SUFFICIENCY CLASS FOR GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS

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Abstract. Let Ω be an open domain of class $\mathbb{C}^3$ contained in $\mathbb{R}^3$, let $(\mathbb{L}^2[\Omega])^3$ be the real Hilbert space of square integrable functions on Ω with values in $\mathbb{R}^3$, and let $\mathbb{H}[\Omega]$ be the completion of the set, $\{\mathbf{u} \in (\mathbb{C}_0^\infty[\Omega])^3 \mid \nabla \cdot \mathbf{u} = 0\}$, with respect to the inner product of $(\mathbb{L}^2[\Omega])^3$. A well-known unsolved problem is the construction of a sufficient class of functions in $\mathbb{H}[\Omega]$ which will allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In this paper, we prove that, under appropriate conditions, there exists a number $\mathbf{u}_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the Stokes operator, such that, for all functions in a dense set $\mathbb{D}$ contained in the closed ball $\mathbb{B}(\Omega)$ of radius $\mathbf{u}_+$ in $\mathbb{H}[\Omega]$, the Navier-Stokes equations have unique, strong, solutions in $\mathbb{C}^1\left((0,\infty), \mathbb{H}[\Omega]\right)$.

Introduction

Let Ω be an open domain of class $\mathbb{C}^k$ contained in $\mathbb{R}^n$, $n \geq 2$, let $(\mathbb{L}^2[\Omega])^n$ be the real Hilbert space of square integrable functions on Ω with values in $\mathbb{R}^n$, let $\mathbf{D}[\Omega]$ be $\{\mathbf{u} \in (\mathbb{C}_0^\infty[\Omega])^n \mid \nabla \cdot \mathbf{u} = 0\}$, let $\mathbb{H}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $(\mathbb{L}^2[\Omega])^n$, and let $\mathbb{V}[\Omega]$ be the completion of

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$\mathbf{D}[\Omega]$ with respect to the inner product of $\mathbb{H}^1[\Omega]$, the functions in $\mathbb{H}[\Omega]$ with weak derivatives in $(\mathbb{L}^2[\Omega])^n$. The global in time classical Navier-Stokes initial-value problem (for $\Omega \subset \mathbb{R}^n$, and all $T > 0$) is to find a functions $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^n$, and $p : [0,T] \times \Omega \to \mathbb{R}$, such that

$$\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T) \times \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \text{ in } (0,T) \times \Omega, \\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T) \times \partial\Omega, \\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.
\end{aligned} \tag{1}$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure p of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient ν in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x}, t)$.

The existence of global weak solutions of (1) was proved by Leray [Le] in 1934, for $\Omega = \mathbb{R}^3$ and later in 1951, Hopf [Ho] solved the problem for a bounded open domain $\Omega \subset \mathbb{R}^n, n \geq 2$, with homogeneous Dirichlet conditions on smooth boundaries, $\partial\Omega$. These results were subsequently extended to include functions $\mathbf{f}(\mathbf{x}, t) \in \mathbb{L}^2[(0,T)\,; \mathbb{V}[\Omega]^{-1}]$, where $\mathbb{V}[\Omega]^{-1} = \mathbb{V}[\Omega]^*$ is the dual of $\mathbb{V}[\Omega]$ (see [Li, T1, vW]). In 1962, Kato and Fujita [KF] proved the existence of strong, global in time, smooth three-dimensional solutions, provided that the body forces are small (in an appropriate sense) and the initial data is small in the Sobolev space $H^{1/2}[\Omega]$ (see also [CH] and Temam [T1, pages 205-208]). (As noted by Temam [T1, see page 344-345], the importance of their work is that it points out the dependence of existence of global solutions on the size of the initial data, the body forces and possibly the spectral properties of these quantities.) In another (related) direction, Raugel and Sell showed that one can get stronger results for thin 3D domains (see [RS]). In this case, they show that the Navier-Stokes equations have strong solutions and that the long-time dynamics has a global attractor.

Purpose

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $(\mathbb{L}^2[\Omega])^n$ onto $\mathbb{H}[\Omega]$ and define the Stokes operator by: $\mathbf{Au} =: -\mathbb{P}\Delta\mathbf{u}$, for $\mathbf{u} \in D(\mathbf{A}) \subset \mathbb{H}^2[\Omega]$, the domain of $\mathbf{A}$. The purpose of this paper is to prove that there exists a number $\mathbf{u}_+$, depending only on $\mathbf{A}$, f, ν and Ω, such that, for all functions in $\mathbb{D} = D(\mathbf{A}) \cap \mathbb{B}(\Omega)$, where $D(\mathbf{A})$ is the domain of $\mathbf{A}$ and $\mathbb{B}(\Omega)$ is the closed ball of radius $\mathbf{u}_+$, in $\mathbb{H}(\Omega)$, the Navier-Stokes equations have unique, strong, solutions in $\mathbf{u} \in L^\infty_{\text{loc}}[[0,\infty); \mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty); \mathbb{H}(\Omega)]$.

Preliminaries

It will be convenient to use the fact that the norms of $\mathbb{V}[\Omega]$ and $\mathbb{V}[\Omega]^{-1}$ are equivalent in their respective graph norms relative to $\mathbb{H}[\Omega]$. It is known that $\mathbf{A}$ is a positive linear operator with compact resolvent. It follows that the fractional powers $\mathbf{A}^{1/2}$ and $\mathbf{A}^{-1/2}$ are well defined. Moreover, it is also known (cf. [SY], [T1]) that the norms $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}$ and $\left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}$ are equivalent to the corresponding norms induced by the Sobolev space $(H^1[\Omega])^n$, so that:

$$\|\mathbf{u}\|_{\mathbb{V}[\Omega]} \equiv \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]} \text{ and } \|\mathbf{u}\|_{\mathbb{V}[\Omega]^{-1}} \equiv \left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}. \tag{2}$$

In addition, it is known that $\mathbf{A}$ is an isomorphism from $D(\mathbf{A}) \xrightarrow{onto} \mathbb{H}[\Omega]$, and from $\mathbb{V}[\Omega] \xrightarrow{onto} \mathbb{V}[\Omega]^{-1}$. Furthermore, the embeddings $\mathbb{V}[\Omega] \to \mathbb{H}[\Omega] \to \mathbb{V}[\Omega]^{-1}$ are compact and the operator $\mathbf{A}^{-1}$ is a bounded compact map from $\mathbb{H}[\Omega]$ onto $D(\mathbf{A})$.

Applying the Leray projection to equation (1), with $\mathbf{B}(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$, we can recast equation (1) in the standard form:

$$\begin{aligned} &\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T)\times\Omega, \\ &\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \end{aligned} \tag{3}$$

where we have used the fact that the orthogonal complement of $\mathbb{H}[\Omega]$ relative to $(\mathbb{L}^2[\Omega])^3$ is $\{\mathbf{v} : \mathbf{v} = \nabla q,\ q \in (H^1[\Omega])^3\}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1,T2]). Theorem 1 below will be used to get our basic estimate in Lemma 2 (see Constantin and Foias [CF]).

Theorem 1. *Let Ω be a bounded open set of class $\mathbb{C}^k$ in $\mathbb{R}^3$. Let $\alpha_i, 1 \le i \le 3$ satisfy $0 \le \alpha_1 \le k$, $0 \le \alpha_2 \le k-1$, $0 \le \alpha_3 \le k$, with $\alpha_1+\alpha_2+\alpha_3 \ge 3/2$ and*

$$(\alpha_1,\alpha_2,\alpha_3) \notin \{(3/2,0,0),(0,3/2,0),(0,0,3/2)\}.$$

Then there is a positive constant $c = c(\alpha_i,\Omega)$ such that

$$\left|\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H}}\right| \leqslant c \left\|\mathbf{A}^{\alpha_1/2}\mathbf{u}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{(1+\alpha_2)/2}\mathbf{v}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{\alpha_3/2}\mathbf{w}\right\|_{\mathbb{H}}.$$

We shall have use of the following interpolation inequality: (See Sell and You [SY], page 363.)

$$\left\|\mathbf{A}^{\gamma}\mathbf{u}\right\|_{\mathbb{H}} \leqslant C \left\|\mathbf{A}^{\alpha}\mathbf{u}\right\|_{\mathbb{H}}^{\theta} \left\|\mathbf{A}^{\beta}\mathbf{u}\right\|_{\mathbb{H}}^{(1-\theta)}$$

for all $\mathbf{u} \in D(\mathbf{A}^{\alpha})$, where $\gamma = \theta\alpha + (1-\theta)\beta$, $\alpha,\beta,\gamma \in \mathbb{R}$, $0 \le \theta \le 1$ and $\beta \leqslant \alpha$.

The following estimate is equation 61.24.1 on page 366, in Sell and You [SY]. If we set $\alpha_1 = 1, \alpha_2 = 1/2$, and $\alpha_3 = 0$ in Theorem 1, along with the interpolation inequality, we get that

$$|\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H}}| \leqslant c \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}} \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}} \|\mathbf{w}\|_{\mathbb{H}}. \tag{4}$$

Lemma 2. *Let $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{H}$, and let $\varepsilon > 0$ be arbitrary. Then for $\delta = 1/4 + \varepsilon/2$, we have that:*

$$\left|\left\langle \mathbf{A}^{-(1+\delta)}\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle_{\mathbb{H}}\right| \leqslant c \|\mathbf{u}\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}} \|\mathbf{w}\|_{\mathbb{H}} \tag{5}$$

Proof. Using the self-adjoint property of $\mathbf{A}$, and integration by parts, we have

$$\left\langle \mathbf{A}^{-\beta}\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle_{\mathbb{H}} = \left\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{A}^{-\beta}\mathbf{w}\right\rangle_{\mathbb{H}} = -\left\langle \mathbf{B}(\mathbf{u},\mathbf{A}^{-\beta}\mathbf{w}),\mathbf{v}\right\rangle_{\mathbb{H}}.$$

It now follows from Theorem 1 that:

$$\left|\left\langle \mathbf{A}^{-\beta}\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle_{\mathbb{H}}\right| \leqslant c \left\|\mathbf{A}^{\alpha_1/2}\mathbf{u}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{-\beta+(1+\alpha_2)/2}\mathbf{w}\right\|_{\mathbb{H}} \left\|\mathbf{A}^{\alpha_3/2}\mathbf{v}\right\|_{\mathbb{H}}.$$

If we set $\beta = 1+\delta,\ \alpha_1 = \alpha_3 = 0,$, we have

$$\left|\left\langle \mathbf{A}^{-(1+\delta)}\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\right\rangle_{\mathbb{H}}\right| \leqslant c \|\mathbf{u}\|_{\mathbb{H}} \|\mathbf{v}\|_{\mathbb{H}} \left\|\mathbf{A}^{(\alpha_2-1-2\delta)/2}\mathbf{w}\right\|_{\mathbb{H}}.$$

With $\delta = 1/4 + \varepsilon/2$, we get that, for the last term to reduce to $\|\mathbf{w}\|_{\mathbb{H}}$, we can set $\alpha_2 = 3/2 + \varepsilon$. It follows that the conditions of Theorem 1 are satisfied if $3/2 + \varepsilon < k - 1$. Thus, it suffices to take k=3, assuming $\varepsilon < 1/2$ (which we will do in the rest of the paper without comment). □

Example 3. *If we use Theorem 1, with $\alpha_1 = 5/4$, $\alpha_2 = 1/4$, and $\alpha_3 = 0$, along with the interpolation inequality, and the fact that $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}$ we have that, for all $\mathbf{u},\mathbf{v} \in D(\mathbf{A})$,*

$$\begin{aligned} \|\mathbf{B}(\mathbf{u},\mathbf{v})\|_{\mathbb{H}} &\leqslant c_1 \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^{3/4} \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}^{1/4} \left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}}^{3/4} \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}^{1/4} \\ &\leqslant c_1 \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}. \end{aligned} \tag{6}$$

A better estimate is possible, but for our use, equation (6) will suffice.

Definition 4. *We say that the operator* $\mathbf{J}(\cdot,t)$ *is (for each* t*)*

(1) *0-Dissipative if* $\langle \mathbf{J}(\mathbf{u},t),\mathbf{u}\rangle_{\mathbb{H}[\Omega]} \le 0$.

(2) *Dissipative if* $\langle \mathbf{J}(\mathbf{u},t)-\mathbf{J}(\mathbf{v},t),\mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}[\Omega]} \le 0$.

(3) *Strongly dissipative if there exists an* $\alpha > 0$ *such that*

$$\langle \mathbf{J}(\mathbf{u},t)-\mathbf{J}(\mathbf{v},t),\mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}[\Omega]} \le -\alpha \left\|\mathbf{u}-\mathbf{v}\right\|_{\mathbb{H}[\Omega]}^2 .$$

(4) *Uniformly dissipative if there exists a strictly monotone increasing function* $a(t)$ *with* $a(0)=0$, $\lim_{t\to\infty} a(t)=\infty$, *and:*

$$\langle \mathbf{J}(\mathbf{u},t)-\mathbf{J}(\mathbf{v},t),\mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}[\Omega]} \le -a\left(\left\|\mathbf{u}-\mathbf{v}\right\|_{\mathbb{H}[\Omega]}\right)\left\|\mathbf{u}-\mathbf{v}\right\|_{\mathbb{H}[\Omega]} .$$

Note that, if $\mathbf{J}(\cdot,t)$ is a linear operator, definitions 1) and 2) coincide. Theorem 5 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35, page 887 in, Vol. IIB], while Theorem 6 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]) .

Theorem 5. *Let* $\mathbb{B}[\Omega]$ *be a closed, bounded, convex subset of* $\mathbb{H}[\Omega]$*. If* $\mathbf{J}(\cdot,t) : \mathbb{B}[\Omega] \to \mathbb{H}[\Omega]$ *is closed and strongly dissipative for each fixed* $t \ge 0$*, then for each* $\mathbf{b} \in \mathbb{B}[\Omega]$*, there is a* $\mathbf{u} \in \mathbb{B}[\Omega]$ *with* $\mathbf{J}(\mathbf{u},t)=\mathbf{b}$ *(e.g., the range, Ran[*$\mathbf{J}(\cdot,t)] \supset \mathbb{B}[\Omega]$*).*

Theorem 6. *Let* $\{\ \mathcal{A}(t), t \in I=[0,\infty)\}$ *be a family of operators defined on* $\mathbb{H}[\Omega]$ *with domains* $D(\mathcal{A}(t))=D$*, independent of* t*. We assume that* $\mathbb{D}=D\cap\mathbb{B}[\Omega]$ *is a closed convex set (in an appropriate topology):*

(1) *The operator* $\mathcal{A}(t)$ *is the generator of a contraction semigroup for each* $t \in I$*.*

(2) *The function* $\mathcal{A}(t)\mathbf{u}$ *is continuous in both variables on* $I \times \mathbb{D}$.

Then, for every $\mathbf{u}_0 \in \mathbb{D}$*, the problem* $\partial_t \mathbf{u}(t,\mathbf{x}) = \mathcal{A}(t)\mathbf{u}(t,\mathbf{x})$*,* $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$*, has a unique solution* $\mathbf{u}(t,\mathbf{x}) \in \mathbb{C}^1(I;\mathbb{D})$.

M-Dissipative Conditions

In the remainder of the paper, $n = 3$, and assume that $\mathbf{f}(t) \in L^\infty[[0,\infty);\mathbb{H}(\Omega)]$ and is Lipschitz continuous in t, with $\|\mathbf{f}(t) - \mathbf{f}(\tau)\|_{\mathbb{H}(\Omega)} \le d\,|t-\tau|^\theta,\ d > 0,\ 0 < \theta < 1$. With δ as in Lemma 2, we can rewrite equation (3) in the form:

$$\begin{aligned} &\partial_t \mathbf{u} = \nu \mathbf{A}^{1+\delta}\mathbf{J}(\mathbf{u},t) \text{ in } (0,T)\times\Omega, \\ &\mathbf{J}(\mathbf{u},t) = -\mathbf{A}^{-\delta}\mathbf{u} - \nu^{-1}\mathbf{A}^{-(1+\delta)}\mathbf{B}(\mathbf{u},\mathbf{u}) + \nu^{-1}\mathbf{A}^{-(1+\delta)}\mathbb{P}\mathbf{f}(t). \end{aligned} \tag{7}$$

Approach

We begin with a study of the operator $\mathbf{J}(\cdot,t)$, for fixed t, and seek conditions depending on $\mathbf{A}$, ν, Ω and $\mathbf{f}(t)$ which guarantee that $\mathbf{J}(\cdot,t)$ is m-dissipative for each t. Clearly $\mathbf{J}(\cdot,t) : D(\mathbf{A}^{(1+\delta)}) \xrightarrow{onto} D(\mathbf{A}^{(1+\delta)})$ and, since $\nu\mathbf{A}^{(1+\delta)} = \nu\mathbb{P}[-\Delta]^{(1+\delta)}$ is a closed positive (m-accretive) operator, so that $-\mathbf{A}^{(1+\delta)}$ generates a linear contraction semigroup, we expect that $\nu\mathbf{A}^{(\mathbf{1}+\delta)}\mathbf{J}(\cdot,t)$ will be m-dissipative for each t.

Theorem 7. *For* $t \in I = [0,\infty)$*, and for each fixed* $\mathbf{u} \in \mathbb{H}[\Omega]$*,* $\mathbf{J}(\mathbf{u},t)$ *is Lipschitz continuous, with* $\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{H}[\Omega]} \le d'\,|t-\tau|^\theta$*, where* $d' = d\nu^{-1}(\lambda_1)^{-(1+\delta)}$*,* d *is the Lipschitz constant for the function* $\mathbf{f}(t)$ *and* λ_1 *is the first eigenvalue of* $\mathbf{A}$.

Proof. For fixed $\mathbf{u} \in \mathbb{H}[\Omega]$,

$$\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{H}[\Omega]} = \nu^{-1}\left\|\mathbf{A}^{-(1+\delta)}[\mathbb{P}\mathbf{f}(t) - \mathbb{P}\mathbf{f}(\tau)]\right\|_{\mathbb{H}[\Omega]}$$

$$\le d\nu^{-1}(\lambda_1)^{-(1+\delta)}\,|t-\tau|^\theta = d'\,|t-\tau|^\theta .$$

We have used the fact that $\mathbf{A}$ is unbounded, and every function $\mathbf{h}(t) \in \mathbb{H}(\Omega)$ has an expansion in terms of the eigenfunctions of $\mathbf{A}$, so that $\mathbf{A}^{-(1+\delta)}\ \mathbf{h}(t) = \sum_{k=1}^{\infty} \lambda_k^{-(1+\delta)} h_k(t)\mathbf{e}^k(\mathbf{x})$, and, from here, it is easy to see that $\left\|\mathbf{A}^{-(1+\delta)}\ \mathbf{h}(t)\right\|_{\mathbb{H}(\Omega)} \leq \lambda_1^{-(1+\delta)} \|\ \mathbf{h}(t)\|_{\mathbb{H}(\Omega)}$. (It is well known that the eigenvalues of $\mathbf{A}$ are positive and increasing (see Temam [T2]).) □

Main Results

Theorem 8. *Let $f = \sup_{t\in\mathbf{R}^+} \|\mathbb{P}\mathbf{f}(t)\|_{\mathbb{H}[\Omega]} < \infty$, then there exists a positive constant $\mathbf{u}_+$, depending only on f, $\mathbf{A}$, ν and Ω, such that for all $\mathbf{u}$, with $\|\mathbf{u}\|_{\mathbb{H}[\Omega]} \leq \mathbf{u}_+$, $\mathbf{J}(\cdot, t)$ is strongly dissipative.*

Proof. The proof of our first assertion has two parts. First, we require that the nonlinear operator $\mathbf{J}(\cdot, t)$ be 0-dissipative, which gives us an upper bound $\mathbf{u}_+$, in terms of the norm (e.g., $\|\mathbf{u}\|_{\mathbb{H}[\Omega]} \leqslant \mathbf{u}_+$). We then use this part, and the fact that $\|\mathbf{u}\|_{\mathbb{H}[\Omega]} \leqslant \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}[\Omega]}$, to show that $\mathbf{J}(\cdot, t)$ is strongly dissipative on the closed ball, $\mathbb{B}_+(\Omega) = \left\{\mathbf{u} \in \mathbb{H}(\Omega) : \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}[\Omega]} \leqslant \mathbf{u}_+\right\}$.

Part 1) From equation (5), we get that

$$
\begin{aligned}
\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{H}[\Omega]} &= -\left\langle \mathbf{A}^{-\delta}\mathbf{u}, \mathbf{u}\right\rangle_{\mathbb{H}} + \nu^{-1}\left\langle -\mathbf{A}^{-(1+\delta)}\mathbf{B}(\mathbf{u}, \mathbf{u}) + \mathbf{A}^{-(1+\delta)}\mathbb{P}\mathbf{f}(t), \mathbf{u}\right\rangle_{\mathbb{H}} \\
&= -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}^2 - \nu^{-1}\left\langle \mathbf{A}^{-(1+\delta)}\mathbf{B}(\mathbf{u}, \mathbf{u}), \mathbf{u}\right\rangle_{\mathbb{H}} + \nu^{-1}\left\langle \mathbf{A}^{-(1+\delta)}\mathbb{P}\mathbf{f}(t), \mathbf{u}\right\rangle_{\mathbb{H}} \quad . \\
&= -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}^2 + \nu^{-1}\left\langle \mathbf{B}(\mathbf{u}, \mathbf{A}^{-(1+\delta)}\mathbf{u}), \mathbf{u}\right\rangle_{\mathbb{H}} + \nu^{-1}\left\langle \mathbb{P}\mathbf{f}(t), \mathbf{A}^{-(1+\delta)}\mathbf{u}\right\rangle_{\mathbb{H}}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{H}[\Omega]} &\leqslant -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}^2 + \nu^{-1}\left|\left\langle \mathbf{B}(\mathbf{u}, \mathbf{A}^{-(1+\delta)}\mathbf{u}), \mathbf{u}\right\rangle_{\mathbb{H}}\right| + \nu^{-1} f\left\|\mathbf{A}^{-(1+\delta)}\mathbf{u}\right\|_{\mathbb{H}} \\
&\leqslant -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}^2 + \nu^{-1} c\,\|\mathbf{u}\|_{\mathbb{H}}^3 + \nu^{-1} f\left\|\mathbf{A}^{-(1+\delta)}\mathbf{u}\right\|_{\mathbb{H}}.
\end{aligned}
$$

In the last line, we used our estimate from Lemma 2. Once again, we use the expansion in eigenfunctions $\{\varphi_i\}$, of $\mathbf{A}$, so that $\mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{\infty} c_i(t)\varphi_i(\mathbf{x})$, and for any number α, $\mathbf{A}^{\alpha}\mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{\infty} \lambda_i^{\alpha} c_i(t)\varphi_i(\mathbf{x})$, where the$\{\lambda_i\}$, are the corresponding eigenvalues of $\mathbf{A}$. Thus, in the last equation, we have that:

$$\langle \mathbf{J}(\mathbf{u},t),\mathbf{u}\rangle_{\mathbb{H}[\Omega]} \leqslant -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}^{2} + +\nu^{-1}c\left\|\mathbf{u}\right\|_{\mathbb{H}}^{3} + (\nu\lambda_1^{(1+\delta)})^{-1} f \left\|\mathbf{u}\right\|_{\mathbb{H}[\Omega]}.$$

Now choose the first eigenvalue λ_n , and number ω such that

(1) $\lambda_n^{-\delta/2}\left\|\mathbf{u}\right\|_{\mathbb{H}} \leqslant \left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \lambda_1^{-\delta/2}\left\|\mathbf{u}\right\|_{\mathbb{H}}$,

(2) $\lambda_1^{-\omega\delta/2}\left\|\mathbf{u}\right\|_{\mathbb{H}} \leqslant \left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \lambda_1^{-\delta/2}\left\|\mathbf{u}\right\|_{\mathbb{H}}$,

and let $\lambda_0^{-\delta/2} = \max\{\lambda_1^{-\omega\delta/2}, \lambda_n^{-\delta/2}\}$. It then follows that $-\lambda_0^{-\delta/2}\left\|\mathbf{u}\right\|_{\mathbb{H}} \geqslant -\left\|\mathbf{A}^{-\delta/2}\mathbf{u}\right\|_{\mathbb{H}}$. Thus, $\mathbf{J}(\cdot,t)$ will be 0-dissipative if

$$-\lambda_0^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}^{2} + \nu^{-1}c\left\|\mathbf{u}\right\|_{\mathbb{H}}^{3} + (\nu\lambda_1^{(1+\delta)})^{-1} f \left\|\mathbf{u}\right\|_{\mathbb{H}[\Omega]} \leqslant 0,$$

so that

$$\left\|\mathbf{u}\right\|_{\mathbb{H}[\Omega]}\left[\nu^{-1}c\left\|\mathbf{u}\right\|_{\mathbb{H}}^{2} - \lambda_0^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} + (\nu\lambda_1^{(1+\delta)})^{-1} f\right] \leqslant 0.$$

Since $\left\|\mathbf{u}\right\|_{\mathbb{H}} > 0$, we have that $\mathbf{J}(\cdot,t)$ is 0-dissipative if

$$\nu^{-1}c\left\|\mathbf{u}\right\|_{\mathbb{H}}^{2} - \lambda_0^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} + (\nu\lambda_1^{(1+\delta)})^{-1} f \leqslant 0.$$

Solving, we get that

$$\mathbf{u}_{\pm} = \tfrac{1}{2}\nu(c\lambda_0^{\delta})^{-1}\left\{1 \pm \sqrt{1 - (4c\lambda_0^{2\delta} f)\Big/(\nu^2\lambda_1^{(1+\delta)})}\right\} = \tfrac{1}{2}\nu(c\lambda_0^{\delta})^{-1}\left\{1 \pm \sqrt{1-\gamma}\right\},$$

where $\gamma = (4c\lambda_0^{2\delta} f)\Big/(\nu^2\lambda_1^{(1+\delta)})$. Since we want real distinct solutions, we must require that

$$\gamma = (4c\lambda_0^{2\delta} f)\Big/(\nu^2\lambda_1^{(1+\delta)}) < 1 \Rightarrow \nu^2\lambda_1^{(1+\delta)} > 4c\lambda_0^{2\delta} f \;\Rightarrow\; \nu > 2\lambda_0^{\delta}\lambda_1^{-(1+\delta)/2}(cf)^{1/2}.$$

It follows that, if $\mathbb{P}\mathbf{f} \neq \mathbf{0}$, then $\mathbf{u}_- < \mathbf{u}_+$, and our requirement that $\mathbf{J}$ is 0-dissipative implies that, since our solution factors as $(\|\mathbf{u}\|_{\mathbb{H}[\Omega]} - \mathbf{u}_+)(\|\mathbf{u}\|_{\mathbb{H}[\Omega]} - \mathbf{u}_-) \leq 0$, we must have that:

$$\|\mathbf{u}\|_{\mathbb{H}[\Omega]} - \mathbf{u}_+ \leq 0, \;\; \|\mathbf{u}\|_{\mathbb{H}[\Omega]} - \mathbf{u}_- \geq 0.$$

This means that whenever $\mathbf{u}_- \leq \|\mathbf{u}\|_{\mathbb{H}[\Omega]} \leq \mathbf{u}_+$, $\langle \mathbf{J}(\mathbf{u},t), \mathbf{u}\rangle_{\mathbb{H}[\Omega]} \leq 0$. (It is clear that when $\mathbb{P}\mathbf{f}(t) = \mathbf{0}, \mathbf{u}_- = \mathbf{0}$, and $\mathbf{u}_+ = \nu(c\lambda_0^{\delta})^{-1}$.)

Part 2): Now, for any $\mathbf{u}, \mathbf{v} \in \mathbb{H}(\Omega)$ with $\max(\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}[\Omega]}, \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}[\Omega]}) \leq \mathbf{u}_+$, we have that

$$\begin{aligned}
&\langle \mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}[\Omega]} = -\left\|\mathbf{A}^{-\delta/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}^2 \\
&- (1/2)\nu^{-1}\left\langle \mathbf{A}^{-(1+\delta)}[\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u}) + \mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v})], (\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{H}} \\
&\leqslant -\lambda_0^{-\delta}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + (1/2)c\nu^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2(\|\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{v}\|_{\mathbb{H}}) \\
&\leqslant -\lambda_0^{-\delta}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + c\nu^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2\mathbf{u}_+ \\
&= -\lambda_0^{-\delta}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + c\nu^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2\left(\tfrac{1}{2}\nu(c\lambda_0^{\delta})^{-1}\left\{1+\sqrt{1-\gamma}\right\}\right) \\
&= -\tfrac{1}{2}\lambda_0^{-\delta}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2\left\{1-\sqrt{1-\gamma}\right\} \\
&= -\alpha\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2, \;\; \alpha = \tfrac{1}{2}\lambda_0^{-\delta}\left\{1-\sqrt{1-\gamma}\right\}.
\end{aligned}$$

□

Theorem 9. *The operator* $\mathcal{A}(t) = \nu\mathbf{A}^{(1+\delta)}\mathbf{J}(\cdot,t)$ *is closed, uniformly dissipative and jointly continuous in* $\mathbf{u}$ *and* t. *Furthermore, for each* $t \in \mathbf{R}^+$ *and* $\beta > 0$, *Ran*$[I - \beta\mathcal{A}(t)] \supset \mathbb{B}[\Omega]$, *so that* $\mathcal{A}(t)$ *is m-dissipative on* $\mathbb{D}$.

Proof. Since $\mathbf{J}(\cdot,t)$ is strongly dissipative and closed on $\mathbb{B}[\Omega]$, it follows from Theorem 5 that *Ran*$[\mathbf{J}(\cdot,t)] \supset \mathbb{B}[\Omega]$.

To show that $\mathcal{A}(t) = \nu \mathbf{A}^{(1+\delta)} \mathbf{J}(\cdot, t)$ is uniformly dissipative, for $\mathbf{u}, \mathbf{v} \in \mathbb{B}_+(\Omega)$, we have

$$\langle \mathcal{A}(t)\mathbf{u} - \mathcal{A}(t)\mathbf{v}, (\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H}} = -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}}^2 - \tfrac{1}{2} \langle [\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u}) + \mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v})], (\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H}}$$

Now, from equation (4),

$$|\langle [\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u}) + \mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v})], (\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H}}| \leqslant c \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}} \left\{ \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}} \right\}.$$

We now have that (using$-\lambda_0^{-\delta} \geqslant -\lambda_1^{1/2}$)

$$\begin{aligned}
\langle \mathcal{A}(t)\mathbf{u} - \mathcal{A}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}} &\leqslant -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}}^2 + \tfrac{1}{2} c \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}} \left\{ \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}} \right\} \\
&= \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \left\{ -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} + \tfrac{1}{2} c \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}} [\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}] \right\} \\
&\leqslant \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}} \left\{ -\nu \lambda_1^{1/2} + \tfrac{1}{2} c [\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}] \right\} \\
&\leqslant \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}} \left\{ -\nu \lambda_0^{-\delta} + \tfrac{1}{2} c [\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}] \right\} \\
&\leqslant \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}} \left\{ -\nu \lambda_0^{-\delta} + \tfrac{1}{2} \nu \lambda_0^{-\delta} \left[1 + \sqrt{1-\gamma} \right] \right\} \\
&= \tfrac{1}{2} \nu \lambda_0^{-\delta} \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}} \left\{ -1 + \sqrt{1-\gamma} \right\} < 0
\end{aligned}$$

If we set $a\left(\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}[\Omega]} \right) = -\tfrac{1}{2} \nu \lambda_0^{-\delta} \left[-1 + \sqrt{1-\gamma} \right] \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}[\Omega]}$, we have that:

$$\langle \mathcal{A}(t)\mathbf{u} - \mathcal{A}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}[\Omega]} \leqslant -a\left(\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}[\Omega]} \right) \|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}[\Omega]}.$$

It follows that $\mathcal{A}(t)$ is uniformly dissipative. Since $-\mathbf{A}^{(1+\delta)}$ is m-dissipative, for $\beta > 0$, $Ran(I + \beta \mathbf{A}^{(1+\delta)}) = \mathbb{H}(\Omega)$. As $\mathbf{J}$ is strongly dissipative, closed, with $Ran[\mathbf{J}] \supset \mathbb{B}[\Omega]$, and $\mathbf{J}(\cdot, t) : \mathbb{D} \xrightarrow{onto} \mathbb{D}$, $\mathcal{A}(t)$ is maximal dissipative, and also closed, so that $Ran[I - \beta \mathcal{A}(t)] \supset \mathbb{B}[\Omega]$. It follows that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{B}[\Omega]$ for each $t \in \mathbf{R}^+$ (since $\mathbb{H}[\Omega]$ is a Hilbert space). To see that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables, let $\mathbf{u}_n, \mathbf{u} \in \mathbb{B}_+(\Omega)$, $\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} \to 0$, with $t_n, t \in I$ and $t_n \to t$. Then (see equation (6))

$$\|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{H}} \leqslant \|\mathcal{A}(t_n)\mathbf{u} - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{H}} + \|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t_n)\mathbf{u}\|_{\mathbb{H}}$$

$$= \|[\mathbb{P}\mathbf{f}(t_n) - \mathbb{P}\mathbf{f}(t)]\|_{\mathbb{H}} + \|\nu\mathbf{A}(\mathbf{u}_n - \mathbf{u}) + [\mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n) + \mathbf{B}(\mathbf{u}, \mathbf{u}_n - \mathbf{u})]\|_{\mathbb{H}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + \|\mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n) + \mathbf{B}(\mathbf{u}, \mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + c_1\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}\,\{\|\mathbf{A}\mathbf{u}_n\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}\}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + 2c_1\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}\,\mathbf{u}_+.$$

It follows that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables. □

Since $\mathbb{B}_+(\Omega)$ is the closure of $\mathbb{D} = D(\mathbf{A}) \cap \mathbb{B}[\Omega]$ equipped with the restriction of the graph norm of $\mathbf{A}$ induced on $D(\mathbf{A})$, it follows that $\mathbb{B}_+(\Omega)$ is a closed, bounded, convex set. We now have:

Theorem 10. *For each $T \in \mathbf{R}^+$, $t \in (0,T)$ and $\mathbf{u}_0 \in \mathbb{D} \subset \mathbb{B}[\Omega]$, the global in time Navier-Stokes initial-value problem in $\Omega \subset \mathbb{R}^3$:*

$$\begin{aligned} &\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T)\times\Omega, \\ &\nabla\cdot\mathbf{u} = 0 \text{ in } (0,T)\times\Omega, \\ &\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega. \end{aligned} \tag{8}$$

has a unique strong solution $\mathbf{u}(t,\mathbf{x})$, which is in $L^2_{loc}[[0,\infty);\mathbb{H}^2(\Omega)]$ and in $L^{\infty}_{loc}[[0,\infty);\mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty);\mathbb{H}(\Omega)]$.

Proof. Theorem 6 allows us to conclude that when $\mathbf{u}_0 \in \mathbb{D}$, the initial value problem is solved and the solution $\mathbf{u}(t,\mathbf{x})$ is in $\mathbb{C}^1[(0,\infty);\mathbb{D}(\Omega)]$. Since $\mathbb{D} \subset \mathbb{H}^2[\Omega]$, it follows that $\mathbf{u}(t,\mathbf{x})$ is also in $\mathbb{V}(\Omega)$, for each $t>0$. It is now clear that for any $T>0$,

$$\int_0^T \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{H}[\Omega]}^2 \, dt < \infty, \text{ and } \sup_{0<t<T} \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{V}[\Omega]}^2 < \infty.$$

This gives our conclusion. □

Discussion

It is clear from our results that the stationary problem also has a unique solution in $\mathbb{B}_+[\Omega]$. It is also known that if $\mathbf{u}_0 \in \mathbb{V}$, and $\mathbf{f}(t)$ is $L^\infty[(0,\infty),\mathbb{H}]$ then there is a time $T>0$, such that a weak solution with this data is uniquely determined on any subinterval of $[0,T)$ (see Sell and You page 396, [SY]). Thus, we also have that:

Corollary 11. *For each $t \in \mathbf{R}^+$ and $\mathbf{u}_0 \in \mathbb{D}$ the Navier-Stokes initial-value problem in $\Omega \subset \mathbb{R}^3$:*

$$\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \ \textit{in} \ (0,T)\times\Omega, \\
&\nabla\cdot\mathbf{u} = 0 \ \textit{in} \ (0,T)\times\Omega, \\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \ \textit{on} \ (0,T)\times\partial\Omega, \\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \ \textit{in} \ \Omega.
\end{aligned} \tag{9}$$

has a unique weak solution $\mathbf{u}(t,\mathbf{x})$, which is in $L^2_{loc}[[0,\infty);\mathbb{H}^2(\Omega)]$ and in $L^\infty_{loc}[[0,\infty);\mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty);\mathbb{H}(\Omega)]$.

Since we require that our initial data be in $\mathbb{H}^2[\Omega]$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $\mathbf{u}_0 \in \mathbb{C}_0^\infty[\Omega]$ (see Giga [G], and references therein). The above Corollary shows that it suffices that $\mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2(\Omega)]$ to insure that the solutions develop no singularities.

If we drop the nonlinear term in equation (3), we get the inhomogeneous Stokes problem

$$\begin{aligned}
&\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T)\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0 \text{ in } (0,T)\times\Omega,\\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.
\end{aligned} \tag{10}$$

In this case, the problem is linear and it is easy to see that $\mathbf{u}_- = (\nu\lambda_1^{(1+\delta)})^{-1}\lambda_0^{\delta} f$ and, for all $\mathbf{u}$ with $\|\mathbf{u}\|_{\mathbb{H}(\Omega)} > \mathbf{u}_-$, we have that

$$\langle \mathbf{J}(\mathbf{u},t),\mathbf{u}\rangle \leqslant \|\mathbf{u}\|_{\mathbb{H}(\Omega)}\Big(\mathbf{u}_- - \|\mathbf{u}\|_{\mathbb{H}(\Omega)}\Big) < 0.$$

It follows that, since $\mathbf{J}(\cdot,t)$ is closed, it is maximal dissipative, and hence m-dissipative. It follows with a little more obvious work that the above problem has a unique solution $\mathbf{u}(t,\mathbf{x}) \in \mathbb{C}^1(I;\mathbb{D})$. Thus, our approach offers advantages even in the linear case.

In closing, we note that for the general semilinear problem

$$\partial_t \mathbf{u} = \mathbf{A}\mathbf{u} + \mathbf{f}(\mathbf{u},t),\ \mathbf{u}(\mathbf{0},\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),$$

where $-\mathbf{A}$ generates a contraction semigroup. We can introduce an "artificial viscosity" ν, and write the same equation as

$$\partial_t \mathbf{u} = \mathbf{A}_\nu \mathbf{J}_\nu(\mathbf{u}, t), \ \mathbf{u}(\mathbf{0}, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}),$$

$$\mathbf{A}_\nu \mathbf{u} = -(I - \nu \mathbf{A})\mathbf{u},$$

$$\mathbf{J}_\nu(\mathbf{u}, t) = -\mathbf{A}(I - \nu \mathbf{A})^{-1}\mathbf{u} + (I - \nu \mathbf{A})^{-1}\mathbf{f}(\mathbf{u}, t).$$

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