20] among others). The relaxation procedure we employ is similar to methods contained in papers mentioned above with minor modifications (and can be found in [19]). When far from feasibility, violated linearized constraints are relaxed enough to guarantee that they form a consistent system of inequalities. This relaxation is obtained by solving a linear programming problem; moreover, the solution to the linear programming problem can then be used as a feasible starting point for the QP. Because O3D is designed to solve either linear or quadratic programming problems, this two-step process can be carried out using one algorithm (and code). This procedure ensures that no "phase-I" or infeasible calculations are needed (i.e. no 'Big-M' method is needed) for the calculation of the SQP descent direction. Details of the constraint perturbation procedure can be found in [19].

To test the efficacy of the constraint relaxation procedure, we solved the problems described in this paper twice, first without the relaxation and then with it. The advantages of relaxing the constraints are shown by the numerical results presented in the following section.

4. Noise distributions. The primary purpose of this paper is to investigate non-Gaussian noise distributions. Following Johnson and Orsak (see [18]), we selected the five densities found in Table 1 (including the Gaussian density for comparison). These densities are graphed in Figure 1, while the associated Kullback-Leibler distances are found in Figure 2.

These densities are chosen to illustrate different possibilities. For example, the Kullback-Leibler distance associated with the Gaussian density is smaller than that of the Laplacian when Δs is small;

Name	Density	$K(\Delta s)$
Gaussian	$\frac{\exp(-\tau^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}$	$\frac{(\Delta s)^2}{2\sigma^2}$
Laplacian	$\frac{\exp(-\lvert\tau\rvert/(\sigma/\sqrt{2}))}{\sqrt{2}\sigma}$	$\frac{\lvert\Delta s\rvert}{\sigma/\sqrt{2}} + \exp(-\frac{\lvert\Delta s\rvert}{\sigma/\sqrt{2}}) - 1$
Hyperbolic Secant	$\frac{\mathrm{sech}(\pi\tau/2\sigma)}{2\sigma}$	$-2\ln(\mathrm{sech}(\pi\Delta s/4\sigma))$
Generalized Gaussian	$\frac{1}{2\Gamma(5/4)A}\exp(-\frac{\tau^4}{A^4})$	$\frac{\Gamma^2(3/4)}{\Gamma^2(1/4)}\left(6\frac{(\Delta s)^2}{\sigma^2} + \frac{(\Delta s)^2}{\sigma^4}\right)$
Cauchy	$\frac{1}{\pi\sigma(1+(\tau/\sigma)^2)}$	$\ln(1+(\Delta s)^2/r\sigma^2)$

TABLE 1
Noise densities and associated Kullback-Leibler distances.

for large Δs, this relationship is reversed. The hyperbolic secant density leads to a distance function that is similar to that of the Gaussian near the origin, but close to that of the Laplacian for large Δs. The Kullback-Leibler distance for the Generalized Gaussian density grows very rapidly with Δs, while that of the Cauchy density grows very slowly.

5. Numerical Tests and Results. In this section we summarize the performance of the SQP algorithms on a suite of problems, corresponding to various choices of the noise distribution, the basis signals, and the number of signals, M. For the purpose of these numerical examples, we fix the length of the signals at $N = 50$, the amplitude at $C = \sqrt{10}$ and the number of basis functions at $K = 2$. The bases used were

$$\left\{\sqrt{\frac{2}{N}}\sin(2\pi\omega_1 n/N), \sqrt{\frac{2}{N}}\sin(2\pi\omega_2 n/N)\right\},$$

and

$$\left\{\sqrt{\frac{2}{N}}\sin(2\pi\omega_1 n/N), \sqrt{\frac{2}{N}}\cos(2\pi\omega_1 n/N)\right\},$$

with $\omega_1 = 10$ and $\omega_2 = 11$.

We use analytic first and second derivatives, and the least-squares estimate of the Lagrange multipliers (see, for example, Gill, Murray, and Wright [16]).

Problems involving the sine-cosine basis are fundamentally more difficult than those involving the sine-sine basis; this is because of the rotational symmetries in the solution space, and also because of the geometry of certain signal sets. For example, when the signal set contains 8 signals, optimality requires that one of the signals lie in the center, with 7 signals on a circle around it (see Figure 5). The signal in the center is actually free to move in a small open set without affecting optimality. Part of the numerical difficulty can be alleviated with a small amount of regularization introduced to the objective function as follows:

$$\min_{t,\alpha} \; -\left(\frac{1}{2}t^2 - \frac{\epsilon}{2}\|s\|_2^2\right) \tag{5.7}$$

$$s.t. \; \sum_{n=0}^{N-1} K_N(s_{m_1}(n) - s_{m_2}(n)) \geq t^2, \; m_1 < m_2 \tag{5.8}$$

$$s_m(n)^2 \leq C^2, \; n = 0, \ldots, N-1, \; m = 0, \ldots, M-1 \tag{5.9}$$

$$t \geq 0. \tag{5.10}$$

A value $\epsilon \approx 10^{-6}$ worked well. The problems where this version was employed are denoted by an asterisk.

In Tables 2 and 3 we report the performance of the algorithm described in [6] without the constraint relaxation. Likewise, Tables 4 and 5 give the performance of the algorithm with constraint relaxation. The first column in the table contains the values for the number of signals M, the number of time

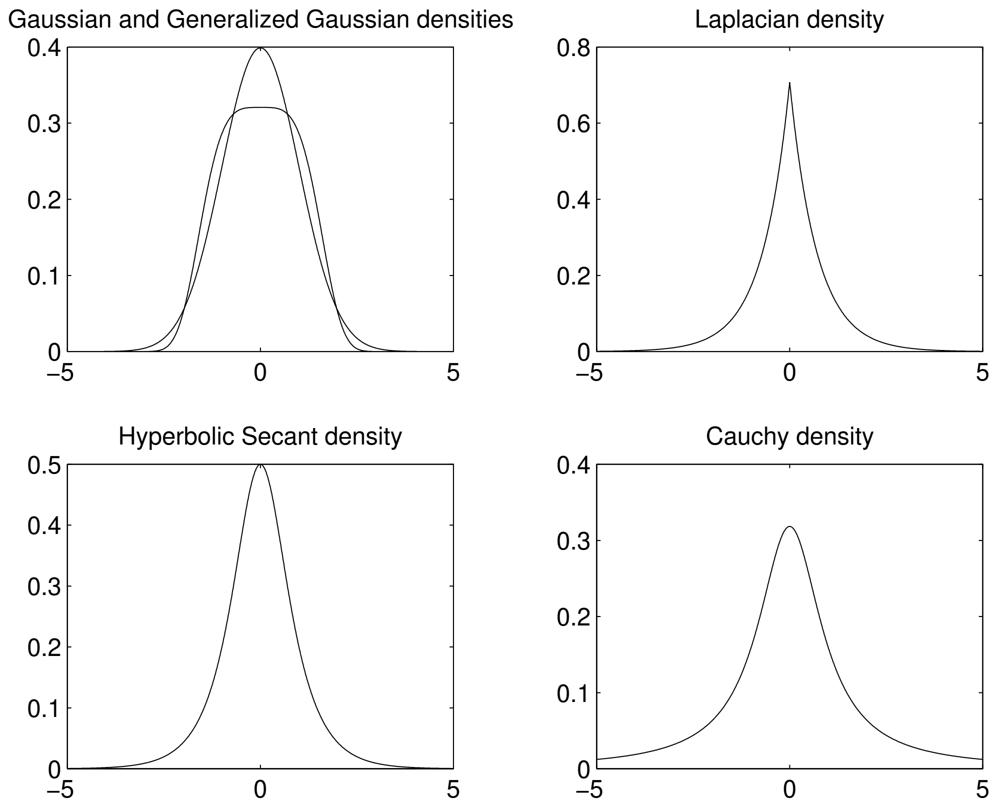

FIG. 1. *Noise densities studied.*

samples N, and the number of basis functions K. The noise distribution is given in the second column, while the number of variables (n) and constraints (m) in the resulting nonlinear program is found in the third column. The number of nonlinear or outer iterations required to find the solution is recorded in the fourth column, while the fifth column gives the number of QP iterations required (with the number of phase I iterations in parentheses). In the sixth column we give the value of the putative global minimum, and in the seventh column the number of times this minimum was found in ten tries, each starting from a randomly generated starting point. In the event that the minimizer was not found in ten tries, we record in parentheses the number of tries it took to find it. Finally, the last column contains the size of the gradient of the Lagrangian at the computed solution. The algorithm halted when either the 2-norm of the gradient of the Lagrangian became less than or equal to 10^{-6} or the 2-norm of the solution to the quadratic subproblem (the SQP step) fell below 10^{-12}.

The optimal constellations for a subset of our test problems are shown in Figures 3, 4, and 5 (with $M = 8$) and 6, 7, and 8 (with $M = 16$). It is interesting to observe that the symmetric nature of constellations conjectured to be present in optimal solutions (see [18]) is apparent in the current estimates of the solutions.

These problems have features that make them difficult to solve numerically. Not only is the number of variables much smaller than the number of constraints, but there are many constraints that are nearly binding at the solution. Many algorithms, including SQP, give rise to rapid local convergence when iterates enter a neighborhood of the solution and the correct collection of active sets has been identified (see the paper by Robinson [21] for a discussion). This rapid local convergence is especially apparent in the event that one can provide an accurate approximation to the true Hessian matrix at every iteration, as is the case with our problem. Even though the notion of active sets is less important to our algorithm because our interior point method quadratic program solver, 03D, does not compute

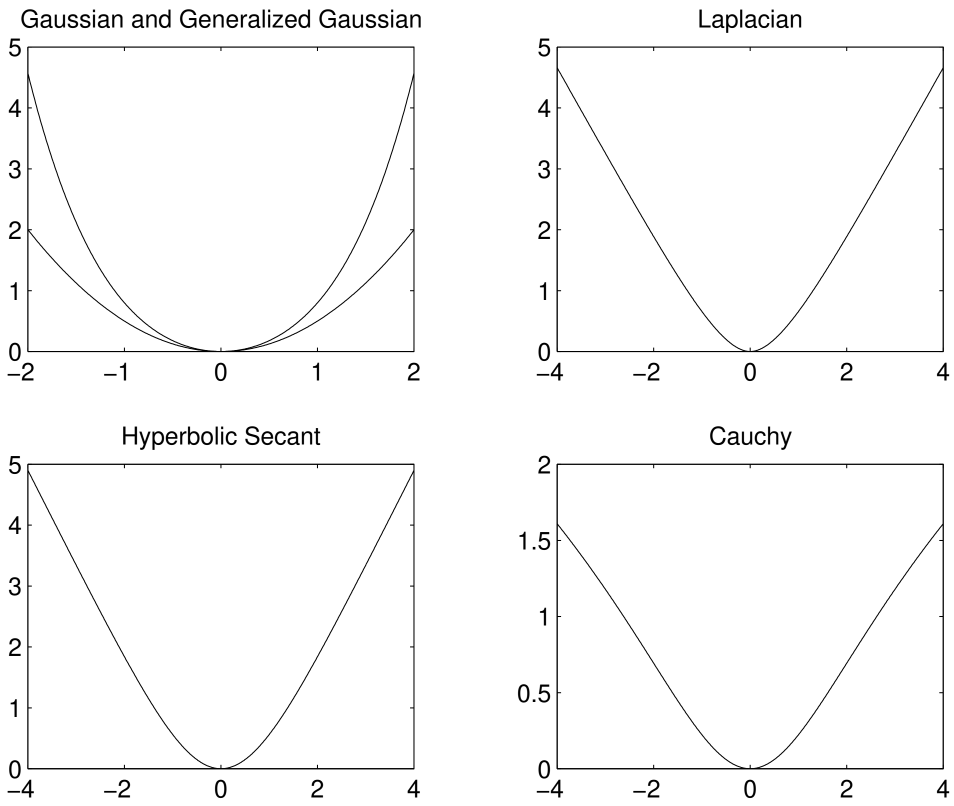

FIG. 2. *Kullback-Leibler distances associated with the various noise densities.*

active sets, the fact that many of the constraints are nearly binding at the solution has an effect on the size of the neighborhood around the solution where fast local convergence is realized.

In these numerical tests the constraint relaxation substantially improved the performance of the algorithm. This is probably due to the fact that the number of inconsistent subproblems encountered was unusually high. It is worth commenting that the additional cost of employing the relaxation procedure is not large.

6. Conclusions. In this paper we have presented an interesting collection of difficult optimization problems and an NLP formulation of them. This formulation allows a broad arsenal of numerical optimization algorithms and modern enhancements to be employed. While these problems are not "large-scale" by modern computing standards, they are, nonetheless, difficult problems to solve efficiently.

Numerical solutions to these problems were located using an SQP method with and without the constraint relaxation procedure described in [19]. Numerous numerical tests (and the summary of these tests that appear in Tables 2, 3, 4, and 5) suggest that this constraint relaxation procedure can significantly improve the performance of this SQP method in the event that linearizations are inconsistent, which may be the case when there are far more constraints than variables.

Because there are so many different algorithms and implementations for the solution of the nonlinear programming problem, there is a need to create an accepted collection of test problems (see the paper by Bongartz, Conn, Gould and Toint [10]). Because of the difficulties it poses, this family of problems is a natural candidate for such a collection.

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(M,N,K)	density	(n,m)	outer	inner	min	#/10	$\|\|\nabla_x L\|\|$
(8,50,2)	Gaussian	(17,429)	16	201 (34)	-69.7937	0 (214)	2e-12
(8,50,2)	Generalized-Gaussian	(17,429)	16	179 (36)	-189.0968	4	7e-10
(8,50,2)	Hyperbolic-Secant	(17,429)	65	730 (94)	-61.0936	5	5e-14
(8,50,2)	Laplacian	(17,429)	10	121 (28)	-63.1230	0 (30)	5e-14
(8,50,2)	Cauchy	(17,429)	36	499 (76)	-22.7308	0 (24)	1e-9
(16,50,2)	Gaussian	(33,937)	27	388 (55)	-29.3142	4	5e-12
(16,50,2)	Generalized-Gaussian	(33,937)	29	478 (80)	-57.8296	3	7e-11
(16,50,2)	Hyperbolic-Secant	(33,937)	10	119 (22)	-29.5770	3	1e-1
(16,50,2)	Laplacian	(33,937)	42	646 (121)	-32.3708	0 (12)	5e-5
(16,50,2)	Cauchy	(33,937)	33	501 (94)	-11.4304	2	6e-2

TABLE 2

Performance of the unperturbed SQP algorithm on a collection of constellation problems (sine-sine basis)

(M,N,K)	density	(n,m)	outer	inner	min	#/10	$\|\|\nabla_x L\|\|$
*(8,50,2)	Gaussian	(17,429)	48	631(22)	-97.5518	0(21)	5.e-11
*(8,50,2)	Generalized-Gaussian	(17,429)	52	679(31)	-264.0240	0(12)	6.e-12
*(8,50,2)	Hyperbolic-Secant	(17,429)	89	1199(30)	-83.1955	0(14)	9.e-4
*(8,50,2)	Laplacian	(17,429)	49	645 (21)	-84.4632	1	1 8.e-11
*(8,50,2)	Cauchy	(17,429)	67	888(19)	-22.7308	1	7.e-10
(16,50,2)	Gaussian	(33,937)	26	343 (44)	-39.7452	1	7e-12
(16,50,2)	Generalized-Gaussian	(33,937)	100	1170 (110)	-76.1390	1	1e-4
(16,50,2)	Hyperbolic-Secant	(33,937)	10	119 (22)	-29.5770	3	1e-1
(16,50,2)	Laplacian	(33,937)	51	611 (121)	-32.3708	0 (12)	5e-5
(16,50,2)	Cauchy	(33,937)	41	417 (31)	-15.6892	1	2e-6

TABLE 3

Performance of the unperturbed SQP algorithm on a collection of constellation problems (sine-cosine basis)

Paul Boggs and Janet Rogers of the National Institute of Standards and Technology, Dr. L. C. Cowsar from Lucent Technologies and Professor Jon Tolle from University of North Carolina for their advice and strong encouragement. Finally, the authors greatly appreciated careful reading of an anonymous referee who suggested numerous beneficial changes to this paper.

(M,N,K)	density	(n,m)	outer	inner	min	#/10	$\|\nabla_x L\|$
(8,50,2)	Gaussian	(17,429)	14	209	-69.7937	1	7e-12
(8,50,2)	Generalized-Gaussian	(17,429)	14	198	-189.0968	1	4e-9
(8,50,2)	Hyperbolic-Secant	(17,429)	59	801	-61.0936	1	1e-9
(8,50,2)	Laplacian	(17,429)	10	133	-63.1230	2	3e-12
(8,50,2)	Cauchy	(17,429)	31	440	-22.7308	2	8e-8
(16,50,2)	Gaussian	(33,937)	25	292	-29.3142	5	4e-11
(16,50,2)	Generalized-Gaussian	(33,937)	27	401	-57.8296	3	5.e-11
(16,50,2)	Hyperbolic-Secant	(33,937)	10	122	-29.5770	3	2.e-13
(16,50,2)	Laplacian	(33,937)	40	537	-32.3708	1	1.e-9
(16,50,2)	Cauchy	(33,937)	30	419	-11.4304	2	1.e-9

TABLE 4

Performance of the constraint perturbed SQP algorithm on a collection of constellation problems (sine-sine basis)

(M,N,K)	density	(n,m)	outer	inner	min	#/10	$\|\nabla_x L\|$
*(8,50,2)	Gaussian	(17,429)	40	123	-69.7937	4	7e-8
*(8,50,2)	Generalized-Gaussian	(17,429)	39	121	-189.0968	4	3.e-8
*(8,50,2)	Hyperbolic-Secant	(17,429)	70	1612	-61.0936	8	2.e-10
*(8,50,2)	Laplacian	(17,429)	46	479	-63.1230	1	4.e-7
*(8,50,2)	Cauchy	(17,429)	58	1333	-22.7308	1	3.e-7
(16,50,2)	Gaussian	(33,937)	25	325	-39.7452	2	6.e-10
(16,50,2)	Generalized-Gaussian	(33,937)	91	1066	-76.1390	4	4.e-8
(16,50,2)	Hyperbolic-Secant	(33,937)	10	122	-29.5770	3	5e-5
(16,50,2)	Laplacian	(33,937)	41	416	-32.3708	1	1.e-6
(16,50,2)	Cauchy	(33,937)	29	410	-15.6892	5	1.e-9

TABLE 5

Performance of the constraint perturbed SQP algorithm on a collection of constellation problems (sine-cosine basis)

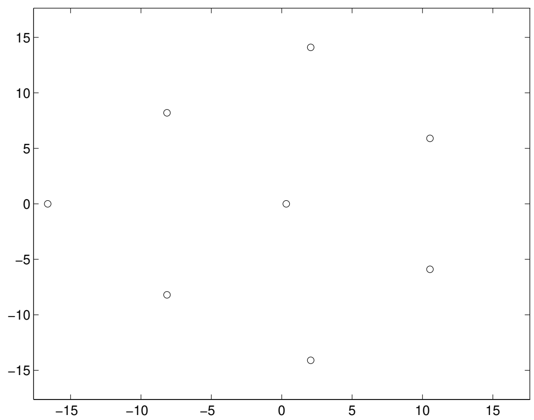

FIG. 3. *Optimal constellation for $M = 8$ with Sine-Sine basis functions using a Gaussian Density*

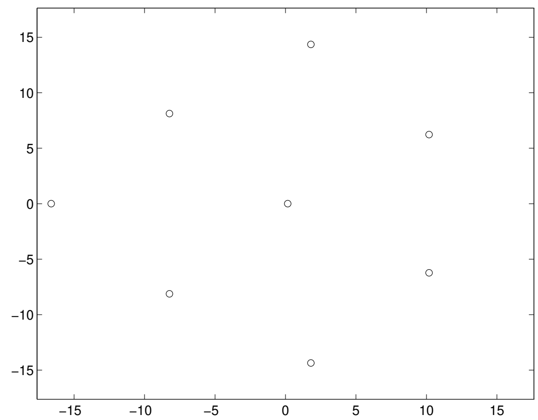

FIG. 4. *Optimal constellation for $M = 8$ with Sine-Sine basis functions using a Generalized Gaussian Density*

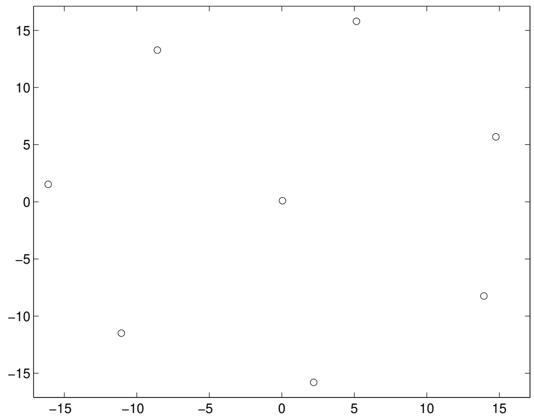

FIG. 5. *Optimal constellation for $M = 8$ with Sine-Cosine basis functions using a Hyperbolic-Secant Density*

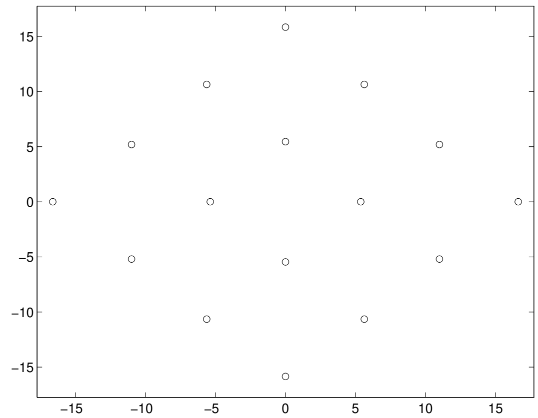

FIG. 6. *Optimal constellation for $M = 16$ with Sine-Sine basis functions using a Cauchy Density*

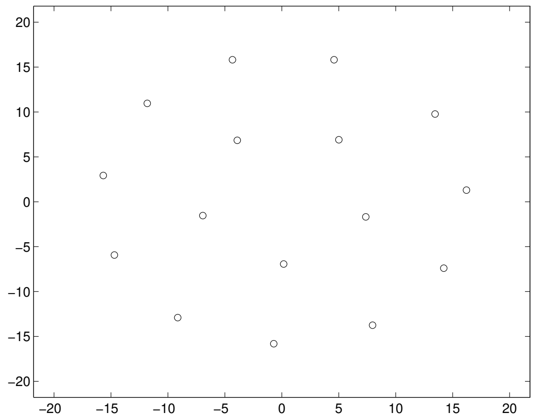

FIG. 7. *Optimal constellation for $M = 16$ with Sine-Cosine basis functions using a Cauchy Density*

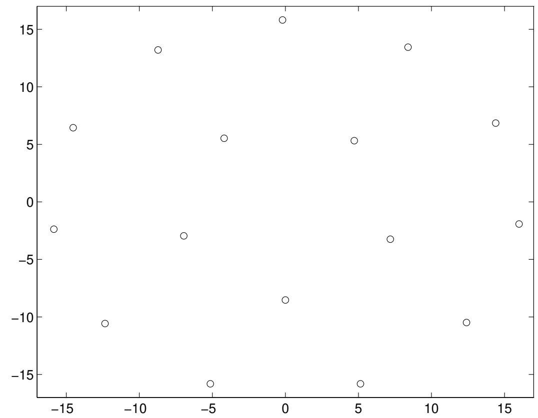

FIG. 8. *Optimal constellation for $M = 16$ with Sine-Cosine basis functions using a Generalized Gaussian Density*

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